3

THEORY AND PARAMETERIZATION OF SOLAR RADIATIVE TRANSFER

Electromagnetic radiation emitted from the sun's photosphere, having an equivalent blackbody temperature of about 6000 K, has its peak energy located at about 0.47 μm, according to Wien's displacement law, which has been discussed at the beginning of Chapter 2. The total solar flux that is available to a planet is commonly represented by the solar constant. The solar constant for the earth has been discussed in Section 1.1. The distribution of solar fluxes averaged over a certain period of time (e.g., one solar day) is referred to as *solar insolation*. Solar insolation is a function of latitude and the characteristics of the earth's orbit around the sun (see Section 6.1).

The solar wavelengths that are significant for the transfer of solar flux range from ~0.2 to 4 μm. Shown in Fig. 3.1 is an observed solar irradiance at the top of the atmosphere (TOA) with a spectral resolution of 20 cm^{-1}. Fluctuations in the ultraviolet (uv) and visible regions are due to absorption of various elements in the solar atmosphere. A temperature of about 6000 K fits the observed curve closely in the visible and near-infrared (ir) regions. In the uv region ($< 0.4\,\mu$m), the solar irradiance spectrum deviates significantly from the 6000 K Planck curve. Variations in the solar irradiance in the uv are due primarily to sunspot variations. The spectral solar irradiance available at the earth's surface in a clear atmosphere without aerosols and clouds is also shown in Fig. 3.1. The depletion of solar irradiance is due to the scattering of molecules and the absorption of various molecules and atoms, including atomic and molecular nitrogen and oxygen in the uv, ozone and molecular oxygen in the visible, and water vapor (and to a lesser degree carbon dioxide) in the near-ir. A detailed discussion of gaseous absorption in the solar spectrum will be given in Section 3.8.

In view of the spectra depicted in Figs. 2.1 and 3.1, the solar and thermal ir spectra may be separated into two independent regions at about 4 μm. The overlap between these two spectra is relatively insignificant. This distinction makes it possible to treat the transfer of solar radiation independent of the transfer of thermal

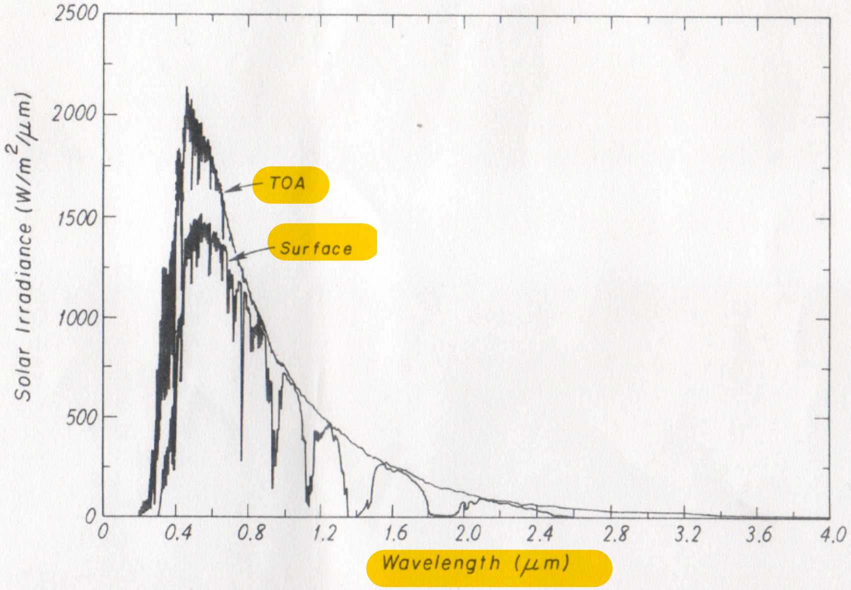

Fig. 3.1 Solar irradiance for a 20 cm^{-1} spectral interval at the top of the atmosphere and at the surface (solar zenith angle 60°) in a clear atmosphere based on the LOWTRAN 7 program (Kneizys et al., 1988).

emission contribution from the earth and the atmosphere can be neglected in the discussion of solar radiative transfer. The one exception to this rule involves the 3.7 μm window, discussed in Subsection 2.2.5.2. If this wavelength is to be used for remote sensing purposes, contributions from solar and thermal ir radiation sources must both be accounted for during daytime.

In the earth's atmosphere, the particulates responsible for scattering range from molecules (~$10^{-4}\,\mu$m), aerosols (~1 μm), water droplets (~10 μm), and ice crystals (~100 μm) to raindrops and hailstones (~1 cm). In view of the ubiquitous nature of aerosols and clouds, scattering plays the dominant role in the transfer of solar radiation. The principles and methodologies for radiative transfer presented in this chapter are primarily developed for plane-parallel atmospheres and an isotropic medium. Subjects relating to radiative transfer in clouds will be comprehensively addressed in Chapter 5.

