Stellar magnetic activity, Spring 2024
Models for exercises II

1. Show that the projected rotational velocity $v \sin i$ of a star will be constant along straight parallel lines over the visible stellar disk if the star rotates as a solid body. How will this change if there is differential rotation? Use the solar surface differential rotation curve as an example.

Solid body rotation

Let us take a star of radius R showing solid body rotation. The angular velocity vector of the star is denoted with $\boldsymbol{\Omega}$ and is rotated along the yz-plane by an angle i away from the xz-plane (see Fig. 1).

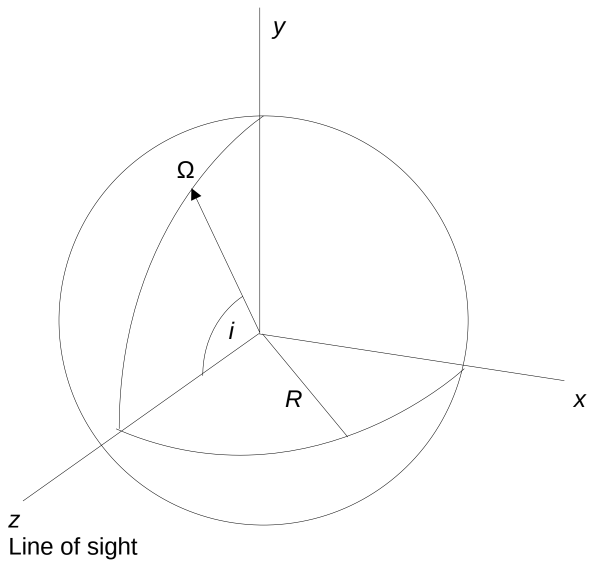


Figure 1

Of any stellar surface element with coordinates (x, y, z), the velocity $\mathbf{v} = (v_x, v_y, v_z)$ is

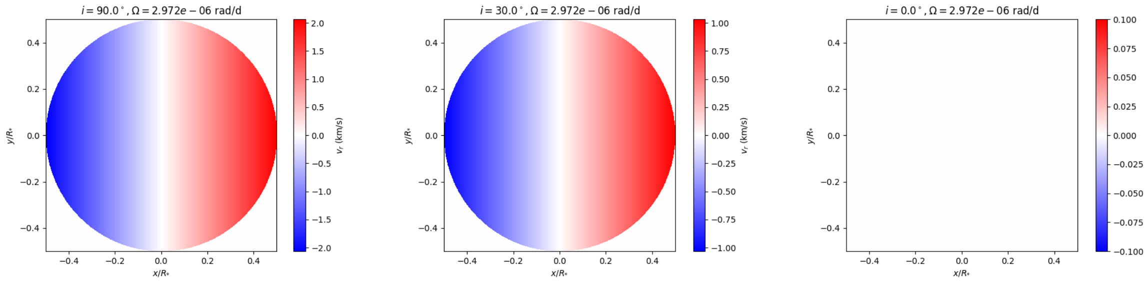
$$\begin{aligned} \mathbf{v} &= \boldsymbol{\Omega} \times \mathbf{R} \\ &= \begin{vmatrix} v_x & v_y & v_z \\ 0 & \Omega \sin i & \Omega \cos i \\ x & y & z \end{vmatrix} \\ &= (z\Omega \sin i - y\Omega \cos i)v_x + x\Omega \cos i v_y - x\Omega \sin i v_z \,. \end{aligned} \tag{1}$$

From Fig. 1, we see that the v_z-component corresponds to the radial velocity v_r. Since we define the radial velocity as being positive if an object moves away from the observer, we change the sign of the v_z component, so that

$$v_r = x\Omega \sin i \,. \tag{2}$$

For a single, solid body rotating star, Ω and i are constants, therefore the radial velocity only changes with x, along parallel straight lines.

The plots in Fig. 2 show the radial velocities for the surface elements on a 1000×1000 grid for inclinations $i = 90°, 30°, 0°$, a solid body rotating star with radius $R_\odot$ and an angular velocity equal to the equatorial angular velocity of the Sun.



(a) (b) (c)

Figure 2: Radial velocities of the surface elements of a solid body rotating $R = R_\odot$ star with $\Omega = 2.972\ \mu\text{rad}\,\text{s}^{-1}$ observed at inclinations (a) $i = 90°$, (b) $i = 30°$, (c) $i = 0°$. Notice the difference in the scale of the images. At $i = 0°$, we observe the star pole-on, so all surface elements have a radial velocity of zero.

Differential rotation

Differential rotation brings in a dependence on the stellar latitude θ_* to the angular velocity, which is often modelled as

$$\Omega(\theta_*) = A + B\sin^2\theta_* + C\sin^4\theta_*\,, \tag{3}$$

where the last term is sometimes excluded in modelling. Setting $\theta_* = 0$, we can see that A is the angular velocity at the equator. For the Sun, $A = 2.972\ \mu\text{rad}\,\text{s}^{-1}$, $B = -0.484\ \mu\text{rad}\,\text{s}^{-1}$, and $C = -0.361\ \mu\text{rad}\,\text{s}^{-1}$ (Snodgrass and Ulrich, 1990).

