Basic dynamo theory concepts

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Learning goals for today

TO UNDERSTAND

- Magnetohydrodynamics concept
- Form of conservation and Maxwell’s equations under this approximation
- Non-dimensional parameters define which form to use
- (In the exercises to develop further understanding of the stellar case)
- Basic mechanisms related to large-scale dynamo action
- Important ingredients and constraints of it
- That large-scale dynamo action is not the full story
- Small-scale dynamo action can have important consequences

Agenda

- First 45 min: Very basic MHD concepts
  - Small exercises in groups
- 5 min break to stretch
- Second 45 min:
  - Mean-field theory (30 minutes)
    - Completing the large-scale dynamo picture
    - Adding SSD
  - Recap in the form of a modified imitation game (15min)

Magnetohydrodynamics: concept

- Roughly 90% of the visible baryonic matter is in a state called ***plasma***: *quasi-neutral ionized gas containing enough free charges to make electromagnetic effects important for its physical behaviour.*
- Reaction of differently charged ions to electromagnetic effects varies; do we need to follow each and every particle to understand dynamos?
- **Luckily not!** If the length scales concerning the dynamo are much larger than any other scale of interest (mean free path, gyroradius, Debye length, …) of the plasma, we can treat it ***macroscopically*** instead.
- We can assume that particles with varying charges can be subsumed into a ***fluid element***, which is neutral in charge, the average physical quantities describing its evolution.

Magnetohydrodynamics: basic equations

$$\frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u},$$

$$\rho\frac{D\boldsymbol{u}}{Dt} = \rho\boldsymbol{f} - \boldsymbol{\nabla}p + \boldsymbol{J}\times\boldsymbol{B} + \mu\boldsymbol{\nabla}^2\boldsymbol{u} + (\xi + \tfrac{1}{3}\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u},$$

Shear

$$\rho\frac{De}{Dt} = -p\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\nabla}\cdot k\boldsymbol{\nabla}\boldsymbol{T} + \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)^2 + \xi(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2 + \rho\eta\mu_0\boldsymbol{J}^2,$$

Bulk

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B} - \eta\mu_0\boldsymbol{J}).$$

$$\nu = \mu/\rho$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$$\eta = (\mu_0\sigma)^{-1}$$

Conductivity, resistivity, magnetic diffusivity

Effect of rotation?

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{f} - \frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u},$$

Replace

$$\frac{D}{Dt} \to \frac{D}{Dt} + \boldsymbol{\Omega} \times$$

$$\boldsymbol{u} \to \frac{D\boldsymbol{r}}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{r} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r}$$

What do you get?
Try out!

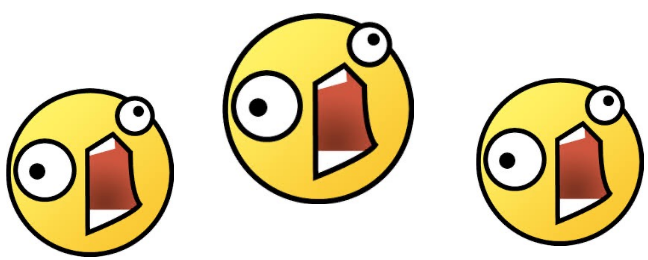
Effect of rotation?

$$\frac{D\boldsymbol{u}}{Dt} \rightarrow \frac{D}{Dt}(\boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r})$$

$$= \frac{D\boldsymbol{u}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

Coriolis Centrifugal

Usually the centrifugal part is omitted



Whatta flip?

Why and when the centrifugal force can be omitted?

Magnetohydrodynamics: non-dimensional parameters

$$\left|\frac{\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}}{\partial\boldsymbol{u}/\partial t}\right| \approx \frac{u^2\tau}{ul} = \frac{u\tau}{l} \equiv \mathrm{St},$$

$$\left|\frac{\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}}{2\,\boldsymbol{\Omega}\times\boldsymbol{u}}\right| \approx \frac{u^2}{2\,\Omega l u} = \frac{u}{2\,\Omega l} \equiv \mathrm{Ro} = \mathrm{Co}^{-1},$$

$$\left|\frac{\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}}{\boldsymbol{g}}\right| \approx \frac{u^2}{lg} \equiv \mathrm{Fr} = \mathrm{Ri}^{-1},$$

$$\left|\frac{\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}}{\nu\boldsymbol{\nabla}^2\boldsymbol{u}}\right| \approx \frac{u^2 l^2}{\nu l u} = \frac{ul}{\nu} \equiv \mathrm{Re},$$

$$\mathrm{Rm} \equiv \frac{ul}{\eta},$$

$$\mathrm{Ma} = \frac{u}{c_s},$$

$$\mathrm{Pm} \equiv \frac{\mathrm{Rm}}{\mathrm{Re}} = \frac{\nu}{\eta}.$$

$$\mathrm{Pr} \equiv \frac{\nu}{\chi},$$

Magnetohydrodynamics: unit systems: cgs and SI

Taulukko 1.2: Perusyksiköitä cgs- ja SI-yksiköissä.

Suure	cgs	SI
pituus	cm	10^{-2}m
massa	g	10^{-3}kg
aika	s	s
magneettikenttä	G	10^{-4}T

Often cgs units are used…

Magnetohydrodynamics: Spitzer formulae

$$\eta = 10^4 \left(\frac{T}{10^6\,\mathrm{K}}\right)^{-3/2} \left(\frac{\ln \Lambda}{20}\right) \mathrm{cm}^2\,\mathrm{s}^{-1}$$

