Scattering matrices of particle ensembles analytically decomposed into pure Mueller matrices

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Abstract

Scattering matrices of particle ensembles are analytically decomposed into sums of pure Mueller matrices. The ensembles are assumed to have equal numbers of nonspherical particles and their mirror particles, both in random orientation. In the general case, there are four pure Mueller matrices in the decomposition. Of these four matrices, there is a single matrix that qualifies as a pure scattering matrix, whereas the remaining three matrices represent other classes of pure Mueller matrices. For ensembles of spherical particles, there are two pure Mueller matrices in the decomposition. Again, there is a single matrix qualifying as a pure scattering matrix, whereas the remaining matrix represents another class of pure matrices.

Keywords: light scattering, scattering phase matrix, Mueller matrix, Jones matrix, decomposition, coherency matrix

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1. Introduction

In Section 2, we describe the ensemble-averaged scattering phase matrix and the related symmetry requirements of the matrix elements. In Section 3, we decompose the scattering phase matrix analytically into a sum of maximum four

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pure Mueller matrices, with insights from the solution for the eigenproblem of the Cloude coherency matrix. In Section 4, we validate the solution by decomposing scattering phase matrices computed numerically and measured experimentally. We provide conclusions and future prospects in Section 5. Appendix A describes an alternative method for obtaining the analytical decomposition.

2. Scattering phase matrix

Consider a 4×4 block-diagonal, ensemble-averaged scattering phase matrix $\mathbf{P}_0 = \mathbf{P}_0(\theta)$, where θ is the scattering angle:

$$\mathbf{P}_0 = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{pmatrix}. \quad (1)$$

We assume the common normalization of

$$\int_{(4\pi)} \frac{d\Omega}{4\pi}\, a_1(\theta) = 1, \quad (2)$$

that is,

$$\frac{1}{2}\int_0^{\pi} d\theta \sin\theta\, a_1(\theta) = 1. \quad (3)$$

The scattering phase matrix is subject to a number of symmetry relations [1]. For arbitrary scattering angles θ,

$$\begin{aligned} |a_j| &\leq a_1, \quad j = 2, 3, 4, \\ |b_j| &\leq a_1, \quad j = 1, 2, \\ (a_3 + a_4)^2 + 4b_2^2 &\leq (a_1 + a_2)^2 - 4b_1^2, \\ |a_3 - a_4| &\leq a_1 - a_2, \\ |a_2 - b_1| &\leq a_1 - b_1, \\ |a_2 + b_1| &\leq a_1 + b_1. \end{aligned} \quad (4)$$

In the forward ($\theta = 0$) and backward scattering directions ($\theta = \pi$),

$$\begin{aligned} a_2(0) &= a_3(0), \\ a_2(\pi) &= -a_3(\pi), \\ a_4(\pi) &= a_1(\pi) - 2a_2(\pi), \\ b_1(0) &= b_2(0) \;=\; b_1(\pi) \;=\; b_2(\pi) \;=\; 0. \end{aligned} \tag{5}$$

3. Mueller matrix decomposition

We model the observed matrix $\mathbf{P}_0$ in full detail with a scattering phase matrix $\mathbf{P} = \mathbf{P}(\theta)$ that is reconstructed from four matrices $\mathbf{U} = \mathbf{U}(\theta)$, $\mathbf{V} = \mathbf{V}(\theta)$, $\mathbf{W} = \mathbf{W}(\theta)$, and $\mathbf{Z} = \mathbf{Z}(\theta)$ [2, 3]:

$$\begin{aligned} \mathbf{P} &= w_U\ \mathbf{U} + w_V\ \mathbf{V} + w_W\ \mathbf{W} + w_Z\ \mathbf{Z}, \\ & 0 \le w_U \le 1, \quad 0 \le w_V \le 1, \quad 0 \le w_W \le 1, \quad 0 \le w_Z \le 1, \\ & w_U + w_V + w_W + w_Z = 1, \end{aligned} \tag{6}$$

where w_U, w_V, w_W, and w_Z are the normalized weights and where the common normalization condition holds for the phase functions:

$$\begin{aligned} \int_{(4\pi)} \frac{d\Omega}{4\pi}\, U_{11}(\theta) &= \int_{(4\pi)} \frac{d\Omega}{4\pi}\, V_{11}(\theta) \;= \\ \int_{(4\pi)} \frac{d\Omega}{4\pi}\, W_{11}(\theta) &= \int_{(4\pi)} \frac{d\Omega}{4\pi}\, Z_{11}(\theta) \;=\; 1. \end{aligned} \tag{7}$$

The matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$ and the coefficients w_U, w_V, w_W, and w_Z follow from the solution of the eigenproblem for the 4×4 Cloude coherency matrix $\mathbf{J} = \mathbf{J}(\theta)$:

$$\mathbf{J}\mathbf{x} = \lambda \mathbf{x}, \tag{8}$$

where $\mathbf{x}$ denotes a complex-valued four-vector and λ denotes the eigenvalue. In the case of the scattering phase matrix in Eq. 1, the matrix $\mathbf{J}$ takes the form

$$\mathbf{J} = \tfrac{1}{4} \cdot$$

$$\begin{pmatrix} a_1+a_2+a_3+a_4 & 2(b_1-\mathrm{i}b_2) & 0 & 0 \\ 2(b_1+\mathrm{i}b_2) & a_1+a_2-a_3-a_4 & 0 & 0 \\ 0 & 0 & a_1-a_2+a_3-a_4 & 0 \\ 0 & 0 & 0 & a_1-a_2-a_3+a_4 \end{pmatrix}. \tag{9}$$

