Interaction of light with cometary dust particles (1)

In general, interaction of electromagnetic wave with a target particle causes two phenomena: scattering and absorption.

However, when describing interaction of light with particle, very often, only the term "light scattering" is being used; whereas, it includes also absorption properties.

Light-scattering properties of a given target particle depend on its morphology, refractive index of the constituent material (it is not necessarily homogeneous and isotropic), and size.

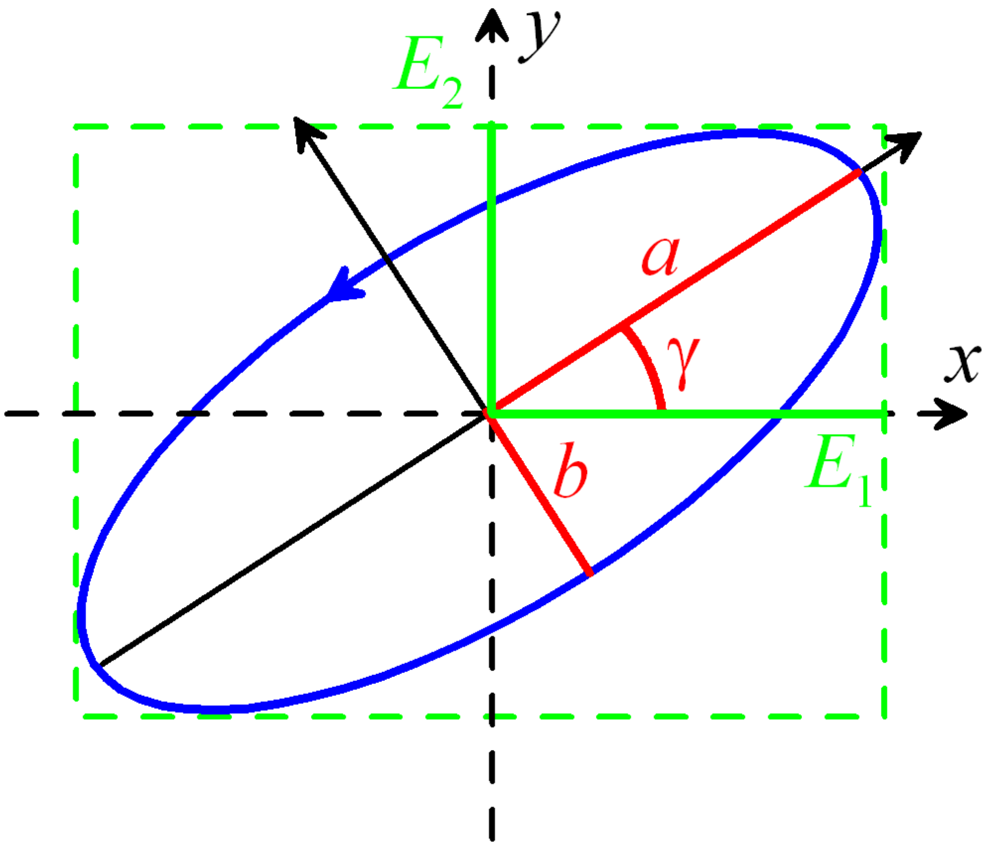
Strictly speaking, light scattering depend not on the size but, on the ratio of particle size to wavelength of incident radiation λ. The ratio can be quantified through the size parameter x:

$$x = 2\pi r/\lambda$$

where r is radius of the particle and λ – wavelength

Light scattering also depends on features of incident radiation.

Electromagnetic radiation is being characterized by intensity and polarization. In general case, it has an elliptical polarization.



The vibrational ellipse for the electric vector polarization.

Elliptically polarized light can be expressed as a superposition of two waves:

$$E_x = E_1\cos((\omega t - (\mathbf{k}\cdot\mathbf{r})) + \delta_1)$$

$$E_y = E_2\cos((\omega t - (\mathbf{k}\cdot\mathbf{r})) + \delta_2)$$

Parameters of these waves E_1, δ_1, E_2, and δ_2 are connected with the ellipse parameters a, b, and γ.

The principal difficulty of such approach is that the parameters E_1, δ_1, E_2, and δ_2 are amplitudes and phases of the electromagnetic waves; whereas, they are non-measurable.

Instead, the values which are proportional to the energy of electromagnetic radiation (i.e., quadratic values of amplitudes, and no absolute phases) can be measured.

Therefore, an alternative description of elliptical polarization of electromagnetic radiation is highly demanded!
A possible solution is through so called Stokes parameters.

Stokes parameters are grouped in a vector as follows:

$$\mathbf{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \quad \begin{bmatrix} \bullet \\ + \\ \times \\ \mathrm{o} \end{bmatrix}$$

Formalism based on Stokes vectors provides two an extremely important advantages:

1. One can express Stokes vector for unpolarized light:

$$\mathbf{S} = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Light scattering can be described through the Stokes vectors for incident and scattered light and, also, some matrix (4×4):

$$\mathbf{S}^{sc} = \mathbf{M} \cdot \mathbf{S}^{inc}$$

Matrix **M** is referred to Muller matrix (or scattering matrix) and it does not depend on property of the incident light.