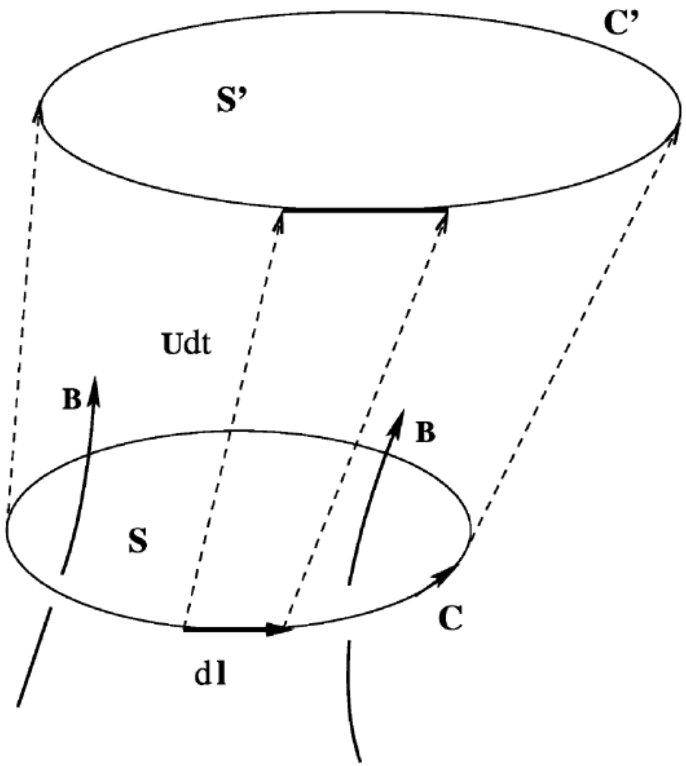
$$\nu = 6.5 \times 10^{22} \left(\frac{T}{10^6\,\mathrm{K}}\right)^{5/2} \left(\frac{n_i}{\mathrm{cm}^{-3}}\right)^{-1} \left(\frac{\ln \Lambda}{20}\right)^{-1} \mathrm{cm}^2\,\mathrm{s}^{-1}$$

$$P_\mathrm{m} \equiv \frac{\nu}{\eta} = 1.1 \times 10^{-4} \left(\frac{T}{10^6\,\mathrm{K}}\right)^{4} \left(\frac{\rho}{0.1\,\mathrm{g\,cm}^{-3}}\right)^{-1} \left(\frac{\ln \Lambda}{20}\right)^{-2}$$

$\ln \Lambda$ is the Coulomb logarithm, usually in the range 5…20. n_i is the number density of certain particle species.

Flux freezing and consequences

$$\nabla \times (\boldsymbol{U} \times \boldsymbol{B}) = -\underbrace{\boldsymbol{U} \cdot \nabla \boldsymbol{B}}_{\text{advection}} + \boxed{\underbrace{\boldsymbol{B} \cdot \nabla \boldsymbol{U}}_{\text{stretching}}} - \underbrace{\boldsymbol{B} \nabla \cdot \boldsymbol{U}}_{\text{compression}}$$



$$\Phi = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\int_S (\nabla \times \eta \boldsymbol{J}) \cdot \mathrm{d}\boldsymbol{S}$$

$$\boxed{\eta \to 0, \mathrm{d}\Phi/\mathrm{d}t \to 0}$$

“Flux is frozen to the fluid”

Brandenburg & Subramanian, 2005, pp 24-25

Flux freezing and consequences

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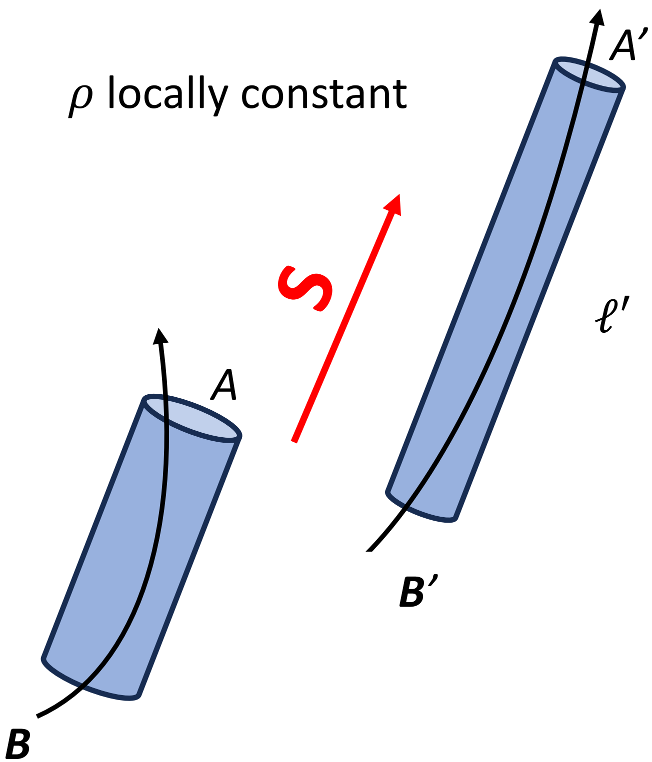
S

ρ locally constant



$\Phi = BA$ is conserved due to flux freezing

$\mathrm{m} = A\, \ell\, \rho$ is conserved due to mass conservation

Fluxtube gets thinner

Magnetic field is amplified proportional to the length increment.

Let us do some stretching!

Consider Cartesian coordinate system *(x,y,z)*

$\boldsymbol{B} = (B_0, 0, 0)$

$\boldsymbol{U} = (0, Sx, 0)$

$\eta = 0$

How is the field amplified?

Discuss and compute! Would this be enough of amplification to bring a primordial field of 10^{-20}G to the solar present-day mean value of 1G with S=1000/Gyr?