The eigenproblem can be solved analytically. In the notation of Eq. 6, we obtain

$$\begin{aligned}
\mathbf{U} &= \begin{pmatrix} U_{11} & U_{12} & 0 & 0 \\ U_{12} & U_{11} & 0 & 0 \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & -U_{34} & U_{33} \end{pmatrix}, \\
\mathbf{V} &= \begin{pmatrix} V_{11} & V_{12} & 0 & 0 \\ V_{12} & V_{11} & 0 & 0 \\ 0 & 0 & V_{33} & V_{34} \\ 0 & 0 & -V_{34} & V_{33} \end{pmatrix}, \\
\mathbf{W} &= \begin{pmatrix} W_{11} & 0 & 0 & 0 \\ 0 & -W_{11} & 0 & 0 \\ 0 & 0 & -W_{11} & 0 \\ 0 & 0 & 0 & W_{11} \end{pmatrix}, \\
\mathbf{Z} &= \begin{pmatrix} Z_{11} & 0 & 0 & 0 \\ 0 & -Z_{11} & 0 & 0 \\ 0 & 0 & Z_{11} & 0 \\ 0 & 0 & 0 & -Z_{11} \end{pmatrix},
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
U_{11} &= \frac{1}{2w_U}\left[\frac{1}{2}(a_1+a_2)+\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}\right], \\
V_{11} &= \frac{1}{2w_V}\left[\frac{1}{2}(a_1+a_2)-\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}\right], \\
\frac{U_{12}}{U_{11}} &= -\frac{V_{12}}{V_{11}} = \frac{b_1}{\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}},
\end{aligned}$$

$$
\begin{aligned}
\frac{U_{33}}{U_{11}} &= -\frac{V_{33}}{V_{11}} = \frac{\frac{1}{2}(a_3+a_4)}{\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}},\\
\frac{U_{34}}{U_{11}} &= -\frac{V_{34}}{V_{11}} = \frac{b_2}{\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}},\\
W_{11} &= \frac{1}{2w_W}\,(a_1-a_2-a_3+a_4),\\
Z_{11} &= \frac{1}{2w_Z}\,(a_1-a_2+a_3-a_4),
\end{aligned}
\tag{11}
$$

with

$$
\begin{aligned}
w_U &= \frac{1}{4}+\frac{1}{8}\int_0^{\pi} d\theta \sin\theta \left[a_2(\theta)+2\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}\right],\\
w_V &= \frac{1}{4}+\frac{1}{8}\int_0^{\pi} d\theta \sin\theta \left[a_2(\theta)-2\sqrt{b_1^2+b_2^2+\frac{1}{4}(a_3+a_4)^2}\right],\\
w_W &= \frac{1}{4}+\frac{1}{8}\int_0^{\pi} d\theta \sin\theta \left[-a_2(\theta)-a_3(\theta)+a_4(\theta)\right],\\
w_Z &= \frac{1}{4}+\frac{1}{8}\int_0^{\pi} d\theta \sin\theta \left[-a_2(\theta)+a_3(\theta)-a_4(\theta)\right].
\end{aligned}
\tag{12}
$$

4. Results and discussion

4.1. Phase matrices of spherical particles

4.2. Phase matrices of Gaussian-random-sphere particles

4.3. Measured phase matrices of feldspar particles

5. Conclusion

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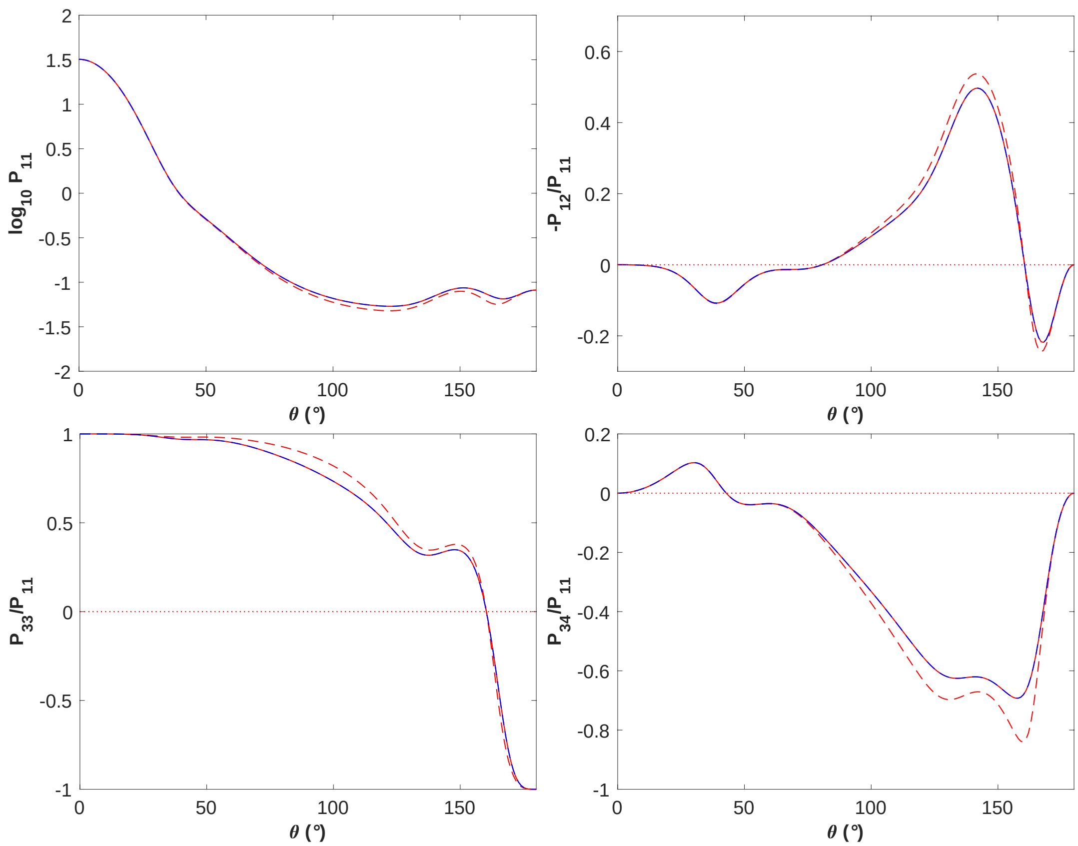

