

3rd exercise set (25.9.)

Background theory

As explained during the lecture, the lower and upper bounds of a $100 \times (1 - \alpha) \%$ two-sided Clopper–Pearson confidence interval for π are defined as the solutions to the equations

$$\sum_{k=y}^n \binom{n}{k} \pi_0^k (1 - \pi_0)^{n-k} = \alpha/2 \quad \text{and} \quad \sum_{k=0}^y \binom{n}{k} \pi_0^k (1 - \pi_0)^{n-k} = \alpha/2$$

for $y = 1, \dots, n - 1$ (lower and upper bound, respectively). If $y = 0$ then the lower bound is simply 0, and likewise if $y = n$ then the upper bound is set at 1. The other limit is still solved from one or the other of the above equations. The bounds have analytical solutions for all values of y .

$100 \times (1 - \alpha) \%$ two-sided mid- p Clopper–Pearson confidence interval endpoints for π are the solutions to the equations

$$\frac{1}{2} \binom{n}{y} \pi_0^y (1 - \pi_0)^{n-y} + \sum_{k=y+1}^n \binom{n}{k} \pi_0^k (1 - \pi_0)^{n-k} = \alpha/2$$

and

$$\sum_{k=0}^{y-1} \binom{n}{k} \pi_0^k (1 - \pi_0)^{n-k} + \frac{1}{2} \binom{n}{y} \pi_0^y (1 - \pi_0)^{n-y} = \alpha/2$$

for $y = 1, \dots, n - 1$. If $y = 0$ then the lower edge is 0, and if $y = n$ then the upper edge is 1. The other edge is still solved from one or the other of the equations above. The edges do not have analytical solutions and they have to be solved numerically in general. Exceptions are cases $y = 0$ and $y = n$. The equation to be solved is then

$$\frac{1}{2} \binom{n}{0} \pi_0^0 (1 - \pi_0)^{n-0} = \frac{1}{2} (1 - \pi_0)^n = \alpha/2$$

(upper bound) or

$$\frac{1}{2} \binom{n}{n} \pi_0^n (1 - \pi_0)^{n-n} = \frac{1}{2} \pi_0^n = \alpha/2$$

(lower bound), respectively. These equations can be solved analytically.¹

¹A. Agresti (2013): *Categorical Data Analysis, 3rd edition*. CUP. Hoboken, NJ. Pages 603–605. A. Agresti and A. Gottard (2007): Nonconservative Exact Small-Sample Inference for Discrete Data. *Computational Statistics & Data Analysis*, 51, 6447–6458. R.G. Newcombe (2013): *Confidence Intervals for Proportions and Related Measures of Effect Size*. CRC. Boca Raton, FL. Pages 63–65.

1. Construction of a Clopper–Pearson confidence interval for a proportion π is explained on pages 603–605 of Agresti’s 2013 book. The true coverage probability $C_n(\pi)$ of a Clopper–Pearson confidence interval is determined by the formula

$$C_n(\pi) = \sum_{k=0}^n I(k, \pi) \binom{n}{k} \pi^k (1 - \pi)^{n-k}$$

(Agresti and Coull 1998)² and is plotted in Figure 16.1 of the book for sample size $n = 25$. Above $I(k, \pi)$ equals 1 if the confidence interval resulting from k successes covers π but takes value 0 otherwise. The reason for the peakedness of the coverage probability in the figure is explored. Both n and π are fixed.

a) There are $n + 1$ different possible confidence intervals. Explain why. (Hint: There are $n + 1$ outcomes associated with Binomial distribution $\text{Bin}(n, \pi)$.)

b) Explain why the above formula for the coverage probability holds.

c) Explain the peaks and smooth sections of the coverage probability in Figure 16.1. (Hint: Alter π slightly. Think what happens when the same confidence intervals cover the altered π and what happens when they do not.)