3.1 Basic equations for solar radiative transfer

In the discussion of the transfer of solar radiation in planetary atmospheres, the plane-parallel assumption, described in Section 2.1, can be followed. The position of the sun, which may be considered as a point light source, must be accounted for in the formulation of the basic radiative transfer equation. The transfer problem

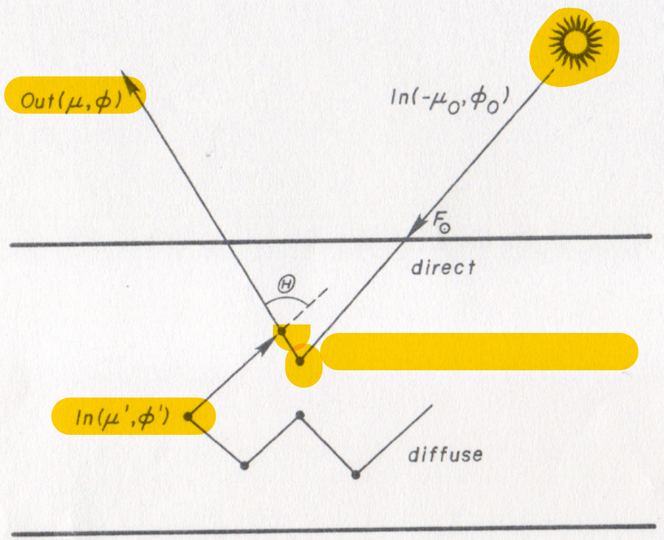

FIG. 3.2 The definitions of direct and diffuse radiation, and the scattering angle, Θ, with respect to solar radiative transfer in plane-parallel atmospheres. The monochromatic solar irradiance at TOA is denoted by $F_\odot$.

direct component is associated with the exponential attenuation of unscattered solar flux in the atmosphere. The diffuse component arises from light beams that undergo multiple-scattering events. In the polar coordinate, the directions defining the incoming and outgoing light beams may be expressed by (μ', ϕ') and (μ, ϕ), respectively, where $\mu = \cos\theta$, θ is the zenith angle and ϕ is the azimuthal angle. Let μ and $-\mu$ denote the upward and downward directions associated with the light beams. Thus the position of the sun may be denoted by $(-\mu_0, \phi_0)$, where $\mu_0 = \cos\theta_0$, and θ_0 and ϕ_0 denote the solar zenith and azimuthal angles, as seen in Fig. 3.2. For simplicity of presentation in this chapter, we shall omit the wavelength subscript, λ, in the radiative parameters.

Based on Eq. (2.1.2) and under the plane-parallel assumption, the basic equation governing the transfer of diffuse solar intensity may be written in the form

$$\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - J(\tau, \mu, \phi). \tag{3.1.1}$$

Three factors contribute to the source function: emission, multiple scattering of diffuse intensity, and single scattering of direct solar irradiance (flux density) at TOA, $F_\odot$, which is attenuated to the level τ. Energy emitted from the earth and the atmosphere with an equilibrium temperature of ~255 K is practically negligible in comparison with that emitted from the sun. Thus, in the solar region, the source

function is given by

$$J(\tau, \mu, \phi) = \frac{\tilde{\omega}}{4\pi} \int_0^{2\pi} \int_{-1}^{1} I(\tau, \mu', \phi') P(\mu, \phi; \mu', \phi')\, d\mu'\, d\phi' + \frac{\tilde{\omega}}{4\pi} F_\odot P(\mu, \phi; -\mu_0, \phi_0) e^{-\tau/\mu_0}, \tag{3.1.2}$$

where P is the scattering phase function (or, simply, the phase function), which represents the angular distribution of scattered energy as a function of direction. The phase function redirects the incoming intensity in the direction (μ', ϕ') to the direction (μ, ϕ), and the integrals account for all possible scattering events within the 4π solid angle. The single-scattering albedo, $\tilde{\omega}$, is defined as the ratio of the scattering cross section σ_s to the extinction (scattering plus absorption) cross section, σ_e; that is,

$$\tilde{\omega} = \frac{\sigma_s}{\sigma_e}. \tag{3.1.3}$$

The single-scattering albedo, phase function, and extinction cross section are fundamental parameters in radiative transfer. These parameters are functions of the incident wavelength, particle size and shape, and refractive index with respect to wavelength. The first and second terms on the right-hand side of Eq. (3.1.2) represent the diffuse (multiple scattering) and direct (single scattering of the direct solar flux) contributions, respectively.

The phase function depends on the incoming and outgoing directions. For spherical particles or nonspherical particles randomly oriented in space, the phase function can be expressed in terms of the scattering angle: the angle defining the incoming and outgoing directions shown in Fig. 3.2. We may express the phase function in terms of a known mathematical function for the purpose of solving Eq. (3.1.2), the first-order differential–integral equation. The Legendre polynomials, by virtue of their unique mathematical properties, have been used extensively in the analysis of radiative transfer problems. In terms of Legendre polynomials, the phase function may be written in the form

$$P(\cos\Theta) = \sum_{\ell=0}^{N} \tilde{\omega}_\ell P_\ell(\cos\Theta). \tag{3.1.4}$$

The Legendre polynomials have the following orthogonal and recurrence properties:

$$\int_{-1}^{1} P_\ell(\mu) P_k(\mu)\, d\mu = \begin{cases} 0, & \ell \neq k \\ \frac{2}{2\ell+1}, & \ell = k \end{cases} \tag{3.1.5}$$

$$\mu P_\ell(\mu) = \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1}. \tag{3.1.6}$$

From the orthogonal equation, the expansion coefficient is given by

$$\tilde{\omega}_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} P(\cos\Theta)P_\ell(\cos\Theta)\,d\cos\Theta, \qquad \ell = 0, 1, \ldots, N. \quad (3.1.7)$$

In the present notation, the phase function is normalized to unity, viz.,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\cos\Theta)\,d\cos\Theta\,d\phi = 1. \quad (3.1.8)$$