Figure 1: Mie.

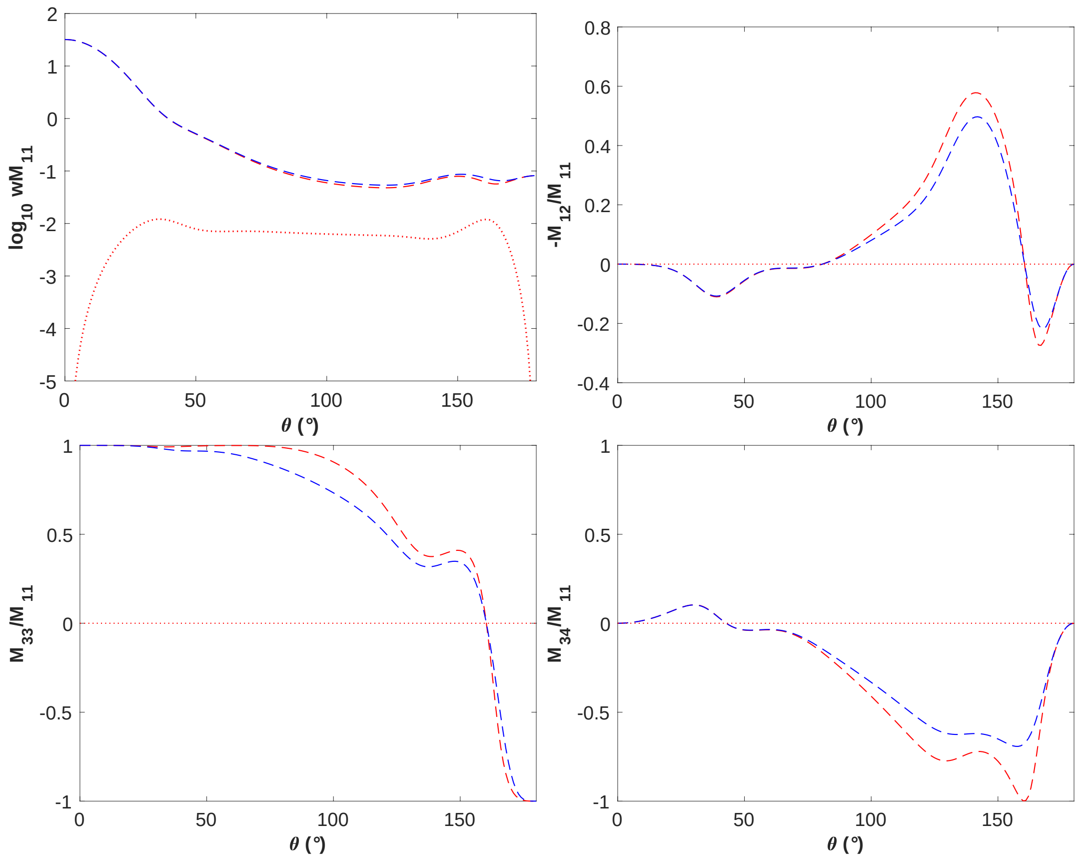
log10 wM11
2
1
0
-1
-2
-3
-4
-5
0
50
100
150
θ (°)
-M12/M11
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
50
100
150
θ (°)
M33/M11
1
0.5
0
-0.5
-1
0
50
100
150
θ (°)
M34/M11
1
0.5
0
-0.5
-1
0
50
100
150
θ (°)

Figure 2: Mie.