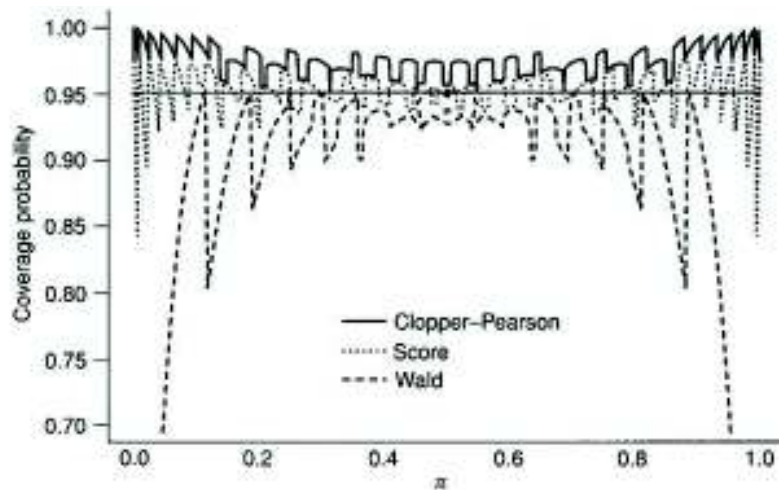


Figure 16.1 Plot of coverage probabilities for nominal 95% confidence intervals for binomial parameter π when $n = 25$.

²A. Agresti and B.A. Coull (1998): Approximate Is Better than "Exact" for Interval Estimation of Binomial Proportions. *American Statistician*, 52, 119–126.

2. $100 \times (1 - \alpha)$ % Clopper–Pearson confidence intervals can be calculated analytically by means of the formulae given in Agresti’s 2013 book (p. 603). One-sided Clopper–Pearson confidence intervals can be calculated especially conveniently if the number of successes or failures equals the number of independent Bernoulli experiments (n).

a) Derive the one-sided $100 \times (1 - \alpha)$ % Clopper–Pearson confidence interval

$$[0, 1 - \alpha^{1/n}]$$

in the case of 0 successes out of n independent Bernoulli experiments. (Hint: Argue that $\binom{n}{0} \pi^0 (1 - \pi)^{n-0} = \alpha$ and solve π .)

b) Derive the one-sided $100 \times (1 - \alpha)$ % Clopper–Pearson confidence interval

$$[\alpha^{1/n}, 1]$$

in the case of n successes out of n independent Bernoulli experiments. (Hint: Now set $\binom{n}{n} \pi^n (1 - \pi)^{n-n} = \alpha$ and solve π .)

3. Empiricists sometimes use the simple *rule of three* to calculate the upper limit of a one-sided 95 % confidence interval for the success probability π in case of no successes in the data. The upper limit is

$$\frac{3}{n}$$

according to the rule. Derive it by calculating a first order Maclaurin approximation (Taylor approximation around zero) for $1 - \alpha^{1/n}$ (the upper limit of the one-sided $100 \times (1 - \alpha)$ % Clopper–Pearson confidence interval in case of 0 successes) with respect to $1/n$ and evaluate the expansion at $\alpha = 0.05$. (Hints: $\partial a^x / \partial x = a^x \ln a$ for $a > 0$. Denote $n^* = 1/n$, and calculate the approximation for $1 - \alpha^{n^*}$.)

4. Palo-Repo (2015) modeled child custody decisions of the Helsinki District Court of Appeal 2003–2006.³ The social services have often conducted a child custody evaluation and given a recommendation which parent should become the custodial parent at whose address the child will be registered at. In Palo-Repos’s data the recommendation was for the mother in 34 cases. The Court decided the custodial dispute accordingly in favour of the mother in all of these cases. The Court does not follow the recommendation of the social services always: The father was recommended in 19 disputes but the decision was in favour of the father in 17 cases only. The 34 cases in which the mother was the recommended custodial parent are focused.

a) Calculate the 95 % Wald confidence interval for the probability that the Court does not decide in favour of the mother if the social service has recommended that the child should be registered to live with the mother. You can use the R code below for the calculation.

```
p <- 0
q <- 1-p
```

³Mari Palo-Repo (2015): Lasten huolto- ja asumisriidat Helsingin hovioikeudessa 2003–2006. Master’s thesis (statistics). Faculty of Social Sciences, University of Helsinki. <https://helda.helsinki.fi/handle/10138/155254> (read 19.9.2015).

```
n <- 34
Wl <- p-1.960*sqrt(p*q/n)
Wu <- p+1.960*sqrt(p*q/n)
Wl
Wu
```

b) Calculate the 95 % score confidence interval for the same probability. You can do it with the command `prop.test(0, 34, correct=FALSE)`, use the script at <http://www.stat.ufl.edu/~aa/cda/R/one-sample/R1/index.html> by Alan Agresti or use the code below.

```
n <- 34
p <- 0
z <- 1.960
Wl <- p*n/(n+z^2)+0.5*(z^2)/(n+z^2)-(z/(n+z^2))
*sqrt(p*(1-p)*n+0.25*(z^2))
Wu <- p*n/(n+z^2)+0.5*(z^2)/(n+z^2)+(z/(n+z^2))
*sqrt(p*(1-p)*n+0.25*(z^2))
Wl
Wu
```

c) Calculate the 95 % Clopper–Pearson confidence interval for the same probability. For the present data it can be obtained from the formula $1 - 0.025^{1/34}$. It can be obtained with the command `binom.test(0, 34)` or with the script at <http://www.stat.ufl.edu/~aa/cda/R/one-sample/R1/index.html> as well.