In general case, all sixteen elements of Muller matrix are non-zero; though, these elements are not independent:

$$\mathbf{M} = \frac{1}{(kR)^2}\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

However, averaging over sample particles and their orientations substantially simplifies the resulting Mueller matrix:

$$\mathbf{M} = \frac{1}{(kR)^2}\begin{pmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{12} & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & M_{34} \\ 0 & 0 & -M_{34} & M_{44} \end{pmatrix}$$

In application to comets, the incident light is emitted by the Sun, which is substantially unpolarized.

Simultaneously, the dust particles appear in huge ensembles.

Therefore, Stokes vector of the scattered light consists of only two non-zero parameters. Under assumption of $I = 1$ in Stokes vector for the incident wave, Stockes vector for the scattered light takes form as follows:

$$\mathbf{S}^{sc} = \frac{1}{(kR)^2}\begin{pmatrix} M_{11} \\ M_{12} \\ 0 \\ 0 \end{pmatrix}$$

The measured values are the intensity of the scattered light $I = (kR)^{-2} M_{11}$ and its degree of linear polarization $P = -M_{12}/M_{11}$. Typically polarization is expressed in percent.

Taking into account the actual expressions for the Mueller matrix elements M_{11} and M_{12}, one can reformulate definitions for the intensity and degree of linear polarization alternatively as follows: $I = I_{\perp} + I_{||}$ and $P = (I_{\perp} - I_{||})/(I_{\perp} + I_{||})$.

Here, $I_{\perp}$ denotes the intensity of the component of scattered light that is polarized perpendicular to the scattering plane; whereas, $I_{||}$ denotes the intensity of the component polarized within the scattering plane.

Intensity I takes positive and non-zero values, degree of linear polarization P may be positive, negative, and equal to zero.

In general, the parameters describing light scattering by a particle can be classified into two groups, sometimes, referred as integral and differential parameters.

Differential parameters are functions of two angles specifying the direction of scattered light to a detector. Intensity I and degree of linear polarization P are the differential parameters. However, in the case of azimuthally symmetric targets, the angular dependence of differential parameters takes a significantly simpler form depending upon only phase angle α or, equivalently, the scattering angle θ.

An essential feature of integral parameters is that they are independent of the conditions of observation. Examples for integral parameters are the cross sections for absorption C_{abs} and extinction C_{ext}, single-scattering albedo ω, asymmetry parameter g, and radiation pressure efficiency Q_{pr}.

Interaction of electromagnetic radiation with particles decreases the energy flux of the incident wave. The total loss of the energy flux can be quantified in terms of area, which is normal to the incident beam and intercepts the lost flux of energy. Such an area is referred to as the cross section for extinction C_{ext}.

In the general case, the interaction of electromagnetic radiation with a particle results in absorption and scattering. The part of the total area that corresponds to loss due to absorption is referred to as the cross section for absorption C_{abs}; whereas, the rest corresponds to the cross section for scattering C_{sca}. These three values are obviously related as follows: $C_{\text{ext}} = C_{\text{abs}} + C_{\text{sca}}$.

Efficiencies for extinction Q_{ext}, absorption Q_{abs}, and scattering Q_{sca} are defined as ratios of the corresponding cross section to the geometric cross section G.

Single-scattering albedo ω determines efficiency of light scattering:

$$\omega = C_{\text{sca}}/C_{\text{ext}} = (C_{\text{ext}} - C_{\text{abs}})/C_{\text{ext}}; \qquad 0 \le \omega \le 1.$$

Asymmetry parameter g indicates the distribution of the scattered electromagnetic energy between forward and backward hemispheres with respect to the direction of the incident beam propagation:

$$g = \frac{\int\limits_{2\pi}\int\limits_{\pi} I(\theta,\varphi)\cos\theta\sin\theta\, d\theta\, d\varphi}{\int\limits_{2\pi}\int\limits_{\pi} I(\theta,\varphi)\sin\theta\, d\theta\, d\varphi}; \qquad -1 \le g \le 1.$$

Here, θ and φ are the scattering and azimuthal angles, $I(\theta,\varphi)$ is the intensity of scattering of unpolarized light. The denominator is equal to the scattering cross section C_{sca}.

The radiation-pressure efficiency Q_{pr} determines the motion of cosmic dust particles:

$$Q_{\mathrm{pr}} = C_{\mathrm{pr}}/G = (C_{\mathrm{ext}} - gC_{\mathrm{sca}})/G.$$

The motion of cosmic dust particles near a star depends on the ratio of the radiation-pressure force to the star's gravitational force, which is designated as β (e.g., Burns et al., 1979; Artymowicz, 1988; Fulle, 2004). Some details on the difference between the orbit of the parent body and an ejected dust particle caused by radiation pressure acting on the particle can be found, e.g., in Augereau and Beust (2006). By definition, the ratio β is in direct proportion to radiation pressure: $\beta \propto Q_{\mathrm{pr}}$ (e.g., Fulle, 2004).