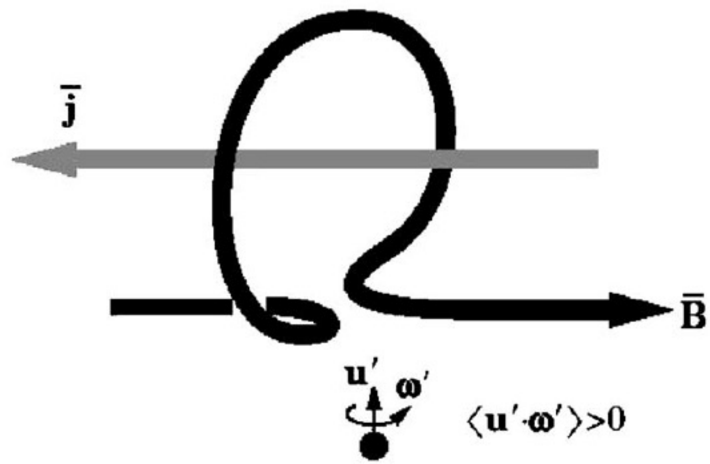
Helicity

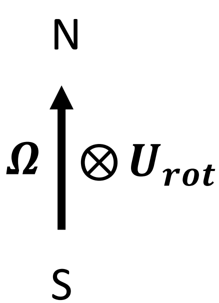
$$H_k = \int \boldsymbol{W} \cdot \boldsymbol{U} dV, \text{ where } \boldsymbol{W} = \boldsymbol{\nabla} \times \boldsymbol{U}$$

Arises in any system with stratification (gravity, buoyancy) and rotation
(pay attention to this in the exercises)

Has a very special role for dynamos, as it provides the needed “twist”
and non-axisymmetry to **crucially complement** the stretching.

Not conserved at the limit $\nu \to 0$.

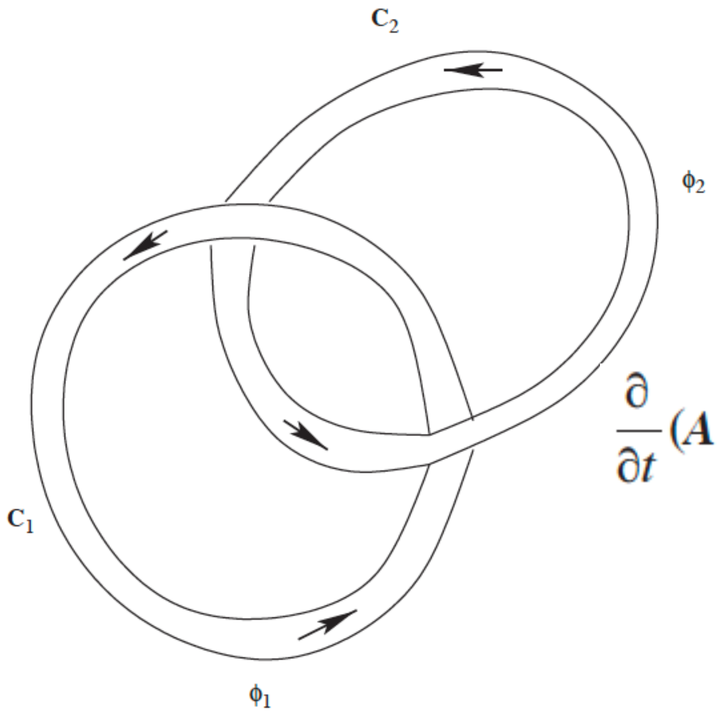




Discuss: on which hemisphere is this magnetic loop? Why?

Magnetic helicity

$H_M = \int \boldsymbol{A} \cdot \boldsymbol{B} dV$, where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$



Topological interpretation as linkage and twist of magnetic loops.

$$H_M = \Phi_1 \oint_{C_1} \mathrm{A} \cdot \mathrm{d}\boldsymbol{l} + \Phi_2 \oint_{C_2} \mathrm{A} \cdot \mathrm{d}\boldsymbol{l} = 2\Phi_1\Phi_2$$

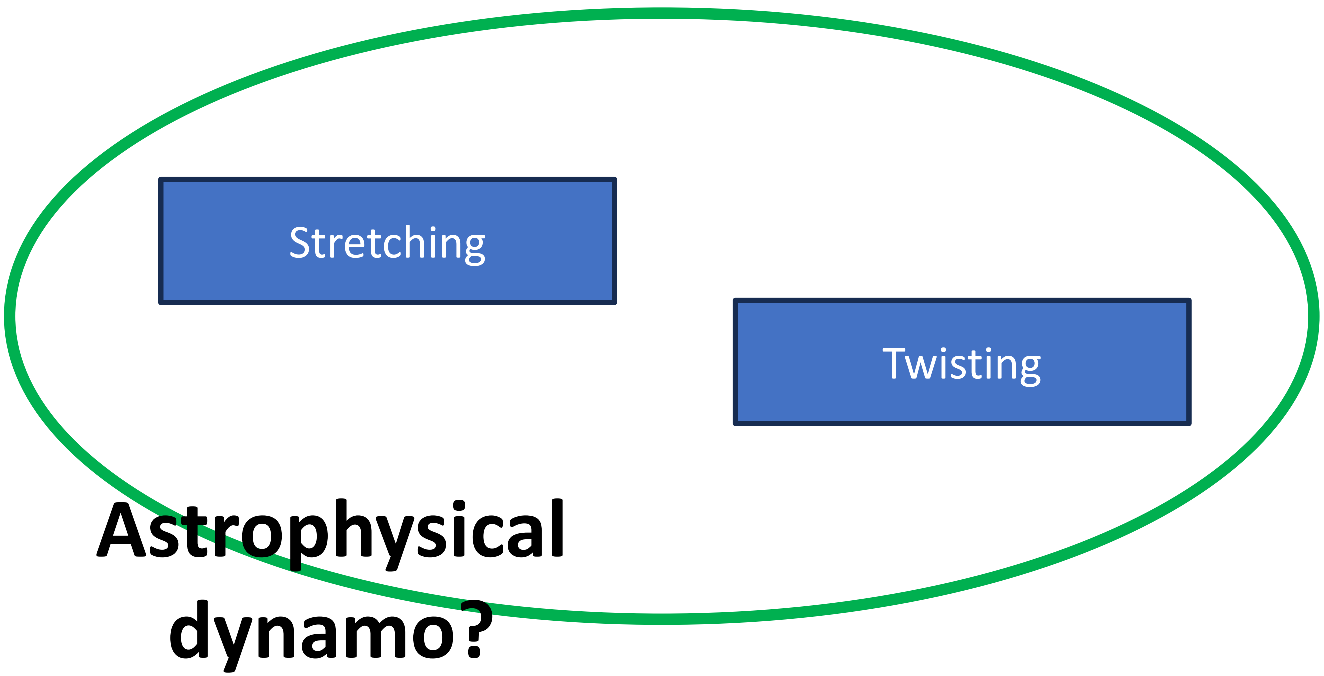
$$\begin{aligned}\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) &= (-\boldsymbol{E} + \nabla\phi) \cdot \boldsymbol{B} + \boldsymbol{A} \cdot (-\nabla \times \boldsymbol{E}) \\ &= -2\boldsymbol{E} \cdot \boldsymbol{B} + \nabla \cdot (\phi\boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{E}) \; . \\ &= -2\eta\mu_0 C\end{aligned}$$