The latitude of a surface element can be calculated with the following steps:

(a) We use the coordinate system from Fig. 1, and change it to spherical coordinates, where the surface element with the highest y-value corresponds to $\theta = 90°$. For simplicity's sake, let us define the system so that $\lambda = 0$ corresponds to the zy-plane with positive z values and $R = 1$.

(b) The coordinates of some surface element (x, y, z) are to be converted to spherical coordinates. By choosing $r = \sqrt{x^2 + y^2 + z^2} = 1$, and keeping in mind the alignment of axes from Fig. 1, we can convert the coordinates of a surface element as

$$\begin{aligned} z &= \cos x \cos y \\ \lambda &= \arctan\left(\frac{x}{z}\right) \\ \theta &= \arccos\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right). \end{aligned} \tag{4}$$

(c) From Fig. 1 and the definition in 1a, the coordinates of the stellar pole are

$$\begin{aligned} \lambda_p &= 0 \\ \theta_p &= i\,. \end{aligned} \tag{5}$$

(d) The angular distance of the surface element of coordinates (λ, θ) from the stellar pole (λ_p, θ_p) is:

$$\Delta\sigma = \arccos(\sin\theta\sin\theta_p + \cos\theta\cos\theta_p\cos\Delta\lambda), \tag{6}$$

where $\Delta\lambda = \lambda - \lambda_p = \lambda$.

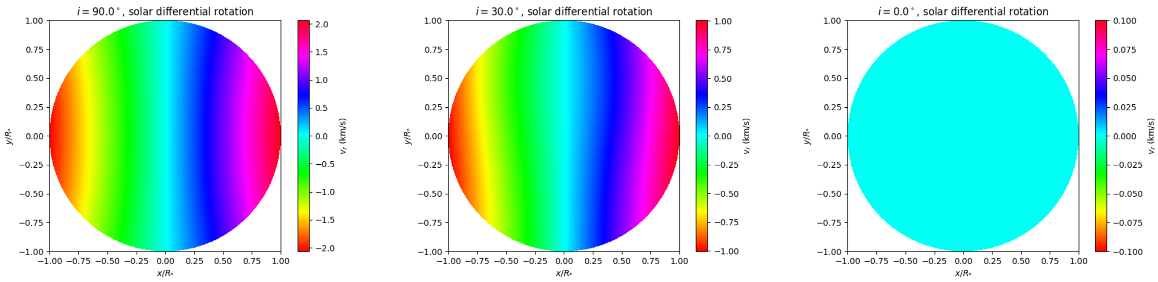
The latitude of each surface element with respect to the stellar pole is

$$\theta_{x,y} = \pi/2 - \Delta\sigma \,. \tag{7}$$

Here $\theta_{x,y}$ is the same as θ_* in Eg. 3. The radial velocity of each surface element is then in an updated form of Eq. 2:

$$v_r = x\,\Omega(\theta_{x,y})\sin i \,. \tag{8}$$

The radial velocities of the surface elements on the Sun, observed at inclinations $i = 90°, 30°, 0°$ are shown on Fig. 3.



(a) (b) (c)

Figure 3: Radial velocities of the surface elements of a differentially rotating star with $R = R_\odot$, $A = 2.972\ \mu\,\mathrm{rad\,s^{-1}}$, $B = -0.484\ \mu\,\mathrm{rad\,s^{-1}}$, and $C = -0.361\ \mu\,\mathrm{rad\,s^{-1}}$, observed at inclinations (a) $i = 90°$, (b) $i = 30°$, (c) $i = 0°$. Similarly as in Fig. 2, at $i = 0°$, the radial velocity of each surface element is zero.

2. The $v \sin i$, the photometric period P_{rot}, the parallax p and the UBV-magnitudes of an active star are known. Describe how you could estimate its radius R_* and inclination i of rotation axis.

   To estimate the radius, we can use the Barnes-Evans relation between the stellar angular diameter and visual surface brightness Barnes et al. (1978). They estimate the surface brightness of a star F_V as

$$F_V = 4.2207 - 0.1V_0 - 0.2\log\phi' \,, \tag{9}$$

   where V_0 is the unreddened apparent magnitude in Johnson's UBV photometric system and ϕ' is the stellar angular diameter in milliarcseconds. F_V can be estimated from the $(B-V)_0$ (unreddened) color index of a star as

$$F_V = \begin{cases} 3.897 - 1.010(B-V)_0\,, & -0.32 \le (B-V)_0 \le -0.10 \\ 3.964 - 0.333(B-V)_0\,, & -0.10 \le (B-V)_0 \le 1.35\,. \end{cases} \tag{10}$$

   Combining Eqs. 9 and 10, we obtain the stellar angular diameter ϕ', which, if the parallax p of the star is known, can be converted to the stellar radius:

$$R_* = \frac{1}{p}\tan\left(\frac{\phi'}{2}\right)\,\mathrm{au}\,. \tag{11}$$

A multiplication with an astronomical unit (au $= 1.496 \times 10^{11}$ m) is necessary due to the way parallaxes are defined:

$$\tan p = \frac{1 \text{ au}}{d}\,, \tag{12}$$

where d is the distance of the star.