Thus we have $\tilde{\omega}_0 = 1$. From spherical geometry, the cosine of the scattering angle can be expressed in terms of the incoming and outgoing directions in the form

$$\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi'-\phi). \quad (3.1.9)$$

Using Eq. (3.1.9), the phase function defined in Eq. (3.1.4) may be written

$$P(\mu,\phi;\mu',\phi') = \sum_{\ell=0}^{N}\tilde{\omega}_\ell P_\ell\left(\mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi'-\phi)\right). \quad (3.1.10)$$

Moreover, from the addition theorem for Legendre polynomials (see, e.g., Liou, 1980), we find

$$P(\mu,\phi;\mu',\phi') = \sum_{m=0}^{N}\sum_{\ell=m}^{N}\tilde{\omega}_\ell^m P_\ell^m(\mu)P_\ell^m(\mu')\cos m(\phi'-\phi), \quad (3.1.11a)$$

where P_ℓ^m denotes the associated Legendre polynomials, and

$$\tilde{\omega}_\ell^m = (2-\delta_{0,m})\,\tilde{\omega}_\ell\frac{(\ell-m)!}{(\ell+m)!}, \qquad \ell = m,\ldots,N, 0 \le m \le N, \quad (3.1.11b)$$

$$\delta_{0,m} = \begin{cases} 1, & m = 0 \\ 0, & \text{otherwise.} \end{cases}$$

In view of the expansion of the phase function, the diffuse intensity may also be expanded in the cosine series in the form

$$I(\tau,\mu,\phi) = \sum_{m=0}^{N} I^m(\tau,\mu)\cos m(\phi_0-\phi). \quad (3.1.12)$$

Substituting Eqs. (3.1.11) and (3.1.12) into Eq. (3.1.2), and using the orthogonality of the associated Legendre polynomials, the equation of transfer splits into $(N+1)$ independent equations, and may be written

$$\mu\frac{dI^m(\tau,\mu)}{d\tau} = I^m(\tau,\mu) - (1+\delta_{0,m})\frac{\tilde{\omega}}{4}\sum_{\ell=m}^{N}\tilde{\omega}_\ell^m P_\ell^m(\mu)$$

$$\times\int_{-1}^{1} P_\ell^m(\mu')I^m(\tau,\mu')\,d\mu'$$

$$-\frac{\tilde{\omega}}{4\pi}\sum_{\ell=m}^{N}\tilde{\omega}_\ell^m P_\ell^m(\mu)P_\ell^m(-\mu_0)F_\odot e^{-\tau/\mu_0},$$

$$m = 0, 1, \ldots N. \quad (3.1.13)$$

Omitting the superscript 0 for simplicity of presentation, for $m = 0$ we have

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\tilde{\omega}}{2}\sum_{\ell=0}^{N}\tilde{\omega}_\ell P_\ell(\mu)\int_{-1}^{1} P_\ell(\mu')I(\tau,\mu')\,d\mu'$$

$$-\frac{\tilde{\omega}}{4\pi}\sum_{\ell=0}^{N}\tilde{\omega}_\ell P_\ell(\mu)P_\ell(-\mu_0)F_\odot e^{-\tau/\mu_0}. \quad (3.1.14)$$

This is the equation of transfer that is independent of the azimuthal angle. From Eq. (3.1.11a), the azimuth-independent phase function may be obtained from

$$P(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\phi;\mu',\phi')\,d\phi'$$

$$= \begin{cases} \sum_{\ell=0}^{N}\tilde{\omega}_\ell P_\ell(\mu)P_\ell(\mu'), & m = 0, \\ 0, & m \neq 0. \end{cases} \quad (3.1.15)$$

Equation (3.1.14) may then be expressed in terms of the azimuth-independent phase function in the form

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\tilde{\omega}}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')\,d\mu'$$

$$-\frac{\tilde{\omega}}{4\pi}P(\mu,-\mu_0)F_\odot e^{-\tau/\mu_0}. \quad (3.1.16)$$

The azimuth-independent phase function defined in Eq. (3.1.15) has the following properties:

$$\frac{1}{2}\int_{-1}^{1} P(\mu,\mu')\,d\mu' = \tilde{\omega}_0 = 1, \quad (3.1.17)$$

$$\frac{1}{2}\int_{-1}^{1} P(\mu,\mu')\mu'\,d\mu' = \tilde{\omega}_1\mu/3. \quad (3.1.18)$$

The monochromatic upward and downward diffuse fluxes at a given optical depth level, τ, are defined by

$$F^{\pm}_{\text{dif}}(\tau) = \int_0^{2\pi}\int_0^{\pm 1} I(\tau,\mu,\phi)\mu\,d\mu\,d\phi, \quad (3.1.19a)$$

tively. Using Eq. (3.1.12), we find

$$F^{\pm}_{\mathrm{dif}}(\tau) = 2\pi \int_0^{\pm 1} I(\tau, \mu)\mu \, d\mu. \tag{3.1.19b}$$

For solar flux computations, it suffices to consider the azimuthally averaged equation for the transfer of diffuse radiation. The direct downward solar flux at level τ is given by the exponential attenuation of the effective solar flux at TOA, $\mu_0 F_\odot$. Thus,

$$F^{-}_{\mathrm{dir}}(\tau) = \mu_0 F_\odot e^{-\tau/\mu_0}. \tag{3.1.20}$$

The total upward and downward fluxes covering the entire solar spectrum, using the height coordinate, may be written

$$F^{+}(z) = \int_0^{\infty} F^{+}_{\mathrm{dif}}(\tau)\, d\lambda, \tag{3.1.21}$$

$$F^{-}(z) = \int_0^{\infty} (F^{-}_{\mathrm{dif}} + F^{-}_{\mathrm{dir}})\, d\lambda. \tag{3.1.22}$$

Thus the net flux is

$$F_s(z) = F^{-}(z) - F^{+}(z). \tag{3.1.23}$$

The heating rate due to the absorption of solar flux in the atmosphere is produced by the divergence of the net solar flux, and is given by

$$\left(\frac{\partial T}{\partial t}\right)_s = \frac{1}{\rho C_p}\frac{dF_s(z)}{dz}, \tag{3.1.24}$$

where ρ is the air density and C_p is the specific heat at constant pressure.