$$C = \int_V \boldsymbol{J} \cdot \boldsymbol{B} \, \mathrm{d}V$$

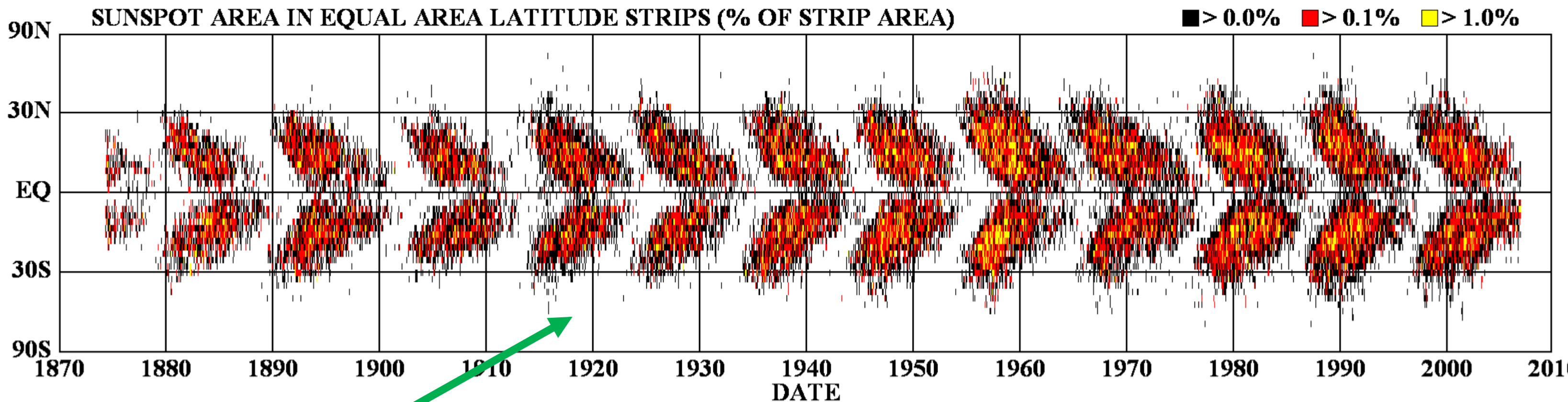
Can be shown to be conserved both at ideal ($\eta = 0$) and large Reynolds number systems ($\eta \rightarrow 0$). Very important constraing for large-scale dynamos!

Note that no velocity in the evolution equation!

Dynamo ingredients found so far…

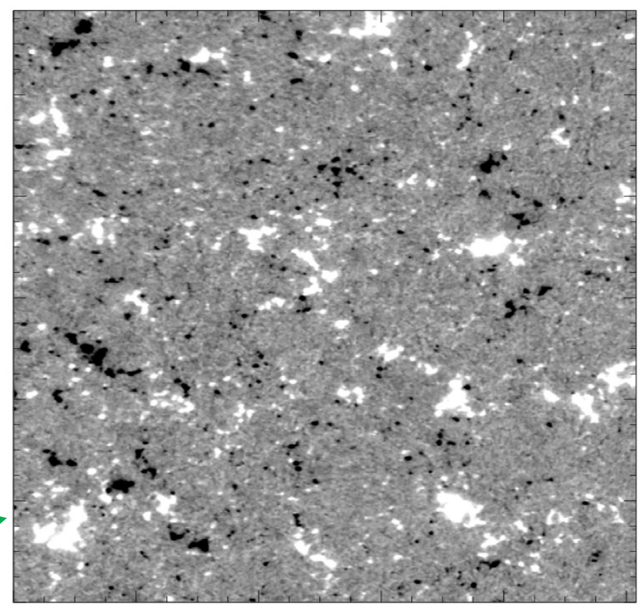


Basic solar observations show…



Coherent, large-scale fields manifested by the behavior of active regions, showing a pattern and a cycle