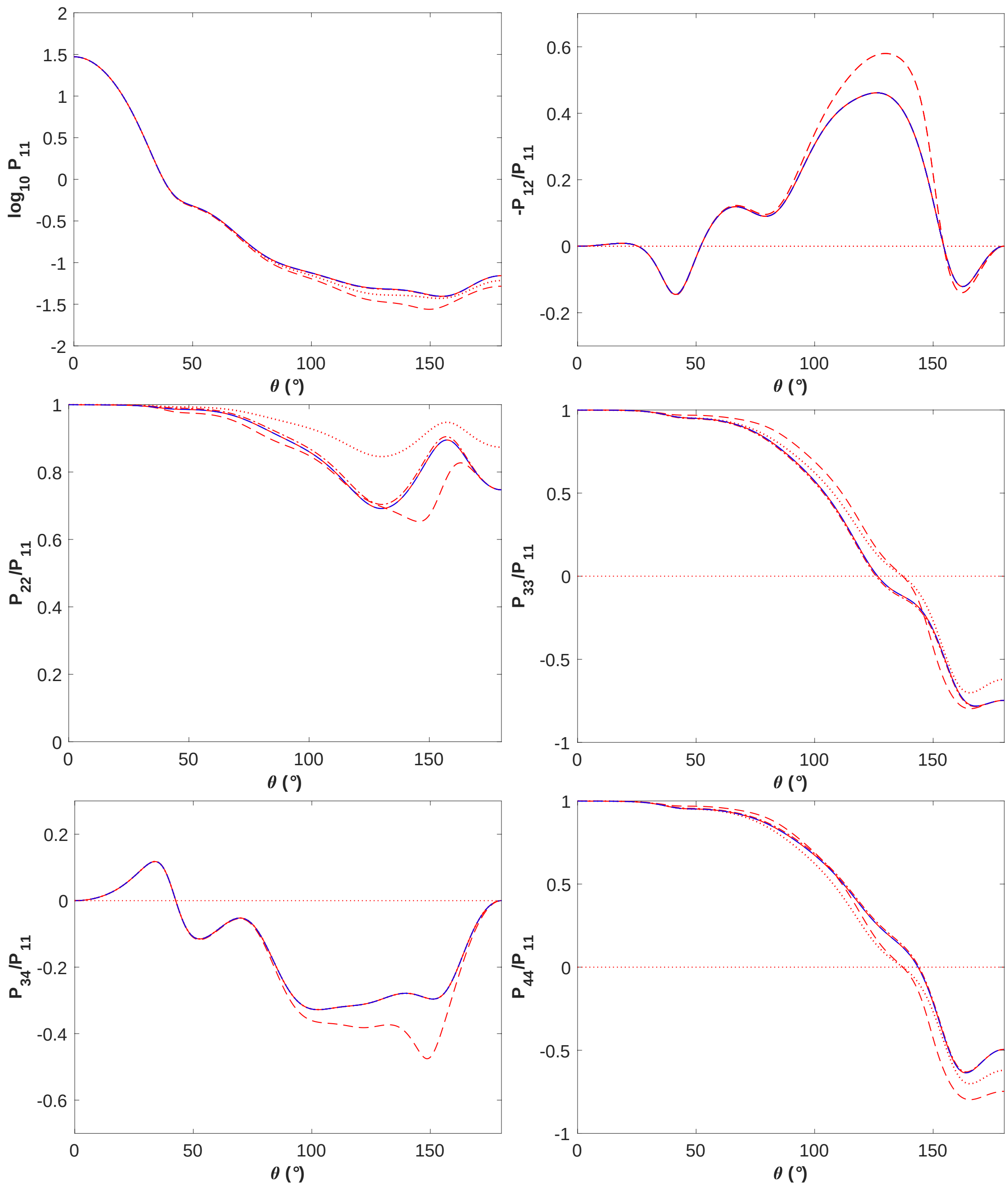

Figure 3: GRS ice.