3.2 Exact methods for radiative transfer

3.2.1 *Discrete-ordinates method*

The discrete-ordinates method for radiative transfer has been elegantly developed by Chandrasekhar (1950) for applications to the transfer of radiation in planetary atmospheres. Liou (1973a) has demonstrated that the discrete-ordinates method is a useful and powerful method for the computation of radiation fields in aerosol and cloudy atmospheres. In essence, the method involves the discretization of the basic radiative transfer equation and the solution of a set of first-order differential equations. With the advance in numerical techniques for solving differential equations, the discrete-ordinates method has been found to be both efficient and accurate for calculations of scattered intensities and fluxes (Stamnes and Swanson, 1981; Stamnes and Dale, 1981).

in Eq. (3.1.13), and rewrite these equations as follows:

$$\mu \frac{dI^m(\tau,\mu)}{d\tau} = I^m(\tau,\mu) - J^m(\tau,\mu). \tag{3.2.1}$$

The source function is given by

$$J^m(\tau,\mu) = (1+\delta_{0,m})\frac{\tilde{\omega}}{4}\sum_{\ell=m}^{N} \tilde{\omega}_\ell^m(\mu) P_\ell^m(\mu) \int_{-1}^{1} P_\ell^m(\mu') I^m(\tau,\mu')\, d\mu'$$
$$+ \frac{\tilde{\omega}}{4\pi}\sum_{\ell=m}^{N} \tilde{\omega}_\ell^m P_\ell^m(\mu) P_\ell^m(-\mu_0) F_\odot e^{-\tau/\mu_0}. \tag{3.2.2}$$

To proceed with the solution of Eq. (3.2.1), we first discretize the equation by replacing μ with μ_i $(i = -n, \ldots, n$, with $n = 1, 2, \ldots)$ and the integral,

$$\int_{-1}^{1} f(\mu)\, d\mu = \sum_{j=-n}^{n} f(\mu_j) a_j,$$

with the weight a_j. The homogeneous solution for the set of first-order differential equations may be written

$$I^m(\tau,\mu_i) = \sum_{j=-n}^{n} L_j^m \phi_j^m(\mu_i) e^{-k_j^m \tau}, \tag{3.2.3}$$

where ϕ_j^m and k_j^m denote the eigenvectors and eigenvalues, respectively, and L_j^m are coefficients to be determined from appropriate boundary conditions. On substituting Eq. (3.2.3) into the homogeneous part of Eq. (3.2.1), the eigenvectors may be expressed by

$$\phi_j^m(\mu_i) = \frac{(1+\delta_{0,m})\tilde{\omega}}{4(1+\mu_j k_j^m)} \sum_{\ell=m}^{N} \tilde{\omega}_\ell^m P_\ell^m(\mu_i) \sum_{\alpha=-n}^{n} a_\alpha P_\ell^m(\mu_\alpha) \phi_j^m(\mu_\alpha). \tag{3.2.4}$$

The particular solution may be written in the form

$$I_p^m(\tau,\mu_i) = Z^m(\mu_i) e^{-\tau/\mu_0}. \tag{3.2.5}$$

From Eq. (3.2.1) we have

$$Z^m(\mu_i) = \frac{\tilde{\omega}}{4(1+\mu_i/\mu_0)} \sum_{\ell=m}^{N} \tilde{\omega}_\ell^m P_\ell^m(\mu_i)$$
$$\times \left(\sum_{\alpha=-n}^{n} a_\alpha P_\ell^m(\mu_\alpha) Z^m(\mu_\alpha) + P_\ell^m(-\mu_0) \frac{F_\odot}{\pi} \right). \tag{3.2.6}$$

Equations (3.2.4) and (3.2.6) are linear equations in ϕ_j^m and Z^m and may be solved numerically. The complete solution for Eq. (3.2.1) is the sum of the general solution for the associated homogeneous system of the differential equations and the particular solution. Thus,

$$I^m(\tau, \mu_i) = \sum_{j=-n}^{n} L_j^m \phi_j^m(\mu_i) e^{-k_j^m \tau} + Z^m(\mu_i) e^{-\tau/\mu_0}, \qquad i = -n, \ldots, n. \tag{3.2.7}$$

In order to determine the unknown coefficients, L_j^m, boundary conditions must be imposed. Assuming that there are no external radiation sources from above or below a layer with an optical depth of τ_*, we have

$$\left.\begin{aligned} I^m(0, -\mu_i) &= 0 \\ I^m(\tau_*, +\mu_i) &= 0 \end{aligned}\right\} \text{for } i = 1, \ldots, n \text{ and } m = 0, \ldots, N. \tag{3.2.8}$$