d) Calculate the 95 % mid-*p* Clopper–Pearson confidence interval for the same probability (Agresti 2013, 605). You can use the numerical routine `midPci` by Anna Gottard⁴ at <http://www.stat.ufl.edu/~aa/cda/R/one-sample/R1/index.html>:

```
midPci <- function(x,n,alpha){
  pp<-seq(0.0001, 1 , 0.0005)
  uplim<-1
  lowlim<-0
  if (x == 0)
    uplim <- 1-alpha^(1/n)
  if (x == n)
    lowlim <- (alpha)^(1/n)
  if (x>0 & x<n){
    a2 <- 0.5*pbinom(x-1, n , pp,lower.tail = T) +
0.5*pbinom(x, n , pp, lower.tail = T, log.p = FALSE)
    uplim=pp[ max(which(a2>(alpha/2))) ]
    lowlim=pp[ min(which(a2<(1-alpha/2))) ]
  }
  c(lowlim,uplim)
}
```

(Hint: Read the script in R, and give the command `midPci(0, 34, 0.05)`.)

e) Calculate the rule of three confidence interval for the same probability.

f) Compare the widths, properties and meaningfulnesses of the intervals for this particular data.

⁴A. Agresti and A. Gottard (2007): Nonconservative Exact Small-Sample Inference for Discrete Data. *Computational Statistics & Data Analysis*, 51, 6447–6458.

P.S. Confidence intervals for a proportion, derived by the above or other means, can be calculated also with R packages `PropCIs` by Ralph Scherer and `binom` by Sundar Dorai-Ra or with the scripts at http://statistics.unl.edu/faculty/bilder/categorical/programs_and_data.html by Chris Bilder and Tom Loughin.

5. Let the null hypothesis be that the parameter of the binomial distribution is $\pi = \pi_0$ and the sample size is n . The MLE of π is $\hat{\pi}$. Rao's score test statistic is then

$$z_s = \frac{\hat{\pi} - \pi_0}{\sqrt{\pi_0(1 - \pi_0)/n}}.$$

a) Prove that the χ^2 test statistic for independence

$$X^2 = \sum_{i=1}^2 \frac{(n_i - \mu_i)^2}{\mu_i}$$

is z_s^2 if X^2 is calculated from a 1×2 table ($n_1 + n_2 = n$).

b) What is the asymptotic distribution of z_s^2 and X^2 ?

c) Does it make any difference, if one or the other statistic is used?

6. The χ^2 test statistic for independence

$$X^2 = \sum_{i=1}^2 \sum_{j=1}^2 \frac{(n_{ij} - \hat{\mu}_{ij})^2}{\hat{\mu}_{ij}}$$

is calculated from the 2×2 table

n_{11}	n_{12}	n_{1+}
n_{21}	n_{22}	n_{2+}
n_{+1}	n_{+2}	n

Above $\hat{\mu}_{ij} = n_{i+}n_{+j}/n$.

a) Prove first that

$$(n_{ij} - \hat{\mu}_{ij})^2 = \frac{(n_{11}n_{22} - n_{12}n_{21})^2}{n^2}.$$

Does the value of $(n_{ij} - \hat{\mu}_{ij})^2$ depend on the indices i or j ? Explain the intuition for your answer. (Hint: The estimates of the expected frequencies sum row- and columnwise to the marginal frequencies.)

b) Prove next that

$$\sum_{i=1}^2 \sum_{j=1}^2 \frac{1}{\hat{\mu}_{ij}} = \frac{n^3}{n_{1+}n_{2+}n_{+1}n_{+2}}.$$

c) Reason now that X^2 can be expressed conveniently this way:

$$X^2 = \frac{n(n_{11}n_{22} - n_{12}n_{21})^2}{n_{1+}n_{2+}n_{+1}n_{+2}}.$$