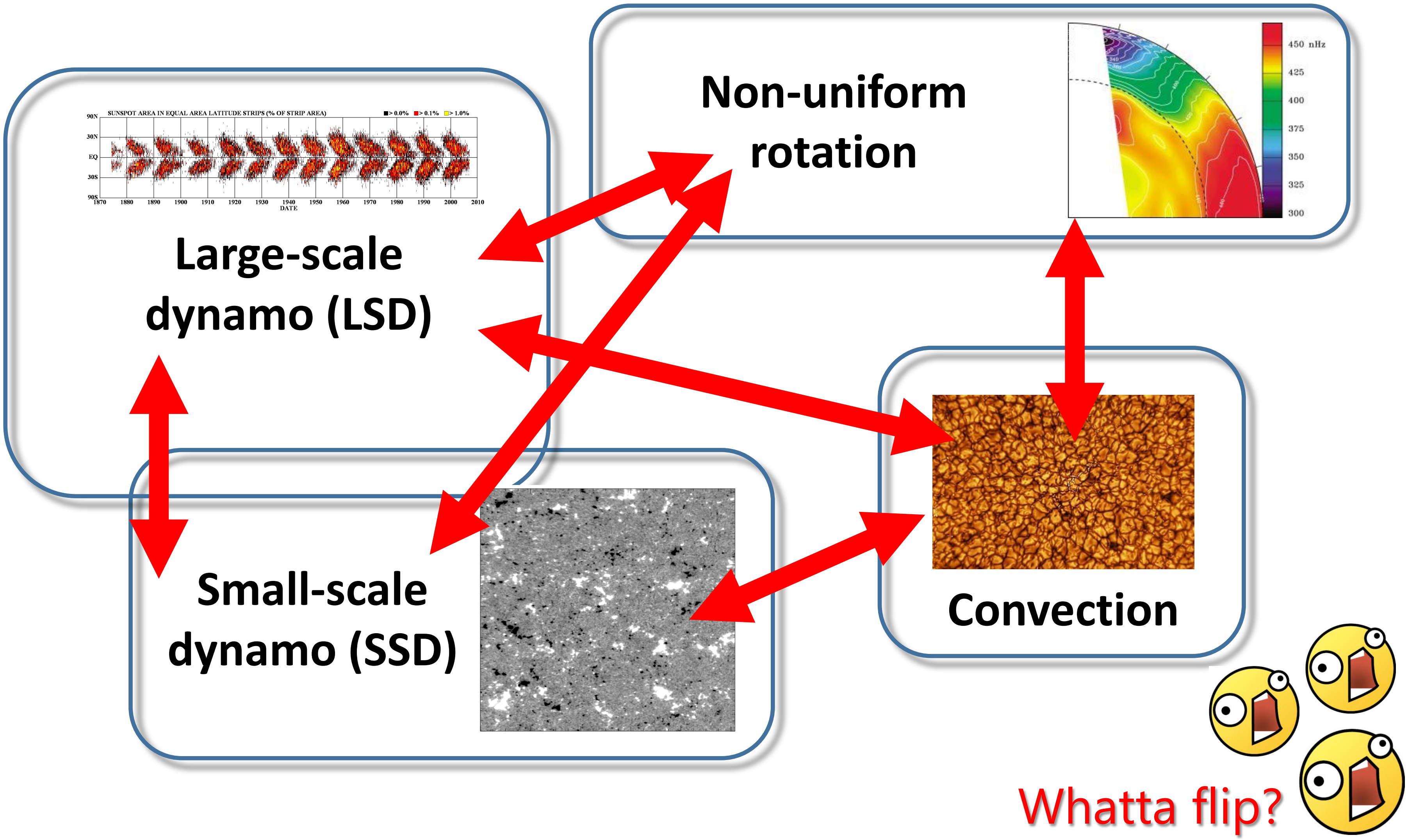
Cycle-independent fluctuations outside active regions



Quiet Sun LOS mgf
HINODE/SOT/NFI 2007

S. Régnier et al. (2008) A&A 484, L47–L50; image size roughly 100Mm, magnetic field strength +/-50G (white/black).

… that the theoretical picture is much more complex



How to approach the dynamo problem?

- Analytical theory is possible in very limited and simplified cases
- Numerical solutions of varying complication level are more useful (but none of these yet in a level of perfection)
  - Mean-field concept and models
  - Direct numerical simulations
  - Large-eddy simulations
  - Nested models
- Comparing data from observations and models to constrain the models
  - Most abundant data for the Sun
  - Less for other stars, but attempts can be made

Mean-field electrodynamics

$$\mathbf{B} = \overline{\mathbf{B}} + \mathbf{b}, \ \mathbf{U} = \overline{\mathbf{U}} + \mathbf{u}$$

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{\mathcal{E}}} - \eta \overline{\boldsymbol{J}} \right), \ \ \overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

- The equation for the large-scale mgf is called **the mean-field (MF) induction equation**
- It is an **exact equation**; no approximations made so far
- One only needs to remember to use correct type of averaging, for which the **Reynolds rules** hold.
  - Other mathematical averaging rules can be employed, but then the equation would not have the same form
  - Reynolds rules have the advantage that they guarantee that the mean and fluctuations have a meaningful definition and separation, and no other conditions have to be met.

The central closure problem

$$\overline{\mathcal{E}} = \overline{u \times b}$$

- How to compute the term involving the mean of the fluctuating fields?
- This term is generally called **the turbulent electromotive force (emf)**
- Without it, axisymmetric dynamo action is not possible (investigated in the exercises).
- If we now assume **scale separation**, that is that the mean fields are varying slowly in space **(L >> l)** and time **(T >> t)** in comparison to the fluctuating field, we can further expand in Taylor series and truncate

$$\overline{\mathcal{E}} = a \cdot \overline{B} + b \cdot \nabla \overline{B} + \ldots,$$

alpha effect **turb. pumping** **turb. diffusion** **shear-current & Rädler effects**

$$\overline{\mathcal{E}} = \alpha \cdot \overline{B} + \gamma \times \overline{B} - \beta \cdot (\nabla \times \overline{B}) - \delta \times (\nabla \times \overline{B}) - \kappa \cdot (\nabla \overline{B})^{(s)}$$

- Due to the truncation of higher gradients of mean mgf, effects at the large scales only are captured by this expression, and an error of the order l/L can be expected.
- **How to obtain the forms and magnitudes of the transport coeffs?**

The standard closure: FOSA/SOCA

$$\mathcal{E} = \boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} + \boldsymbol{\gamma}\times\overline{\boldsymbol{B}} - \boldsymbol{\beta}\cdot(\nabla\times\overline{\boldsymbol{B}}) - \boldsymbol{\delta}\times(\nabla\times\overline{\boldsymbol{B}}) - \boldsymbol{\kappa}\cdot(\nabla\overline{\boldsymbol{B}})^{(s)}$$

- The quest in closure models is to find analytical expressions for the turbulent transport coefficients going into the turbulent emf.
- The next task is to write down the equations for the fluctuating magnetic field, and decide what is reasonable to do.