A mathematical procedure from which the eigenvalues k_j^m may be calculated from a recurrence characteristic equation has been developed by Chandrasekhar (1950). The eigenvectors $\phi_j^m(\mu_i)$ may be expressed in terms of known functions, which contain the eigenvalues, and the particular solution is related to a known mathematical function, the so-called H function. The characteristic equation for the eigenvalues derived by Chandrasekhar is mathematically, as well as numerically, ambiguous. The method is unstable for highly peaked phase functions, as pointed out by Liou (1973a), who discovered that the solution of the characteristic equation may be formulated as an algebraic eigenvalue problem. Further, Asano (1975) has shown that the degree of the characteristic equation for the eigenvalues can be reduced by a factor of two because the solution for the eigenvalues may be obtained by solving a characteristic polynomial of degree n for k^2. Both of these authors have expanded the matrix in polynomial form to solve the characteristic equation for the eigenvalues corresponding to the associated homogeneous system of the differential equations. However, the expansion in polynomial form is not a stable numerical scheme for obtaining eigenvalues. To solve the algebraic eigenvalue problem, a well-developed numerical subroutine found in the IMSL User's Manual (1987) may be used to compute the eigenvalues and eigenvectors of a real general matrix in connection with the discrete-ordinates method. Stamnes and Dale (1981) have shown that azimuthally dependent scattered intensities may be computed accurately and efficiently using numerical methods.

In the discrete-ordinates method for radiative transfer, analytical solutions for the diffuse intensity are explicitly given for any optical depth. Thus the internal radiation field can be evaluated without additional computational effort. Moreover, useful approximations can be developed from this method for flux calculations.

We now confine our discussion to the transfer of solar fluxes, and consider the azimuth-independent component in the diffuse intensity component. On replacing the integral with a summation, Eq. (3.1.16) may be written in the form

$$\begin{aligned} \mu_i \frac{dI(\tau, \mu_i)}{d\tau} &= I(\tau, \mu_i) - \frac{\tilde{\omega}}{2} \sum_{j=-n}^{n} I(\tau, \mu_j) P(\mu_i, \mu_j) a_j \\ &\quad - \frac{\tilde{\omega}}{4\pi} F_\odot P(\mu_i, -\mu_0) e^{-\tau/\mu_0}, \qquad i = -n, \ldots, n, \end{aligned} \tag{3.2.9}$$

where we may select the quadrature weights and points that satisfy $a_{-j} = a_j (\sum_j a_j = 2)$ and $\mu_{-j} = -\mu_j$. To simplify this equation, we may define

$$c_{i,j} = \frac{\tilde{\omega}}{2} a_j P(\mu_i, \mu_j) = \frac{\tilde{\omega}}{2} a_j \sum_{\ell=0}^{N} \tilde{\omega}_\ell P_\ell(\mu_i) P_\ell(\mu_j), \qquad j = -n, \ldots -0, \ldots, n, \tag{3.2.10}$$

and

$$I(\tau, -\mu_0) = e^{-\tau/\mu_0} F_\odot / 2\pi, \tag{3.2.11}$$

where we set $a_{-0} = 1$ and the notation -0 is used to be consistent with the definition, $\mu_{-0} = -\mu_0$. By virtue of the definiton of Legendre polynomials, we have

$$c_{i,-j} = c_{-i,j}, \qquad c_{-i,-j} = c_{i,j}, \qquad i \neq -0. \tag{3.2.12}$$

Moreover, we may define

$$b_{i,j} = \begin{cases} c_{i,j}/\mu_i, & i \neq j \\ (c_{i,j} - 1)/\mu_i, & i = j. \end{cases} \tag{3.2.13}$$

It follows that $b_{i,j} = -b_{-i,-j}$, and $b_{i,-j} = -b_{-i,j}$. Using the preceding definitions, Eq. (3.2.9) becomes

$$\frac{dI(\tau, \mu_i)}{d\tau} = \sum_j b_{i,j} I(\tau, \mu_j). \tag{3.2.14}$$

We may separate the upward and downward intensities in the forms

$$\frac{dI(\tau, \mu_i)}{d\tau} = \sum_{j=1}^{n} b_{i,j} I(\tau, \mu_i) + \sum_{j=0}^{n} b_{i,-j} I(\tau, -\mu_j), \tag{3.2.15a}$$

$$\frac{dI(\tau, -\mu_i)}{d\tau} = \sum_{j=1}^{n} b_{-i,j} I(\tau, \mu_j) + \sum_{j=0}^{n} b_{-i,-j} I(\tau, -\mu_j). \tag{3.2.15b}$$

In terms of a matrix representation for the homogeneous part, we write

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{I}^+\\ \mathbf{I}^-\end{bmatrix} = \begin{bmatrix}\mathbf{b}^+ & \mathbf{b}^-\\ -\mathbf{b}^- & -\mathbf{b}^+\end{bmatrix}\begin{bmatrix}\mathbf{I}^+\\ \mathbf{I}^-\end{bmatrix}, \tag{3.2.16}$$

where

$$\mathbf{I}^{\pm} = \begin{bmatrix} I(\tau, \pm\mu_1)\\ I(\tau, \pm\mu_2)\\ \cdot\\ \cdot\\ \cdot\\ I(\tau, \pm\mu_n)\end{bmatrix}, \tag{3.2.17}$$

and $\mathbf{b}^{\pm}$ denotes the elements associated with $b_{i,j}$ and $b_{i,-j}$. Since Eq. (3.2.16) is a first-order differential equation, we may seek a solution in the form

$$\mathbf{I}^{\pm} = \phi^{\pm} e^{-k\tau}. \tag{3.2.18}$$

Substituting Eq. (3.2.18) into Eq. (3.2.16) leads to

$$\begin{bmatrix}\mathbf{b}^+ & \mathbf{b}^-\\ -\mathbf{b}^- & -\mathbf{b}^+\end{bmatrix}\begin{bmatrix}\phi^+\\ \phi^-\end{bmatrix} = -k\begin{bmatrix}\phi^+\\ \phi^-\end{bmatrix}. \tag{3.2.19}$$