The geometric albedo A describes the ratio of the intensity backscattered by the particle to that scattered by a white disk of the same geometric cross-section G in accordance with Lambert's law (e.g., Hanner et al., 1981; Hanner, 2003):

$$A = \frac{M_{11}(0°)\pi}{k^2 G}$$

Here, $M_{11}(0°)$ is the corresponding element of the Mueller matrix at backscattering α = 0º and k is the wavenumber.

The geometric albedo A equates to the backscattering efficiency of target particles.

Depending on value of size parameter x, one can distinguish three regimes of light scattering:

$x << 1$ – light scattering by particles much smaller than wavelength (other names: Rayleigh scattering, electrostatic approximation)

$x \approx 1 – 100$ – light scattering by particles comparable with wavelength (other name: resonant scattering)

$x >> 100$ – light scattering by particles much larger than wavelength (other name: geometric optics approximation (GOA))

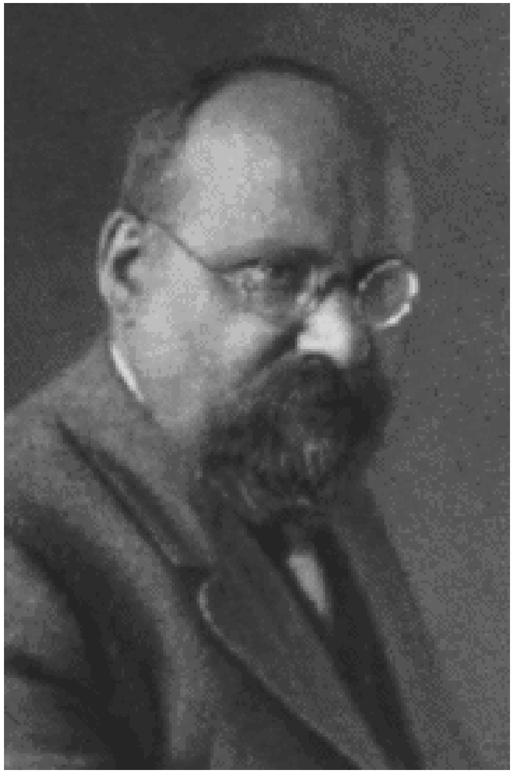
Maxwell equations remain to be valid in all three regimes. However, on practice, technique of computation depends on regime of light scattering.

The most difficult for the consideration is the case of particles comparable with wavelength (i.e., $x \approx 1 - 100$).

The number of problems of light scattering by particles comparable with wavelength which have been successfully resolved is a quite limited. The most famous case is so-called Mie theory.

Mie theory describes light scattering by a perfect sphere with arbitrary size parameter x and refractive index m. Though sphere is a rough approximation for cometary dust particles, it is still widely-used in the literature, at least, for approximate estimations.

Computation of light scattering by realistic models of cometary dust requires a numerical solution of Maxwell equations.
One famous approach discrete dipole approximation (DDA).



Gustav Mie
29·09·1869 –
13·02·1957

Mie theory

What is it?

It is an analytical solution of the problem of light scattering by a sphere.

Why is this solution called as Mie theory?

In honor of Gustav Mie, one of scientists who obtained the solution (1908).

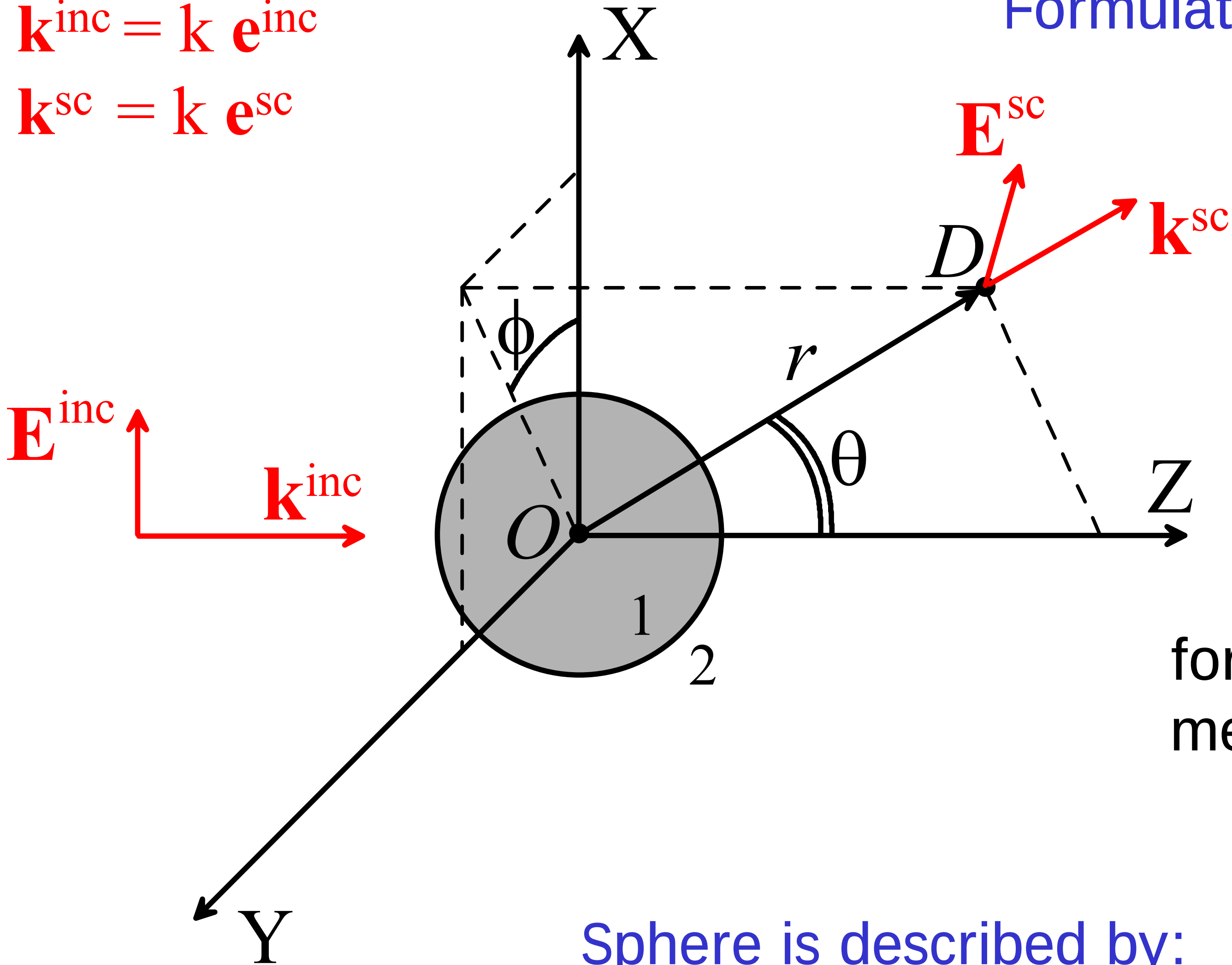
How was the goal attained?