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla}\times\left(\cancel{\overline{\boldsymbol{U}}\times\boldsymbol{b}} + \boldsymbol{u}\times\overline{\boldsymbol{B}} + \cancel{\boldsymbol{u}\times\boldsymbol{b}} - \cancel{\overline{\boldsymbol{\mathcal{E}}}} - \eta\boldsymbol{j}\right),$$

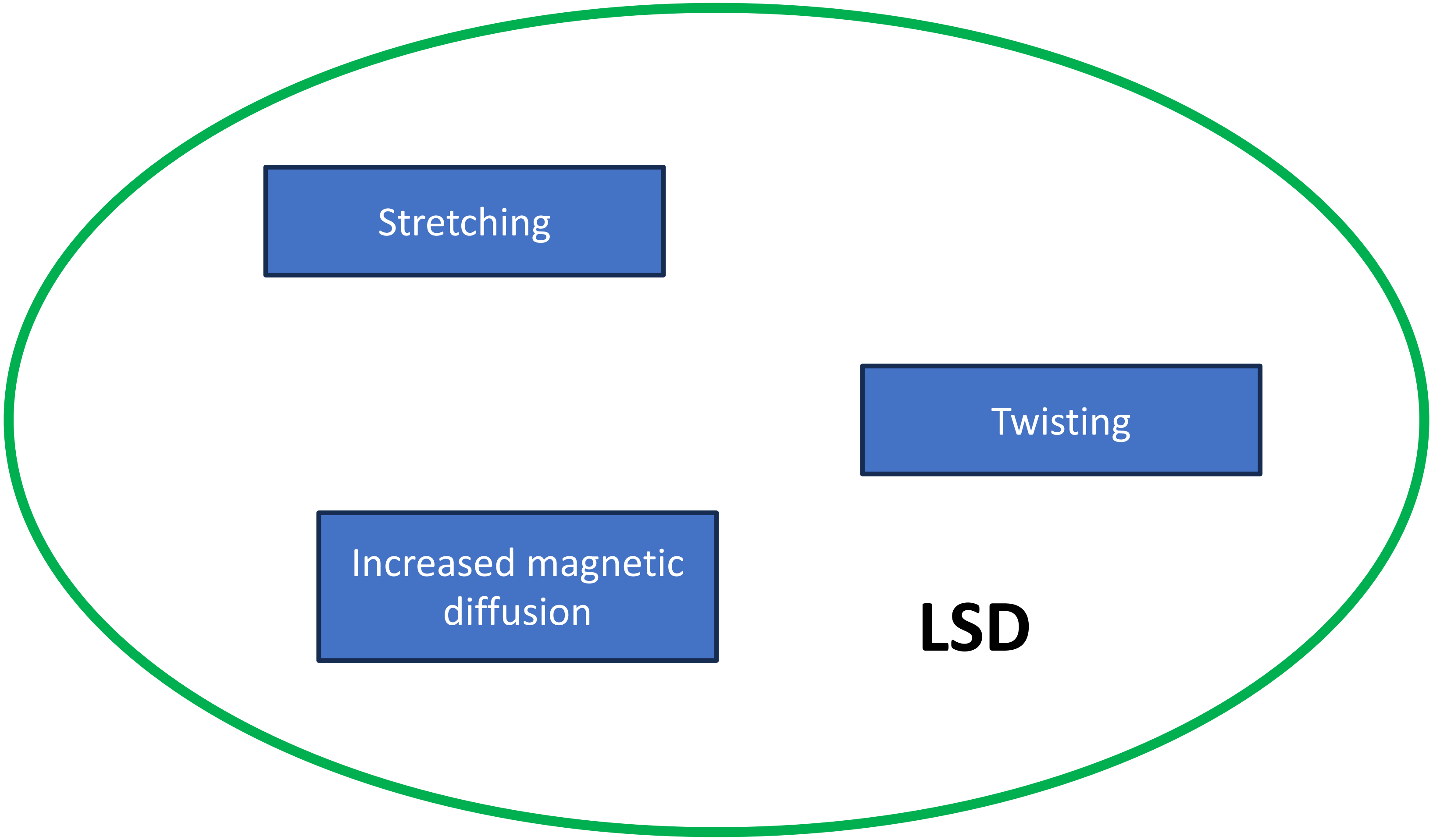
- In the so-called **first-order smoothing** or **quasi-linear approximation**,
  - non-linear terms in b are neglected; *b must remain small for all times*
- **Valid for small Rm or short correlation times, min(Rm,St) << 1. Exercises: try to judge whether these assumptions are good or bad**

$$\overline{\boldsymbol{\mathcal{E}}}(t) = \int_0^t \left[\hat{\alpha}(t-t')\overline{\boldsymbol{B}}(t') - \hat{\eta}_{\mathrm{t}}(t-t')\overline{\boldsymbol{J}}(t')\right]\,\mathrm{d}t',$$

$$\hat{\alpha}(t-t') = -\tfrac{1}{3}\overline{\boldsymbol{u}(t)\cdot\boldsymbol{\omega}(t')} \qquad \hat{\eta}_{\mathrm{t}}(t-t') = \tfrac{1}{3}\overline{\boldsymbol{u}(t)\cdot\boldsymbol{u}(t')}$$

Kinetic helicity **Intensity of turbulence**

Big kinematic picture



Higher-order closures

- Eddy Damped Quasi Normal Markovian statistical closure (**EDQNM**); fourth order moments damped with a relaxation term, to be presentable with the third order moments (Orzag; Pouquet, Frisch, Leorat)
- **Minimal tau approximation** emerged as a simplification of this approach; triple correlations presented with a relaxation term.