Equation (3.2.19) may be solved as a standard eigenvalue problem. In the discrete-ordinates method for radiative transfer, the eigenvalues associated with the differential equations are all real and occur in pairs ($\pm k$) because of the symmetry of the $\mathbf{b}$ matrix. Thus the rank of the matrix may be reduced by a factor of 2. To accomplish this reduction, we rewrite Eq. (3.2.19) in the forms

$$\mathbf{b}^+\phi^+ + \mathbf{b}^-\phi^- = -k\phi^+, \tag{3.2.20a}$$

$$\mathbf{b}^-\phi^+ + \mathbf{b}^+\phi^- = k\phi^-, \tag{3.2.20b}$$

Adding and subtracting these two equations yield

$$(\mathbf{b}^+ - \mathbf{b}^-)(\mathbf{b}^+ + \mathbf{b}^-)(\phi^+ + \phi^-) = k^2(\phi^+ + \phi^-). \tag{3.2.21}$$

Hence, the eigenvectors of the original system, $\phi^{\pm}$, can now be obtained from the reduced system, $(\phi^+ + \phi^-)$, in terms of the eigenvalue k^2. As discussed by Chandrasekhar (1950), the Gaussian quadrature formula for the complete angular range, $-1 < \mu < 1$, is efficient and accurate for the discretization of the basic radiative transfer equation. However, the Gaussian quadrature can also be applied separately to the half-ranges, $-1 < \mu < 0$ and $0 < \mu < 1$, which are referred to as the *double-Gauss quadrature* and appear to offer numerical advantages when upward and downward radiation streams are treated separately.

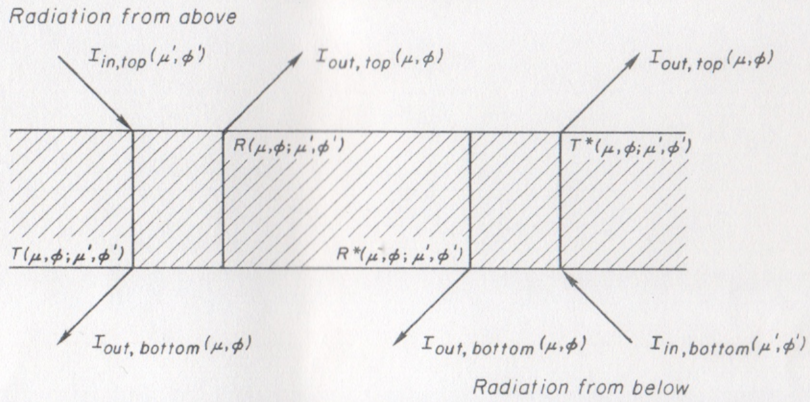

FIG. 3.3 Configurations for radiation incident from above and below, and the definitions of the reflection and transmission functions.

3.2.2 *Adding method*

The adding method has been demonstrated to be a powerful tool for multiple-scattering calculations. The principle for the method was stated by Stokes (1862) in a problem dealing with reflection and transmission by glass plates. Peebles and Plesset (1951) have developed the adding method theory for application to gamma-ray transfer. van de Hulst (1980) has presented a set of adding equations for multiple scattering that is now commonly used.

To introduce the adding method for radiative transfer, the reflection function R and the transmission function T must first be defined. Consider a light beam incident from above, as represented in Fig. 3.3. The reflected and transmitted intensities of this beam are expressed in terms of the incident intensity in the form

$$I_{\text{out,top}}(\mu, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\mu, \phi; \mu', \phi') I_{\text{in,top}}(\mu', \phi')\mu'\, d\mu'\, d\phi', \tag{3.2.22}$$

$$I_{\text{out,bottom}}(\mu, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T(\mu, \phi; \mu', \phi') I_{\text{in,top}}(\mu', \phi')\mu'\, d\mu'\, d\phi', \tag{3.2.23}$$

Likewise, if the light beam comes from below, as is also shown in Fig. 3.3, we write

$$I_{\text{out,bottom}}(\mu, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R^*(\mu, \phi; \mu', \phi') I_{\text{in,bottom}}(\mu', \phi')\mu'\, d\mu'\, d\phi', \tag{3.2.24}$$

$$I_{\text{out,top}}(\mu, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T^*(\mu, \phi; \mu', \phi') I_{\text{in,bottom}}(\mu', \phi')\mu'\, d\mu'\, d\phi', \tag{3.2.25}$$

where R^* and T^* are so defined, and the superscript $*$ signifies that radiation comes from below.