Through the separation of variables in the corresponding wave equation.

Formulation of the problem

$\mathbf{k}^{\mathrm{inc}} = k\ \mathbf{e}^{\mathrm{inc}}$

$\mathbf{k}^{\mathrm{sc}} = k\ \mathbf{e}^{\mathrm{sc}}$



for simplicity,
medium 2 – vacuum

Sphere is described by:

(a) size parameter x ($x = 2\pi r/\lambda$)

(b) permittivity ε, magnetic permeability μ, conductivity σ

Irradiation of the sphere with some electromagnetic wave excites electromagnetic waves in both media (1 and 2). The electric and magnetic fields in outer medium $\mathbf{E}_2$, $\mathbf{H}_2$ can be expressed as sum of two parts: the incident field and the rest. The latter part is referred to the scattered electric field:

$$\mathbf{E}^{\mathrm{sc}} = \mathbf{E}_2 - \mathbf{E}^{\mathrm{inc}}$$
$$\mathbf{H}^{\mathrm{sc}} = \mathbf{H}_2 - \mathbf{H}^{\mathrm{inc}}$$

Any solution starts from the Maxwell equations:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{D} = 4\pi\rho$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \text{(CGS)}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{j} = \sigma \mathbf{E}$$

$\mathbf{E}$, $\mathbf{H}$ – electric and magnetic fields

$\mathbf{D}$, $\mathbf{B}$ – electric displacement field and magnetic induction

ρ, $\mathbf{j}$ – free charge and current densities

Maxwell equations need to be simplified:

1. Suppose that electromagnetic field oscillates harmonically, i.e., each parameter characterizing that field depends on time as follows:

$$\mathbf{A}(r,t)=\mathbf{A}(r)\exp(-i\omega t)$$

$$\nabla\times\mathbf{E}=i\frac{\omega}{c}\mu\mathbf{H} \qquad \nabla\cdot\mathbf{E}=\frac{4\pi}{\varepsilon}\rho$$

$$\nabla\times\mathbf{H}=-i\frac{\omega}{c}\left(\varepsilon+i\frac{4\pi\sigma}{\omega}\right)\mathbf{E} \qquad \nabla\cdot\mathbf{H}=0$$

Maxwell equations need to be simplified:

2. Suppose that sphere and surrounding medium are electrically neutral, i.e., their total electric charge is zero:

$$\rho = 0.$$

However, it does not necessarily mean that $\mathbf{j} \neq 0$.

$$\nabla \times \mathbf{E} = i\frac{\omega}{c}\mu\mathbf{H} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = -i\frac{\omega}{c}\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right)\mathbf{E} \qquad \nabla \cdot \mathbf{H} = 0$$

Maxwell equations need to be simplified:

3. Because sphere and surrounding medium are homogeneous (i.e., within each of them $\varepsilon = \mathrm{const}$, $\mu = \mathrm{const}$), the Maxwell equations can be replaced with two wave equations and boundary conditions.

$$\nabla^2\mathbf{E} + \mathrm{k}^2\mathbf{E} = 0 \qquad \nabla^2\mathbf{H} + \mathrm{k}^2\mathbf{H} = 0$$

$$\mathrm{k}^2 = \left(\frac{\omega}{c}\right)^2 \left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right)\mu$$

$$\mathbf{E}_1 \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n} \qquad \mathbf{H}_1 \times \mathbf{n} = \mathbf{H}_2 \times \mathbf{n}$$