$$\frac{\partial \overline{\boldsymbol{\mathcal{E}}}}{\partial t} = \overline{\boldsymbol{u} \times \dot{\boldsymbol{b}}} + \overline{\dot{\boldsymbol{u}} \times \boldsymbol{b}}, \qquad \ldots \qquad \frac{\partial \overline{\boldsymbol{\mathcal{E}}}}{\partial t} = \tilde{\alpha}\, \overline{\boldsymbol{B}} - \tilde{\eta}_{\mathrm{t}}\, \overline{\boldsymbol{J}} - \frac{\overline{\boldsymbol{\mathcal{E}}}}{\tau},$$

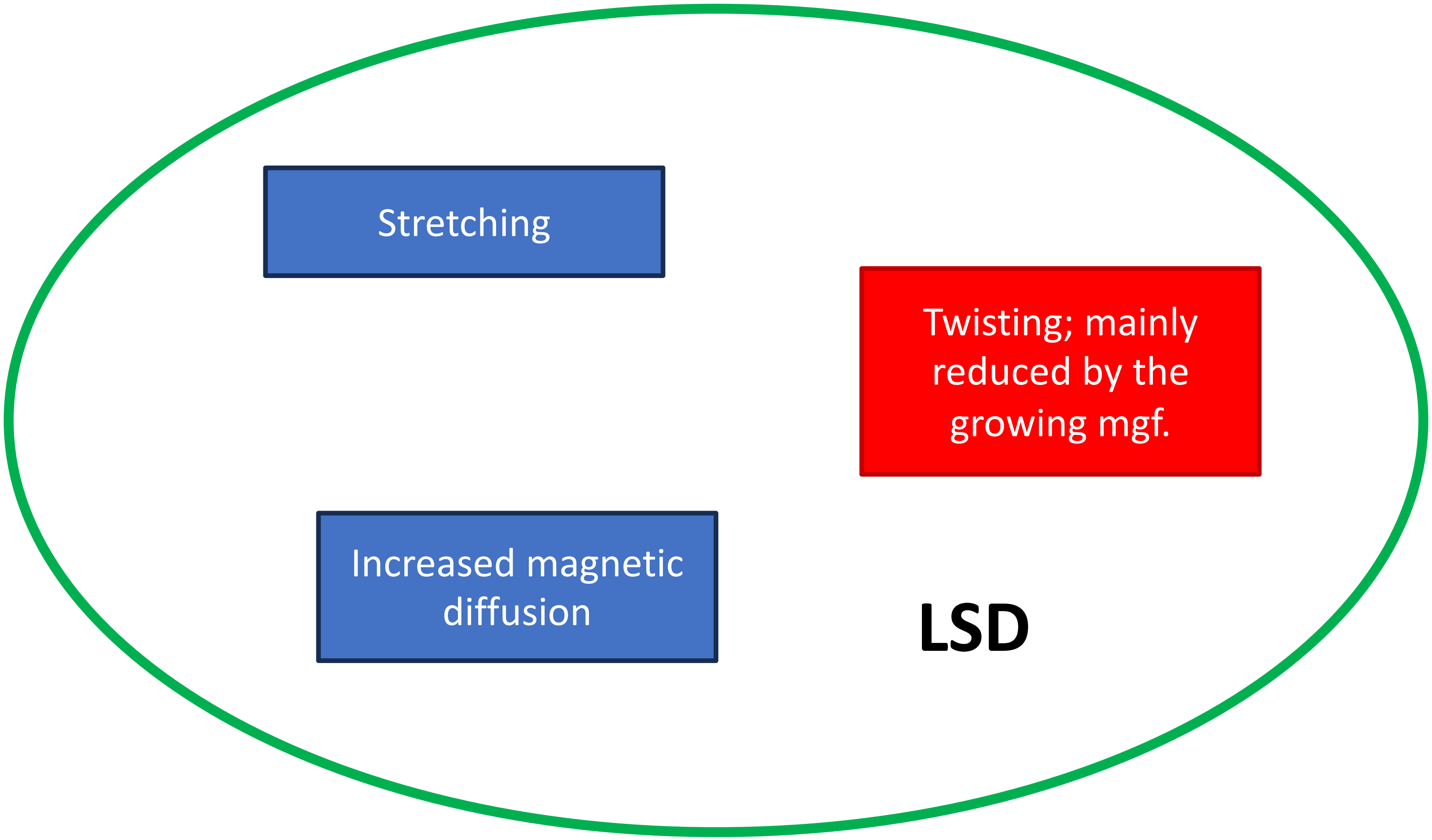
- **Extended validity; Rm, Re >> 1,** higher order moments retained
- **Criticism**: requires that the second order correlations should not vary over the correlation time of turbulence
- This formalism has the interesting consequence of leading to the **magnetic quenching of the alpha-effect**, when Lorentz force feedback is included

$$\tilde{\alpha} = -\tfrac{1}{3}\left(\overline{\boldsymbol{\omega} \cdot \boldsymbol{u}} - \overline{\boldsymbol{j} \cdot \boldsymbol{b}}\right), \quad \text{and} \quad \tilde{\eta}_{\mathrm{t}} = \tfrac{1}{3}\overline{\boldsymbol{u}^2},$$