Consider the transfer of monochromatic solar radition. The incident solar intensity, in the present notation, may be written in the form

$$I_{\mathrm{in,top}}(-\mu_0, \phi_0) = \delta(\mu' - \mu_0)\delta(\phi' - \phi_0)F_{\odot}, \tag{3.2.26}$$

where δ is the Dirac delta function. Using Eq. (3.2.26), the reflection and transmission functions defined in Eqs. (3.2.22) and (3.2.23) are given by

$$R(\mu, \phi; \mu_0, \phi_0) = \pi I_{\mathrm{out,top}}(\mu, \phi)/\mu_0 F_{\odot}, \tag{3.2.27}$$

$$T(\mu, \phi; \mu_0, \phi_0) = \pi I_{\mathrm{out,bottom}}(\mu, \phi)/\mu_0 F_{\odot}. \tag{3.2.28}$$

Under the single-scattering approximation, the source function defined in Eq. (3.1.2) may be written in the form

$$J(\tau, \mu, \phi) = \frac{\tilde{\omega}}{4\pi} F_{\odot} P(\mu, \phi; -\mu_0, \phi_0) e^{-\tau/\mu_0}. \tag{3.2.29}$$

Assuming that there are no diffuse intensities from the top and bottom of the layer with an optical depth $\Delta\tau$, then the radiation boundary conditions are as follows:

$$I_{\mathrm{in,top}}(\mu, \phi) = 0$$
$$I_{\mathrm{in,bottom}}(\mu, \phi) = 0. \tag{3.2.30}$$

With these boundary conditions, the reflected and transmitted diffuse intensities due to single scattering can be derived directly from the basic radiative transfer equation. Thus the solutions for the reflection and transmission functions for an optical depth $\Delta\tau$ are given by

$$R(\mu, \phi; \mu_0, \phi_0) = \frac{\tilde{\omega}}{4(\mu + \mu_0)} P(\mu, \phi; -\mu_0, \phi_0) \left\{ 1 - \exp\left[-\Delta\tau \left(\frac{1}{\mu} + \frac{1}{\mu_0} \right) \right] \right\}, \tag{3.2.31}$$

$$T(\mu, \phi; \mu_0, \phi_0) = \begin{cases} \dfrac{\tilde{\omega}}{4(\mu - \mu_0)} P(-\mu, \phi; -\mu_0, \phi_0)(e^{-\Delta\tau/\mu} - e^{-\Delta\tau/\mu_0}), & \mu \neq \mu_0 \\ \dfrac{\tilde{\omega}\Delta\tau}{4\mu_0^2} P(-\mu, \phi; -\mu_0, \phi_0) e^{-\Delta\tau/\mu_0}, & \mu = \mu_0. \end{cases} \tag{3.2.32}$$

If we consider a layer in which $\Delta\tau$ is very small (e.g., $\Delta\tau \approx 10^{-8}$), Eqs. (3.2.31) and (3.2.32) may further be simplified in the forms

$$R(\mu, \phi; \mu_0, \phi_0) = \frac{\tilde{\omega}\Delta\tau}{4\mu\mu_0} P(\mu, \phi; -\mu_0, \phi_0), \tag{3.2.33}$$

$$T(\mu, \phi; \mu_0, \phi_0) = \frac{\tilde{\omega}\Delta\tau}{4\mu\mu_0} P(-\mu, \phi; -\mu_0, \phi_0). \tag{3.2.34}$$

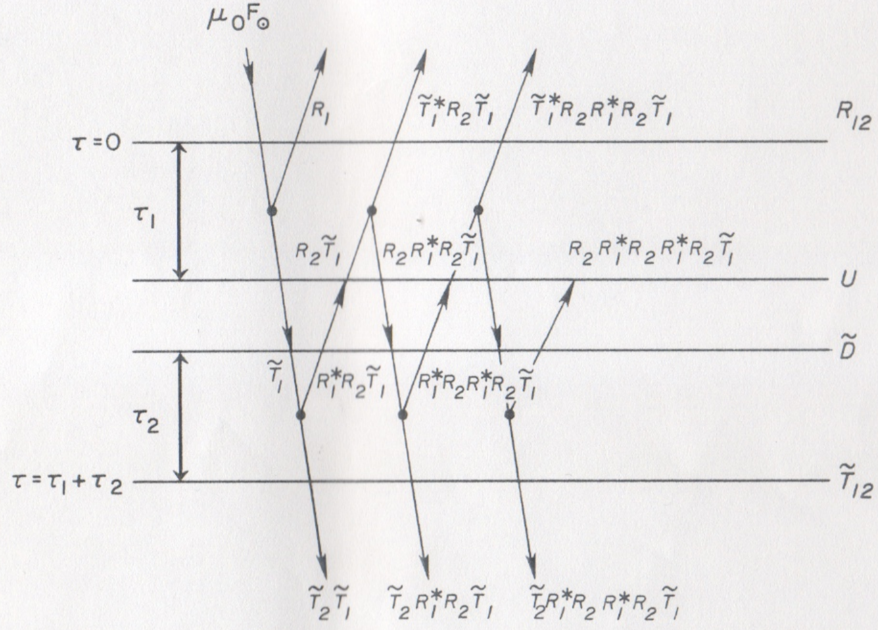

FIG. 3.4 Configuration of the adding method. The two layers of optical depths τ_1 and τ_2 are rendered, for convenient illustration, as though they were physically separated.

For a thin homogeneous layer, the reflection and transmission functions will be the same regardless of whether the light beam is incident from above or below. Thus, $R^* = R$ and $T^* = T$. However, when we proceed with the adding principle for radiative transfer, the reflection and transmission functions for combined layers will depend on the direction of the incoming light beam.