The stellar radius can also be estimated using Stefan-Boltzmann's law for black body radiation combined with the absolute magnitude and the colour dependent bolometric correction of the star.

The rotation period and rotation velocity v are connected by $v = \frac{2\pi}{P_{\text{rot}}} R_*$.

Given the known stellar radius R_*, the rotational period P_{rot}, and the projected rotational velocity $v \sin i$, the inclination i has to fulfil

$$\sin i = \frac{P_{\text{rot}} v \sin i}{2\pi R_*}. \tag{13}$$

3. In the lecture slides (lecture 4, slide 6.4.5) additional constraints for Doppler imaging are discussed. Make a suggestion for a "penalty function", which would prevent too high and too low values (eg. surface temperatures) in the solution.

   In order to constrain the solution between two values, any solution with values outside this range should be penalised. Taking temperatures as an example, we want a function that is 0 when the temperature of each surface element $T_i \in [T_{\text{min}}, T_{\text{max}}]$ and increases with the "distance" on both sides when T_i is outside this range. Since the inversion is based on numerical minimisation methods, it is preferable to have a smooth transition at the minimum and maximum values of the range. An example of such a function is

$$f_p(T) = \begin{cases} c\Sigma_i (T_i - T_{\text{min}})^2\,, & T < T_{\text{min}} \\ 0\,, & T_{\text{min}} \le T \le T_{\text{max}} \\ c\Sigma_i (T_i - T_{\text{max}})^2\,, & T > T_{\text{max}}\,, \end{cases} \tag{14}$$

   where c is a coefficient to weight the penalty function and Σ_i signifies the summing over all surface elements. The transition could also be bridged with a more linear growth/decline, or an integral of a double-sided Heaviside step function (Arfken, 1985) as suggested by (Hackman et al., 2001), but the latter may lead to numerical instabilities because of multiplication of very high numbers with numbers close to 0.

4. Why do ground based photometric light curves of stars with strong spot activity usually show a maximum of two clear minima?

   If we assume a star with uneven spot distribution, the observer can often divide its surface to two halves: a brighter (less spotted) and a dimmer (more strongly spotted) hemisphere. As the dimmer hemisphere rotates into view towards the observer, the star's brightness decreases. This results in one clear dip in brightness for every rotation of the star.

   If the spot structure of the star is more complex, the light curve of the star may show more than one dip per rotation. But the recovery of more than two clear minima would be challenging for ground based light curves, since the accuracy and resolution usually does not allow for the detection of smaller brightness changes. The only way a spot structure would influence the light curve for only a short phase interval, is if it is on the hemisphere of the non-visible rotation pole and near to the stellar limb. In this case, even a large spot structure will not cause a major dip in the light curve because of two reasons: Its projected size will still be small and the limb darkening will reduce its effect.

A fast spot evolution, where spots appear and disappear within the time they are on the visible hemisphere may result in more than two clear minima, but the average lifetime of spot groups is generally longer than one stellar rotation.

An analytical estimate for the minimum phase separation of two minima is presented by (Lehtinen et al., 2011). The result is that for typical ground based photometry, spots have to be separated by at least $\Delta\phi \sim 0.3$ to cause separate minima in the light curve. Thus three light curve minima are in theory possible, but unlikely.

References

G. Arfken. *Mathematical Methods for Physicists*. Academic Press, Inc., San Diego, third edition, 1985.

T. G. Barnes, D. S. Evans, and T. J. Moffett. Stellar angular diameters and visual surface brightnesses - III. An improved definition of the relationship. *Monthly Notices of the Royal Astronomical Society*, 183:285–304, May 1978. doi: 10.1093/mnras/183.3.285.

T. Hackman, L. Jetsu, and I. Tuominen. Surface imaging of HD 199178 (V1794 Cygni). , 374:171–181, July 2001. doi: 10.1051/0004-6361:20010682.

J. Lehtinen, L. Jetsu, T. Hackman, P. Kajatkari, and G. W. Henry. The continuous period search method and its application to the young solar analogue HD 116956. *Astronomy and Astrophysics*, 527:A136, Mar. 2011. doi: 10.1051/0004-6361/201015454.

H. B. Snodgrass and R. K. Ulrich. Rotation of Doppler Features in the Solar Photosphere. *Astrophysical Journal*, 351:309, Mar. 1990. doi: 10.1086/168467.