Astrophysical light scattering problems, spring 2023 (PAP316, 5 cr)
Exercise 2/3

Please note that your answers are due by **April 13, 2023**. Send the answers in pdf form by email to the course assistant `mikko.vuori@helsinki.fi`.
Background literature: C. F. Bohren & D. R. Huffman, Absorption and Scattering of Light by Small Particles, Wiley, 2008, Sects. 3.2 and 3.3 (B&H).

1. Consider a 2×2 Jones amplitude scattering matrix with nonzero diagonal elements S_1, S_2 and vanishing off-diagonal elements S_3, S_4 (see B&H). The elements are functions of the scattering angle θ that we omit for the sake of brevity. In that case, the nonzero 4×4 Mueller scattering matrix elements are

$$\begin{aligned}
S_{11} &= \frac{1}{2}(|S_1|^2 + |S_2|^2) = S_{22}, \\
S_{12} &= \frac{1}{2}(-|S_1|^2 + |S_2|^2) = S_{21}, \\
S_{33} &= \mathrm{Re} S_1 S_2^* = S_{44}, \\
S_{34} &= -\mathrm{Im} S_1 S_2^* = -S_{43}.
\end{aligned}$$

In matrix form,

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{11} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & -S_{34} & S_{33} \end{pmatrix},$$

which is the common form of the Mie scattering matrix for spherical particles. Assume that only the relative electromagnetic phase is relevant for S_1, S_2 at each scattering angle so that S_1 can be assumed real-valued. Show that the inverse relation of the Jones matrix elements as a function of the Mueller matrix elements can be written as

$$\begin{aligned}
S_1 &= \sqrt{S_{11} - S_{12}}, \\
S_2 &= \frac{1}{S_1}(S_{33} + i S_{34}).
\end{aligned}$$

(3 points)

2. Consider next a hypothetical Jones amplitude scattering matrix with nonzero off-diagonal elements S_3, S_4 and vanishing diagonal elements S_1, S_2. Now the nonzero hypothetical Mueller scattering matrix elements are

$$\begin{aligned}
S_{11} &= \frac{1}{2}(|S_3|^2 + |S_4|^2) = -S_{22}, \\
S_{12} &= \frac{1}{2}(-|S_3|^2 + |S_4|^2) = -S_{21}, \\
S_{33} &= \mathrm{Re} S_3 S_4^* = -S_{44}, \\
S_{34} &= -\mathrm{Im} S_3 S_4^* = S_{43}.
\end{aligned}$$

or, in matrix form,

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ -S_{12} & -S_{11} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & S_{34} & -S_{33} \end{pmatrix}.$$

Assuming, again, that only the relative electromagnetic phase is relevant for S3 , S4 , the element S3 can be assumed real-valued. Show that

$$S_3 = \sqrt{S_{11} - S_{12}},$$
$$S_4 = \frac{1}{S_3}(S_{33} + iS_{34}).$$

How does this compare to the result in question 1?
(3 points)

3-4. Consider the analytical spectral model by Shkuratov et al. (*Icarus* **137**, 235, 1999; reprint available on the course home page). Starting from their Eqs. 9a and 9b, derive the inverse relation (Eq. 13) giving the imaginary part of the complex refractive index as a function of particle size assuming that the geometric albedo is available from the observations and that the real part of the refractive index as well as the regolith porosity are known.
(6 points)

5. Using the analytical spectral model in question 4 at the wavelength of 0.55 microns, estimate the imaginary part of the complex refractive index of the material for a dust-covered asteroid with geometric albedo of 0.05 (typical for dark, C-type asteroids), porosity of 60%, and real part of the refractive index of 1.6, when the particle size (diameter) is assumed to be 50 microns. What would be the imaginary part if the albedo were 0.60 (possible for bright, E-type asteroids) instead of 0.05? How does the change in the imaginary part compare to the change in the albedo?
(3 points)