Thus, the solution of light scattering problem can be find from the wave equations:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \qquad \nabla^2\mathbf{H} + k^2\mathbf{H} = 0$$

However, their solutions have to satisfy the Maxwell equations as well; in particular, the pair:

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{H} = 0$$

But, the divergence of some vector is always equal to zero only if this vector, in turn, is a curl of a vector:

$$\nabla\cdot(\nabla\times\mathbf{A}) \equiv 0$$

Using a scalar function ψ and constant vector $\mathbf{c}$, we construct a vector function $\mathbf{M}$:

$$\mathbf{M} = \nabla \times (\mathbf{c}\psi)$$

Obviously, such a function satisfies equation:

$$\nabla \cdot \mathbf{M} = 0$$

On the other hand, the vector function $\mathbf{M}$ will satisfy the vector wave equation if ψ is a solution to the scalar wave equation:

$$\nabla^2\psi + k^2\psi = 0$$

So far, we were considering only one vector function $\mathbf{M}$. But, in the electromagnetic theory, we have two different vectors $\mathbf{E}$ and $\mathbf{H}$.

Each of these vectors satisfies the wave equation and, simultaneously, their divergence is equal to zero. Therefore, $\mathbf{M}$ is related to either $\mathbf{E}$ or $\mathbf{H}$.

It means that we need to define a complimentary vector function to the function $\mathbf{M}$. We can do that with help of one of the Maxwell equations.

We suppose that the function $\mathbf{M}$ is associated with electric field $\mathbf{E}$. Then, the complimentary vector function $\mathbf{N}$ can be constructed with help of equation:

$$\nabla \times \mathbf{E} = i\frac{\omega}{c}\mu\mathbf{H}$$

Let us define the relationship between the vector functions $\mathbf{M}$ and $\mathbf{N}$ as follows:

$$\nabla \times \mathbf{M} = \mathrm{k}\mathbf{N}$$

Such a relationship is completely symmetric for the functions $\mathbf{M}$ and $\mathbf{N}$. Indeed:

$$\nabla \times \mathbf{M} = \mathrm{k}\mathbf{N}$$

$$\nabla \times \nabla \times \mathbf{M} = \mathrm{k}(\nabla \times \mathbf{N})$$

$$\nabla \times \nabla \times \mathbf{M} = \nabla(\nabla \cdot \mathbf{M}) - \nabla^2 \mathbf{M} = \mathrm{k}^2 \mathbf{M}$$

$$\mathrm{k}^2 \mathbf{M} = \mathrm{k}(\nabla \times \mathbf{N})$$

$$\mathrm{k}\mathbf{M} = \nabla \times \mathbf{N}$$

Now, we can return to the scalar function ψ, which has to satisfy to the scalar wave equation:

$$\nabla^2\psi + k^2\psi = 0$$

Scalar wave equation in spherical polar coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0$$

Solution of such an equation has form

$$\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

Then, initial scalar wave equation can be separated in three equations:

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\mathrm{k}^2 r^2 - n(n+1)\right)R = 0$$

Here, m and n – the separation constants.

Solving these three equations, we can construct the function ψ. There are two types of ψ:

even – $$\psi_{emn} = \cos(m\phi) P_n^m(\cos\theta) z_n(\mathrm{k}r)$$

odd – $$\psi_{omn} = \sin(m\phi) P_n^m(\cos\theta) z_n(\mathrm{k}r)$$

Here, $P_n^m(\cos\theta)$ – the associated Legendre functions, $z_n(\mathrm{k}r)$ – any of four spherical Bessel functions j_n, y_n, $h_n^{(1)}$, and $h_n^{(2)}$.

Before construction of the vector function $\mathbf{M}$ and $\mathbf{N}$ we have to decide what is the arbitrary vector $\mathbf{c}$ in the defintion of $\mathbf{M}$:

$$\mathbf{M} = \nabla \times (\mathbf{c}\psi)$$

The choice is not obvious, so, let us do it in the way which will, at least, simplify $\mathbf{M}$.

$$\mathbf{M} = \nabla \times (\mathbf{c}\psi) = \nabla\psi \times \mathbf{c}$$

In other words, $\mathbf{M}$ is perpendicular to the vector $\mathbf{c}$:

$$(\mathbf{M} \cdot \mathbf{c}) = 0$$

Therefore, choosing radius vector $\mathbf{r}$ instead of $\mathbf{c}$, we make a transverse the vector functions $\mathbf{M}$ and $\mathbf{N}$.

$$\mathbf{M} = \nabla \times (\mathbf{r}\psi)$$

Note that the functions $\mathbf{M}$ and $\mathbf{N}$ are called as vector spherical harmonics. Using functions ψ_{emn} and ψ_{omn}, we can express $\mathbf{M}$ and $\mathbf{N}$ as follows:

$$\mathbf{M}_{emn} = \nabla \times (\mathbf{r}\psi_{emn}) \qquad \mathbf{M}_{omn} = \nabla \times (\mathbf{r}\psi_{omn})$$

$$\mathbf{N}_{emn} = \frac{\nabla \times \mathbf{M}_{emn}}{\mathrm{k}} \qquad \mathbf{N}_{omn} = \frac{\nabla \times \mathbf{M}_{omn}}{\mathrm{k}}$$

In general case, electric and magnetic fields have to be expanded into series of four sets of vector spherical harmonic: $\mathbf{M}_{emn}$, $\mathbf{M}_{omn}$, $\mathbf{N}_{emn}$, and $\mathbf{N}_{omn}$.