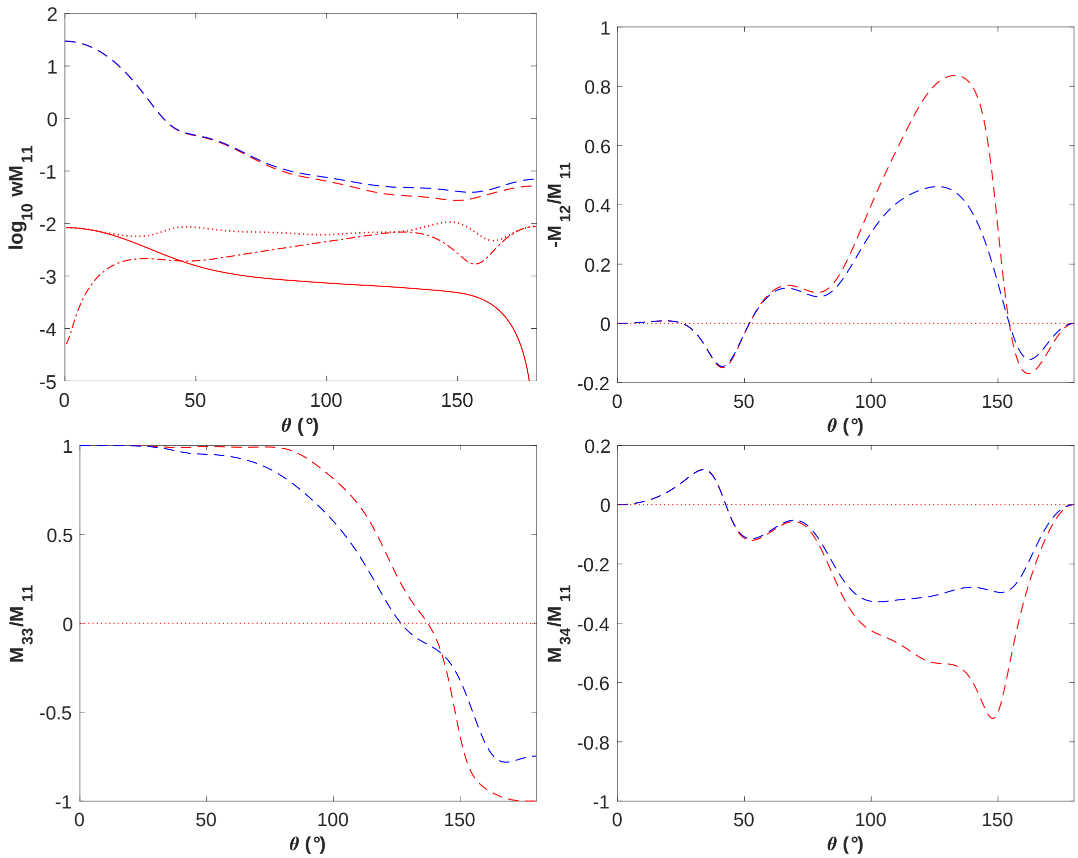

Figure 4: GRS ice.

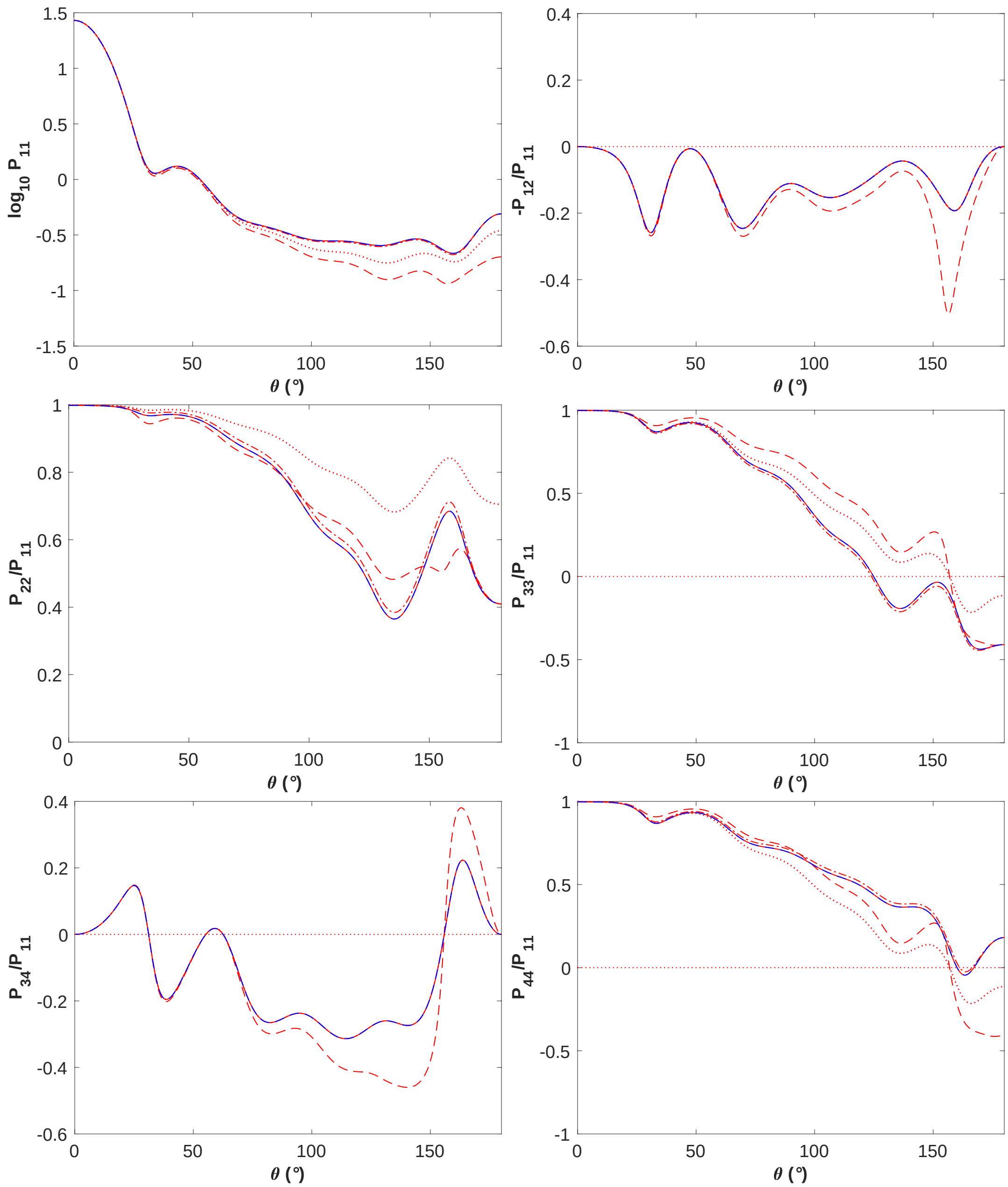

Figure 5: GRS silicate.