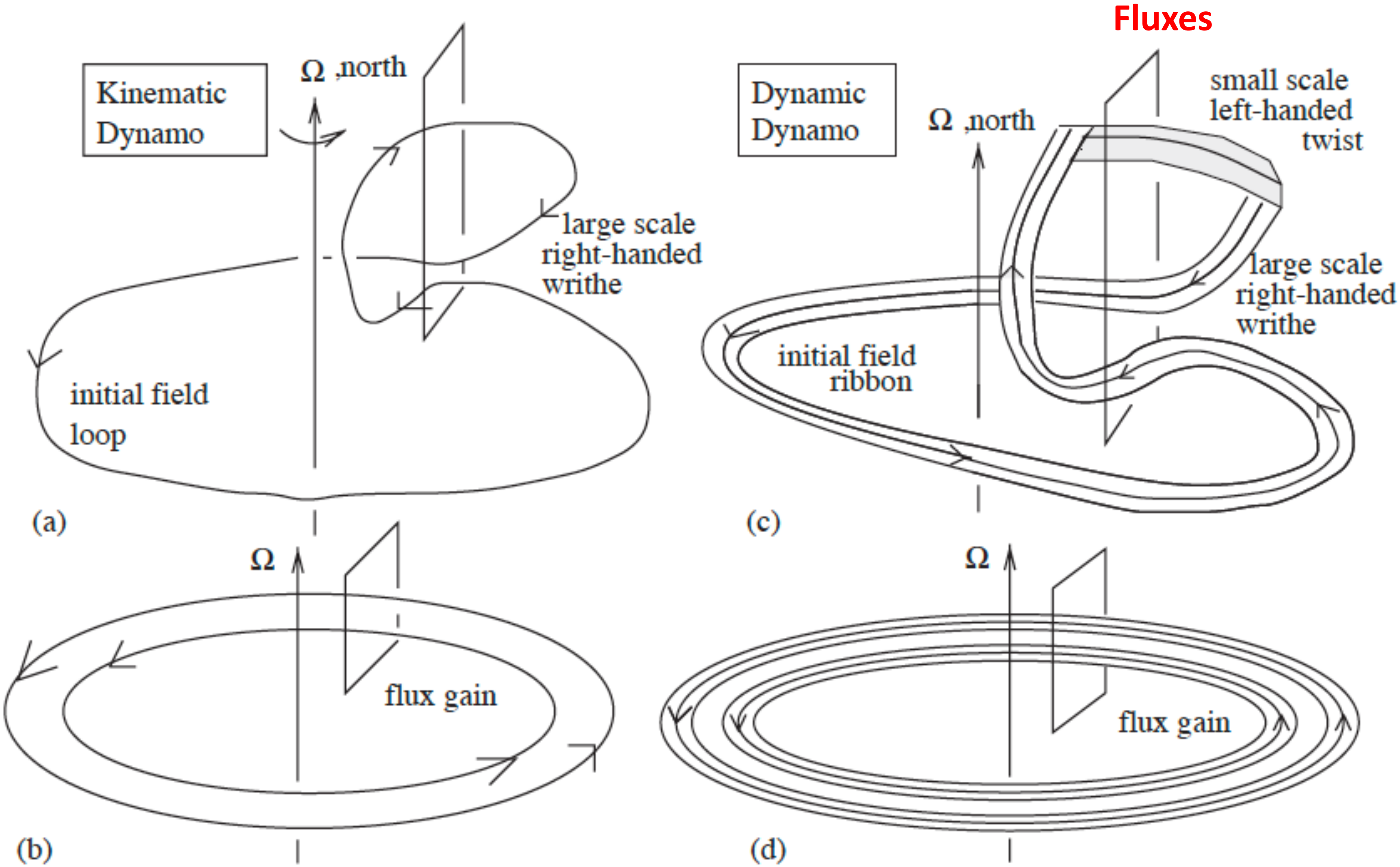
Quenching

- **Minimal tau approximation** already hints towards the need to modify the turbulent transport coefficients with the growing magnetic field.
- The nonlinear saturation of a dynamo is usually called the **quenching problem**: how to modify the inductive terms to describe the magnetic backreaction?
- The dynamo taking into account the nonlinear effects in call the dynamic dynamo
- They should obviously reduce (quench) by the growing magnetic field. How to describe this properly?
- Here the magnetic helicity conservation comes to play an important role: **whenever a helical dynamo operates, it is forced to generate large- and small-scale helicities of different signs to guarantee conservation.**
- **Small-scale magnetic helicity will quench the large-scale dynamo action in Rm-dependent manner, if no helicity fluxes occur.**

Big dynamic picture



Schematic dynamic dynamo



Dynamic dynamo with equation

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times [\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \alpha \overline{\boldsymbol{B}} - (\eta + \eta_t)\overline{\boldsymbol{J}}] \ ,$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -2\eta_t k_f^2 \left(\frac{\alpha \langle \overline{\boldsymbol{B}}^2 \rangle - \eta_t \langle \overline{\boldsymbol{J}} \cdot \overline{\boldsymbol{B}} \rangle}{B_{eq}^2} + \frac{\alpha - \alpha_K}{\tilde{R}_m} \right)$$

No flux

$$\frac{\partial \alpha}{\partial t} = -2\eta_t k_f^2 \left(\frac{\alpha \overline{\boldsymbol{B}}^2 - \eta_t \overline{\boldsymbol{J}} \cdot \overline{\boldsymbol{B}} + \frac{1}{2} k_f^{-2} \nabla \cdot \overline{\mathscr{F}}_C}{B_{eq}^2} + \frac{\alpha - \alpha_K}{R_m} \right)$$

Fluxes

Based on analytics, virtually all this quantities, especially the fluxes, remain unknown.

What about the SSD?

- A completely different kind of dynamo instability to LSD.
- Small-scale phenomenon that **does not** require **kinetic helicity** at all.
- Does not suffer from the **magnetic helicity constraint**
- Operates on much faster timescale than LSD
- Harder to excite than LSD, requires higher Rms to operate.
- Hard to excite especially in low Pm flows.
- Has been claimed to kill LSD, but this was an erraneous conclusion due to overlooking magnetic helicity conservation
- Hecticly studied; has the potential to solve problems that we are currently facing (next lectures).

Time to recap!

Rules

- Maarit asks a question
- Thomas/András answers it
- Students vote whether they think the answer was correct or not
- Maarit randomly picks up a student to justify their vote
- We discuss together what was the correct answer

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Next lectures

- Lecture 2: we develop MF theory further to the two currently prevailing dynamo paradigms
  - Discuss observational evidence against and for both
  - Discuss challenges in theory and how could they be resolved
  - Discuss theory of stellar differential rotation
  - Extend the solar dynamo to stellar dynamos
- Lecture 3: Full MHD simulations of various kinds
  - Discuss how they have alleviated the theoretical challenges
  - Discuss how they have advanced the understading of stellar magnetism
  - Remaining problems
  - Ways ahead