In Cartesian system of coordinates, the incident electric field can be written as follows:

$$\mathbf{E}_i = E_0 \exp(i\mathrm{k}r\cos\theta)\hat{\mathbf{e}}_x$$

where the unit vector $\mathbf{e}_x$ in spherical system of coordinates takes form:

$$\hat{\mathbf{e}}_x = \hat{\mathbf{e}}_r \sin\theta\cos\phi + \hat{\mathbf{e}}_\theta \cos\theta\cos\phi - \hat{\mathbf{e}}_\phi \sin\phi$$

When expanding such an incident electric field into series of vector spherical harmonics, we obtain the following result:

$$\mathbf{E}^{inc} = E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(\mathbf{M}_{o1n}^{(1)} - i\mathbf{N}_{e1n}^{(1)} \right)$$

Corresponding magnetic field takes form:

$$\mathbf{H}^{inc} = \frac{-\mathrm{k}}{\omega\mu} E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(\mathbf{M}_{e1n}^{(1)} + i\mathbf{N}_{o1n}^{(1)} \right)$$

Vector spherical harmonics $\mathbf{M}_{o1n}$ and $\mathbf{N}_{e1n}$

In order to reduce number of formulae, we will focus on electric field only.

$$\mathbf{M}_{o1n} = \cos\phi \frac{P_n^1(\cos\theta)}{\sin\theta} z_n(\rho)\hat{\mathbf{e}}_\theta - \sin\phi \frac{d}{d\theta} P_n^1(\cos\theta) z_n(\rho)\hat{\mathbf{e}}_\phi$$

$$\mathbf{N}_{e1n} = n(n+1)\cos\phi\, P_n^1(\cos\theta)\frac{z_n(\rho)}{\rho}\hat{\mathbf{e}}_r +$$

$$\cos\phi \frac{d}{d\theta} P_n^1(\cos\theta)\frac{(\rho z_n(\rho))'}{\rho}\hat{\mathbf{e}}_\theta - \sin\phi \frac{P_n^1(\cos\theta)}{\sin\theta}\frac{(\rho z_n(\rho))'}{\rho}\hat{\mathbf{e}}_\phi$$

As it was mentioned, $z_{\mathrm{n}}(\rho)$ denotes any of four spherical Bessel functions $j_n(\rho)$, $y_n(\rho)$, $h_n^{(1)}(\rho)$, and $h_n^{(2)}(\rho)$.

The choice of spherical Bessel function depends on their behavior. For instance, near the center of sphere (i.e., at $\rho \to 0$), only $j_n(\rho)$ takes finite values. Therefore, only this function contributes to spherical harmonics $\mathbf{M}_{o1n}$ and $\mathbf{N}_{e1n}$, forming the incident and internal electric fields $\mathbf{E}^{\mathrm{inc}}$ and $\mathbf{E}_1$. Such a choice we denote with superscript (1).

Note, that the argument $\rho = \mathrm{k}r$ is different for the incident and internal electric fields because the wavenumber k is different for a material of sphere and surrounding space.

Vector spherical harmonics $\mathbf{M}_{o1n}$ and $\mathbf{N}_{e1n}$

For scattered field $\mathbf{E}^{\mathrm{sc}}$, we could use one of two linear combinations $h_n^{(1)}(\rho)=j_n(\rho)+i\,y_n(\rho)$ or $h_n^{(2)}(\rho)=j_n(\rho)-i\,y_n(\rho)$. The choice will be clear if we consider an asymptotic behavior of these functions at $\rho \to 0$:

$$h_n^{(1)}(\rho) \sim \exp(i\mathrm{k}r)$$

$$h_n^{(2)}(\rho) \sim \exp(-i\mathrm{k}r)$$

For time dependence $\sim \exp(-i\omega t)$, $h_n^{(1)}(\rho)$ presents the wave propagating from sphere to outer space.

Such a choice we denote with superscript (3).

Expansion of the internal and scattered fields

We have found that the incident electric wave is based on only spherical harmonics $\mathbf{M}_{o1n}$ and $\mathbf{N}_{e1n}$.

However, it means that this incident electric wave will induce only electric waves which are based on the same vector spherical harmonics. Thus, the internal and scattered fields $\mathbf{E}_1$ and $\mathbf{E}^{\mathrm{sc}}$ are as follows:

$$\mathbf{E}_1 = E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(c_n \mathbf{M}_{o1n}^{(1)} - i d_n \mathbf{N}_{e1n}^{(1)} \right)$$

$$\mathbf{E}^{sc} = -E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(b_n \mathbf{M}_{o1n}^{(3)} - i a_n \mathbf{N}_{e1n}^{(3)} \right)$$