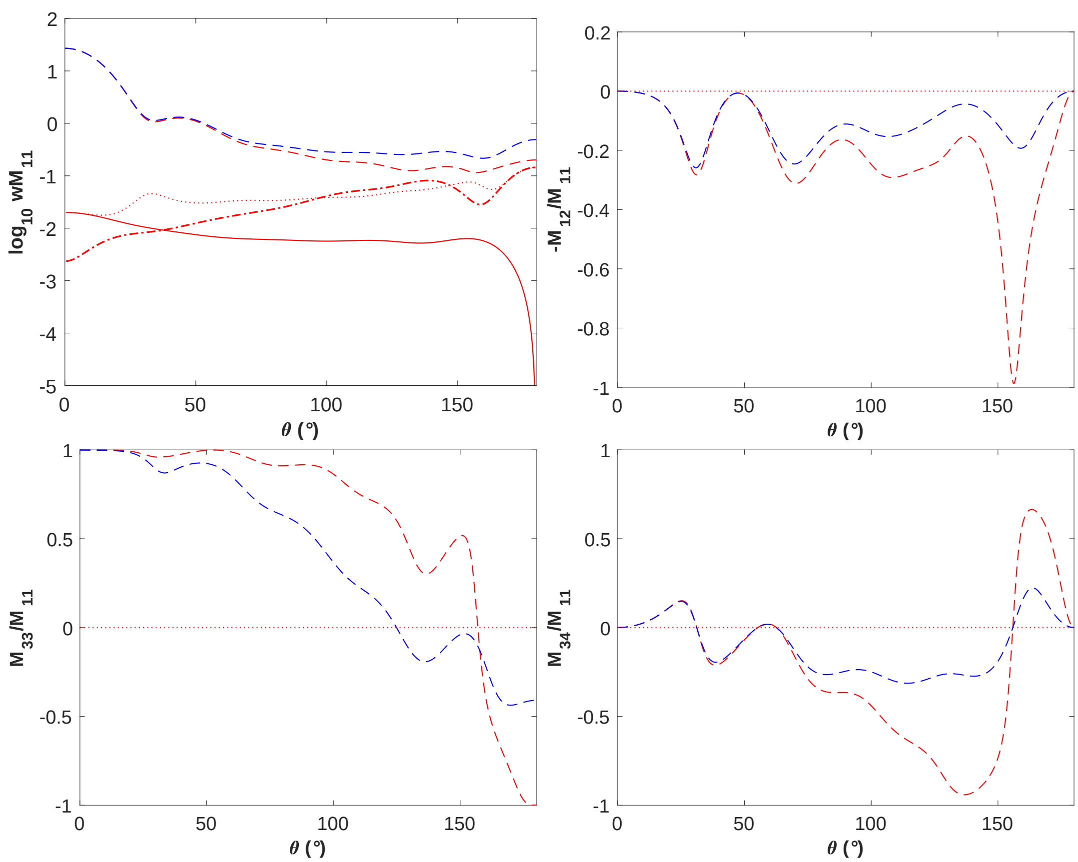

Figure 6: GRS silicate.