Coefficient of the scattered and internal fields

In order to find the coefficients of the scattered and internal fields, we need to consider boundary conditions at the surface of sphere:

$$\mathbf{E}_1 \times \mathbf{n} = (\mathbf{E}^{inc} + \mathbf{E}^{sc}) \times \mathbf{n} \qquad \mathbf{H}_1 \times \mathbf{n} = (\mathbf{H}^{inc} + \mathbf{H}^{sc}) \times \mathbf{n}$$

These boundary conditions are expressed in spherical coordinate system in four equations:

$$E_{1\theta} = E_\theta^{inc} + E_\theta^{sc} \qquad H_{1\theta} = H_\theta^{inc} + H_\theta^{sc}$$

$$E_{1\phi} = E_\phi^{inc} + E_\phi^{sc} \qquad H_{1\phi} = H_\phi^{inc} + H_\phi^{sc}$$

Coefficient of the scattered fields a_n and b_n

After exhausting mathematical exercises we obtain the coefficients of the scattering as follows:

$$a_n = \frac{m\psi_n(mx)\psi_n'(x) - \psi_n(x)\psi_n'(mx)}{m\psi_n(mx)\xi_n'(x) - \xi_n(x)\psi_n'(mx)}$$

$$b_n = \frac{\psi_n(mx)\psi_n'(x) - m\psi_n(x)\psi_n'(mx)}{\psi_n(mx)\xi_n'(x) - m\xi_n(x)\psi_n'(mx)}$$

where

$$x = \frac{2\pi r_{sph}}{\lambda}$$ – size parameter

$$m = \frac{k_1}{k} = \sqrt{\mu\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right)}$$ – refractive index

Coefficient of the scattered fields a_n and b_n

$\psi_n(\rho)$ and $\xi_n(\rho)$ – Riccati–Bessel functions

$$\psi_n(\rho) = \rho \; j_n(\rho) \qquad \xi_n(\rho) = \rho \; h_n^{(1)}(\rho)$$

Recurrence relations of spherical Bessel functions

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$$

$$h_0^{(1)}(\rho) = \frac{-i\exp(i\rho)}{\rho} \qquad h_1^{(1)}(\rho) = \frac{-\exp(i\rho)}{\rho}\left(1 + \frac{i}{\rho}\right)$$

$$z_{n+1}(\rho) = \frac{2n+1}{\rho} z_n(\rho) - z_{n-1}(\rho) \qquad z_n'(\rho) = z_{n-1}(\rho) - \frac{n+1}{\rho} z_n(\rho)$$

Some values of the coefficients a_n and b_n

Sphere with $x = 5$ and $m = 1.5 + 0.01i$ gives the first five coefficients a_n and b_n as follows:

n	a_n	b_n
1	0.517973+0.437219*i*	0.357583+0.444909*i*
2	0.585369+0.459639*i*	0.538009+0.427248*i*
3	0.663224+0.422361*i*	0.874567+0.291533*i*
4	0.940888+0.107549*i*	0.769533+0.386314*i*
5	0.461418-0.465229*i*	0.722904-0.336354*i*

Full scattered electric field

$$\mathrm{E}_r^{\mathrm{sc}} = \frac{\cos\phi}{(\mathrm{k}r)^2} E_0 \sum_{n=1}^{\infty} i^{(n+1)} (2n+1) a_n \xi_n(\mathrm{k}r) P_n^1(\cos\theta)$$

$$\mathrm{E}_\theta^{\mathrm{sc}} = \frac{\cos\phi}{\mathrm{k}r} E_0 \sum_{n=1}^{\infty} i^{(n+1)} \frac{2n+1}{n(n+1)} \left(a_n \frac{d\xi_n(\mathrm{k}r)}{d(\mathrm{k}r)} \frac{dP_n^1(\cos\theta)}{d\theta} + ib_n \xi_n(\mathrm{k}r) \frac{P_n^1(\cos\theta)}{\sin\theta} \right)$$

$$\mathrm{E}_\phi^{\mathrm{sc}} = -\frac{\sin\phi}{\mathrm{k}r} E_0 \sum_{n=1}^{\infty} i^{(n+1)} \frac{2n+1}{n(n+1)} \left(a_n \frac{d\xi_n(\mathrm{k}r)}{d(\mathrm{k}r)} \frac{P_n^1(\cos\theta)}{\sin\theta} + ib_n \xi_n(\mathrm{k}r) \frac{dP_n^1(\cos\theta)}{d\theta} \right)$$

r, θ and φ – spherical polar coordinates; k – wavenumber;
$P_n^1(\cos\theta)$ – the associated Legendre functions;
$\xi_n(\mathrm{k}r) = (\mathrm{k}r) \cdot h_n^{(1)}(\mathrm{k}r)$, here $h_l^{(1)}(\mathrm{k}r)$ – spherical Hankel function;
a_n and b_n – coefficients of scattering.

Recurrence relations of the associated Legendre functions

$$P_0(\cos\theta) = 1 \qquad P_1(\cos\theta) = \cos\theta$$

$$P_{n+1}(\cos\theta) = \frac{2n+1}{n+1}\cos\theta\ P_n(\cos\theta) - \frac{n}{n+1}P_{n-1}(\cos\theta)$$

the associated Legendre functions:

$$P_n^1(\cos\theta) = \frac{1}{\sin\theta}\left(P_{n-1}(\cos\theta) - \cos\theta\ P_n(\cos\theta)\right)$$

$$\frac{dP_n^1(\cos\theta)}{d\theta} = \frac{\cos\theta}{\sin\theta}\left(n(n+1)\frac{P_n(\cos\theta)}{\sin\theta} - P_n^1(\cos\theta)\right)$$

The scattered electric field in far zone

At large distance from the sphere, full scattered electric field can be simplified as follows:

(a) Omitting of radial component of the field, i.e., $\mathrm{E}_r^{\mathrm{sc}}=0$;

(b) The function $\xi_n(\mathrm{k}r)$ takes very simple form at very large values of $(\mathrm{k}r)$.

$$\xi_n(\mathrm{k}r) \approx (-i)^{n+1}\exp(i\mathrm{k}r) \qquad \frac{d\xi_n(\mathrm{k}r)}{d(\mathrm{k}r)} \approx (-i)^n \exp(i\mathrm{k}r)$$

The scattered electric field in far zone

$$\mathrm{E}_{\theta}^{\mathrm{sc}} = i\cos\phi \frac{\exp(i\mathrm{k}r)}{\mathrm{k}r} E_0 \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n \frac{dP_n^1(\cos\theta)}{d\theta} + b_n \frac{P_n^1(\cos\theta)}{\sin\theta} \right)$$

$$\mathrm{E}_{\phi}^{\mathrm{sc}} = -i\sin\phi \frac{\exp(i\mathrm{k}r)}{\mathrm{k}r} E_0 \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n \frac{P_n^1(\cos\theta)}{\sin\theta} + b_n \frac{dP_n^1(\cos\theta)}{d\theta} \right)$$

The problem of light scattering by an arbitrary sphere has been solved.

In order to study light scattering by sphere, we have to construct the corresponding Mueller matrix.

Therefore, we need to study a scattering of two independent beams having mutually perpendicular states of polarization.

So far, we have result for only one polarization of the incident wave. However, the symmetry of sphere allows us to derive result for another incident wave (which is perpendicularly polarized in respect to the first one) without an additional computation.

Indeed, at azimuth angle ϕ = 0, the total scattering field in far zone is defined by only one component $\mathrm{E}^{\mathrm{sc}} = \mathrm{E}_{\theta}{}^{\mathrm{sc}}$ because $\mathrm{E}_{\phi}{}^{\mathrm{sc}}$ = 0. At that, light scattering of the X-polarized incident wave happens in the plane XOZ.

At azimuth angle ϕ = 90$^{\mathrm{o}}$, the total scattering field in far zone is defined by only component $\mathrm{E}^{\mathrm{sc}} = \mathrm{E}_{\phi}{}^{\mathrm{sc}}$ because $\mathrm{E}_{\theta}{}^{\mathrm{sc}}$ = 0. At that, light scattering of the X-polarized incident wave happens in the plane YOZ. Due to symmetry, it is equal to the scattering the Y-polarized incident wave in the plane XOZ.

One can rewrite the solution on far zone

$$\mathrm{E}_{\theta}^{\mathrm{sc}} = i\cos\phi \frac{\exp(i\mathrm{k}r)}{\mathrm{k}r} E_0 \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n \frac{dP_n^1(\cos\theta)}{d\theta} + b_n \frac{P_n^1(\cos\theta)}{\sin\theta} \right)$$

$$\mathrm{E}_{\phi}^{\mathrm{sc}} = -i\sin\phi \frac{\exp(i\mathrm{k}r)}{\mathrm{k}r} E_0 \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n \frac{P_n^1(\cos\theta)}{\sin\theta} + b_n \frac{dP_n^1(\cos\theta)}{d\theta} \right)$$

as follows:

$$\mathrm{E}_{\theta}^{\mathrm{sc}} = E_0 \frac{\exp(i\mathrm{k}r)}{-i\mathrm{k}r} \cos\phi\, S_2(\cos\theta)$$

$$\mathrm{E}_{\phi}^{\mathrm{sc}} = -E_0 \frac{\exp(i\mathrm{k}r)}{-i\mathrm{k}r} \sin\phi\, S_1(\cos\theta)$$

Using functions $S_1(\cos\theta)$ and $S_2(\cos\theta)$, we can construct amplitude scattering matrix (also known as Jones matrix) as follows:

$$\begin{pmatrix} \mathrm{E}_{\mathrm{ll}}^{\mathrm{sc}} \\ \mathrm{E}_{\perp}^{\mathrm{sc}} \end{pmatrix} = \frac{\exp(i\mathrm{k}r)}{-i\mathrm{k}r} \begin{pmatrix} S_2 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} \mathrm{E}_{\mathrm{ll}}^{\mathrm{inc}} \\ \mathrm{E}_{\perp}^{\mathrm{inc}} \end{pmatrix}$$

Now, one can obtain Mueller matrix:

$$\mathbf{M} = \frac{1}{(kR)^2} \begin{pmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{12} & M_{11} & 0 & 0 \\ 0 & 0 & M_{33} & M_{34} \\ 0 & 0 & -M_{34} & M_{33} \end{pmatrix}$$

As one can see, number of independent non-zero elements in Mueller matrix is only four.

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