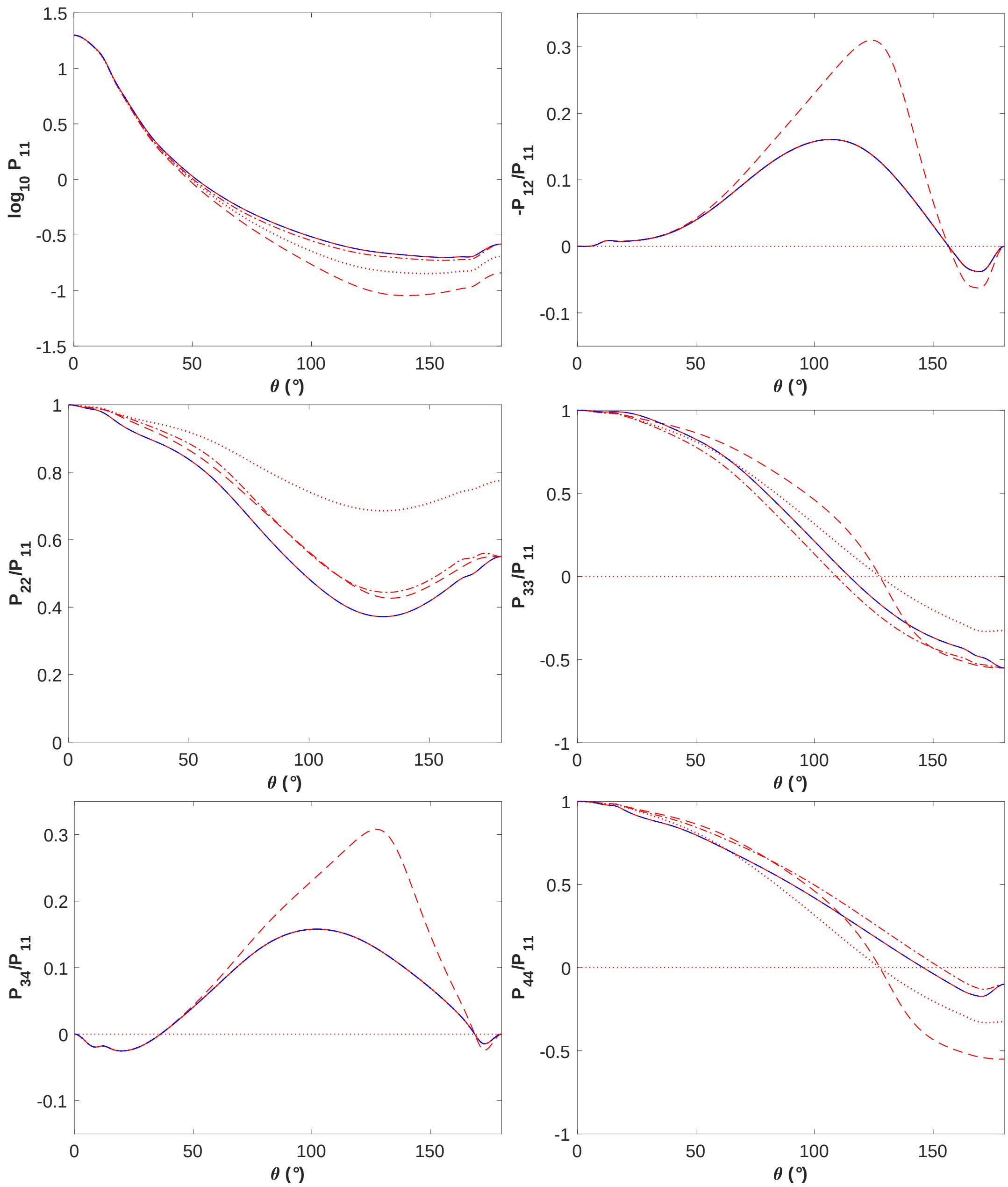

Figure 7: Feldspar.

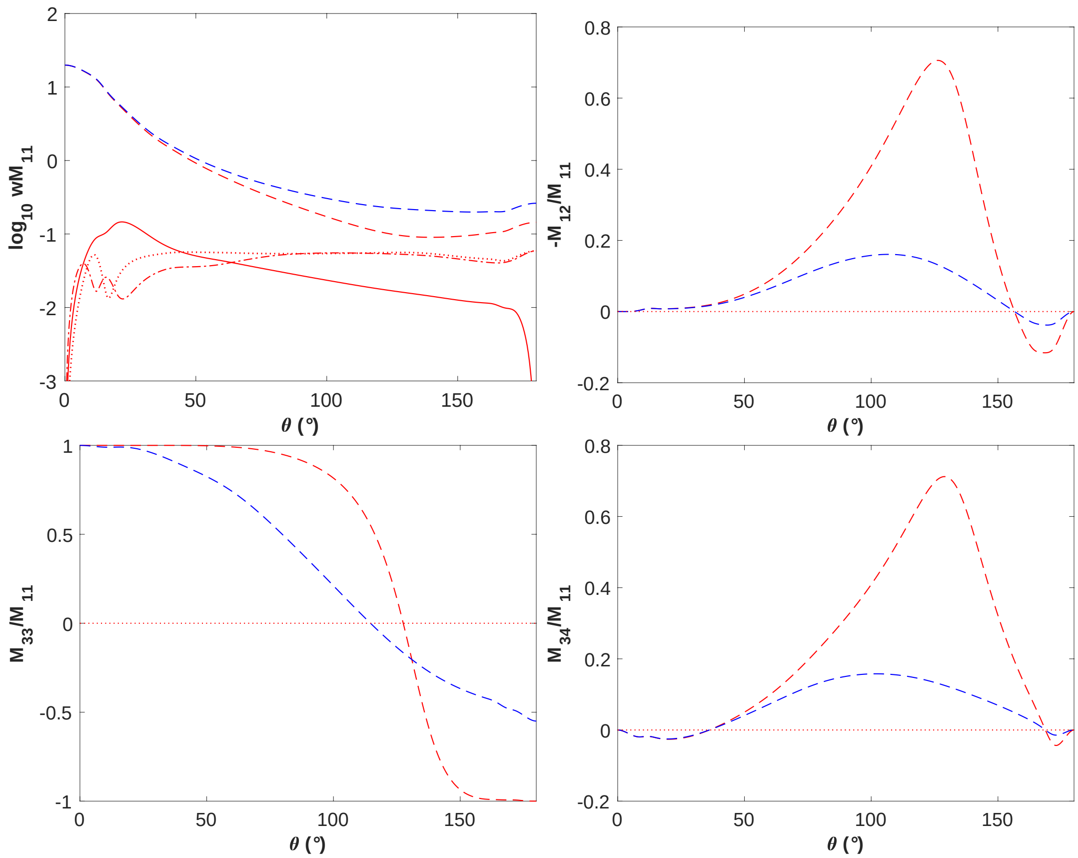

Figure 8: Feldspar.

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URL https://www.sciencedirect.com/science/article/pii/S0022407321001151

Appendix A. Alternative analytical derivation

Alternatively, the decomposition can be derived directly from Eqs. 1 and 6 by assuming that the matrices **U**, **V**, **W**, and **Z** are of the form depicted in Eq. 10 and assuming that the matrices **U** and **V** are closely related (cf. Eq. 11):

$$\frac{V_{12}}{V_{11}} = -\frac{U_{12}}{U_{11}}, \qquad \frac{V_{33}}{V_{11}} = -\frac{U_{33}}{U_{11}}, \qquad \frac{V_{34}}{V_{11}} = -\frac{U_{34}}{U_{11}}. \tag{A.1}$$

Now the task is to determine the functions U_{11}, U_{12}, U_{33}, U_{34}, V_{11}, V_{12}, V_{34}, W_{11}, and Z_{11}, and the weights w_U, w_V, w_W, and w_Z (recalling the normalization conditions in Eq. 6) so that $\mathbf{P}_0$ is modelled by $\mathbf{P}$. Since **U** and **V** are pure Mueller matrices, we have the conditions

$$\begin{aligned} U_{11}^2 &= U_{12}^2 + U_{33}^2 + U_{34}^2, \\ V_{11}^2 &= V_{12}^2 + V_{33}^2 + V_{34}^2. \end{aligned} \tag{A.2}$$

The equivalent condition is a priori enforced for **W** and **Z** by their definition in Eq. 10.

Let us start with the explicit equations resulting from the model in Eq. 6:

$$\begin{aligned} w_U\, U_{11} + w_V\, V_{11} + w_W\, W_{11} + w_Z\, Z_{11} &= a_1, \\ w_U\, U_{11} + w_V\, V_{11} - w_W\, W_{11} - w_Z\, Z_{11} &= a_2, \\ w_U\, U_{33} + w_V\, V_{33} - w_W\, W_{11} + w_Z\, Z_{11} &= a_3, \end{aligned}$$

$$
\begin{aligned}
w_U\; U_{33} + w_V\; V_{33} + w_W\; W_{11} - w_Z\; Z_{11} &= a_4, \\
w_U\; U_{12} + w_V\; V_{12} &= b_1, \\
w_U\; U_{34} + w_V\; V_{34} &= b_2.
\end{aligned}
\tag{A.3}
$$

We make the *educated guess* based on the analyses of the Cloude coherency matrix that the matrices **U** and **V** are interrelated as depicted in Eq.¨A.1. It follows that

$$
\begin{aligned}
w_U\; U_{11} + w_V\; V_{11} &= \frac{1}{2}\,(a_1 + a_2), \\
w_W\; W_{11} + w_Z\; Z_{11} &= \frac{1}{2}\,(a_1 - a_2), \\
w_W\; W_{11} - w_Z\; Z_{11} &= \frac{1}{2}\,(-a_3 + a_4). \\
(w_U U_{11} - w_V V_{11})\; U_{33} &= \frac{1}{2}\,(a_3 + a_4)\; U_{11}, \\
(w_U U_{11} - w_V V_{11})\; U_{12} &= b_1\; U_{11}, \\
(w_U U_{11} - w_V V_{11})\; U_{34} &= b_2\; U_{11}.
\end{aligned}
\tag{A.4}
$$

This leads to

$$
\begin{aligned}
W_{11} &= \frac{1}{2w_W}\,(a_1 - a_2 - a_3 + a_4), \\
Z_{11} &= \frac{1}{2w_Z}\,(a_1 - a_2 + a_3 - a_4),
\end{aligned}
\tag{A.5}
$$

where, from the normalization requirement in Eq. 7,

$$
\begin{aligned}
w_W &= \frac{1}{4} + \frac{1}{8}\int_0^{\pi} d\theta \sin\theta\; \left[-a_2(\theta) - a_3(\theta) + a_4(\theta)\right], \\
w_Z &= \frac{1}{4} + \frac{1}{8}\int_0^{\pi} d\theta \sin\theta\; \left[-a_2(\theta) + a_3(\theta) - a_4(\theta)\right],
\end{aligned}
\tag{A.6}
$$

which completes the derivation of the matrices **W** and **Z** and their weights.

The elements U_{11} and V_{11} and the weights w_U, w_V follow from Eqs. A.2 and A.4 in a straightforward way:

$$
\begin{aligned}
U_{11} &= \frac{1}{2w_U}\left[\frac{1}{2}(a_1 + a_2) + \sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2}\right], \\
V_{11} &= \frac{1}{2w_V}\left[\frac{1}{2}(a_1 + a_2) - \sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2}\right]
\end{aligned}
\tag{A.7}
$$

and

$$w_U = \frac{1}{4} + \frac{1}{8} \int_0^\pi d\theta \sin\theta \left[a_2(\theta) + 2\sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2} \right],$$

$$w_V = \frac{1}{4} + \frac{1}{8} \int_0^\pi d\theta \sin\theta \left[a_2(\theta) - 2\sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2} \right], \quad \text{(A.8)}$$

Finally, the matrix elements U_{12}, U_{33}, and U_{34} as well as V_{12}, V_{33}, and V_{34} are determined with the help of Eqs. A.4, A.1, and A.7:

$$\frac{U_{12}}{U_{11}} = \frac{b_1}{\sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2}},$$

$$\frac{U_{33}}{U_{11}} = \frac{\frac{1}{2}(a_3 + a_4)}{\sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2}},$$

$$\frac{U_{34}}{U_{11}} = \frac{b_2}{\sqrt{b_1^2 + b_2^2 + \frac{1}{4}(a_3 + a_4)^2}}. \quad \text{(A.9)}$$