In reference to Fig. 3.4, consider two layers, one on top of the other. Let the reflection and total (direct and diffuse) transmission functions be denoted R_1 and $\tilde{T}_1$ for the first layer and as R_2 and $\tilde{T}_2$ for the second layer, respectively. We define $\tilde{D}$ and U for the combined total transmission and reflection functions between layers 1 and 2. In principle, the light beam may undergo an infinite number of scattering events. Upon tracing the light beam in the two layers, we find the combined reflection and total transmission functions as follows:

$$\begin{aligned} R_{12} &= R_1 + \tilde{T}_1^* R_2 \tilde{T}_1 + \tilde{T}_1^* R_2 R_1^* R_2 \tilde{T}_1 + \tilde{T}_1^* R_2 R_1^* R_2 R_1^* R_2 \tilde{T}_1 + \cdots \\ &= R_1 + \tilde{T}_1^* R_2 \left[1 + R_1^* R_2 + (R_1^* R_2)^2 + \cdots \right] \tilde{T}_1 \\ &= R_1 + \tilde{T}_1^* R_2 (1 - R_1^* R_2)^{-1} \tilde{T}_1, \end{aligned} \tag{3.2.35}$$

$$\begin{aligned} \tilde{T}_{12} &= \tilde{T}_2 \tilde{T}_1 + \tilde{T}_2 R_1^* R_2 \tilde{T}_1 + \tilde{T}_2 R_1^* R_2 R_1^* R_2 \tilde{T}_1 + \cdots \\ &= \tilde{T}_2 \left[1 + R_1^* R_2 + (R_1^* R_2)^2 + \cdots \right] \tilde{T}_1 \\ &= \tilde{T}_2 (1 - R_1^* R_2)^{-1} \tilde{T}_1. \end{aligned} \tag{3.2.36}$$

Likewise, the expressions for U and $\tilde{D}$ are given by

$$\begin{aligned} U &= R_2\tilde{T}_1 + R_2R_1^*R_2\tilde{T}_1 + R_2R_1^*R_2R_1^*R_2\tilde{T}_1 + \cdots \\ &= R_2\left[1 + R_1^*R_2 + (R_1^*R_2)^2 + \cdots\right]\tilde{T}_1 \\ &= R_2(1 - R_1^*R_2)^{-1}\tilde{T}_1, \end{aligned} \tag{3.2.37}$$

$$\begin{aligned} \tilde{D} &= \tilde{T}_1 + R_1^*R_2\tilde{T}_1 + R_1^*R_2R_1^*R_2\tilde{T}_1 + \cdots \\ &= \left[1 + R_1^*R_2 + (R_1^*R_2)^2 + \cdots\right]\tilde{T}_1 \\ &= (1 - R_1^*R_2)^{-1}\tilde{T}_1. \end{aligned} \tag{3.2.38}$$

In Eqs. (3.2.35)–(3.2.38), the infinite series is replaced by a single inverse function. We may define an operator in the form

$$S = R_1^*R_2(1 - R_1^*R_2)^{-1}. \tag{3.2.39}$$

Thus, $(1 - R_1^*R_2)^{-1} = 1 + S$. From the preceding adding equations, we have

$$R_{12} = R_1 + \tilde{T}_1^*U, \tag{3.2.40a}$$

$$\tilde{T}_{12} = \tilde{T}_2\tilde{D}, \tag{3.2.40b}$$

$$U = R_2\tilde{D}. \tag{3.2.40c}$$

At this point, we wish to separate the diffuse and direct components of the total transmission function, which is defined by

$$\tilde{T} = T + e^{-\tau/\mu'}, \tag{3.2.41}$$

where $\mu' = \mu_0$ when transmission is associated with the incident solar beam, and $\mu' = \mu$ when it is associated with the emergent light beam in the direction μ. Using Eq. (3.2.41), we may separate the direct and diffuse components in Eqs. (3.2.38) and (3.2.40b) to obtain

$$\begin{aligned} \tilde{D} &= D + e^{-\tau_1/\mu_0} = (1 + S)(T_1 + e^{-\tau_1/\mu_0}) \\ &= (1 + S)T_1 + Se^{-\tau_1/\mu_0} + e^{-\tau_1/\mu_0}, \end{aligned} \tag{3.2.42}$$

$$\begin{aligned} \tilde{T}_{12} &= (T_2 + e^{-\tau_2/\mu})(D + e^{-\tau_1/\mu_0}) \\ &= e^{-\tau_2/\mu}D + T_2e^{-\tau_1/\mu_0} + T_2D \\ &\quad + \exp\left[-\left(\frac{\tau_1}{\mu_0} + \frac{\tau_2}{\mu}\right)\right]\delta(\mu - \mu_0), \end{aligned} \tag{3.2.43}$$

where D, T_1, and T_2 denote the diffuse components only, and a delta function is added to the exponential term to signify that the direct transmission function is a function of μ_0 only. On the basis of the preceding analysis, a set of iterative

equations for the computation of diffuse transmission and reflection for the two layers may be written in the forms

$$Q = R_1^*R_2, \tag{3.2.44a}$$

$$S = Q(1 - Q)^{-1}, \tag{3.2.44b}$$

$$D = T_1 + ST_1 + Se^{-\tau_1/\mu_0}, \tag{3.2.44c}$$

$$U = R_2D + R_2e^{-\tau_1/\mu_0}, \tag{3.2.44d}$$

$$R_{12} = R_1 + e^{-\tau_1/\mu} + T_1^*U, \tag{3.2.44e}$$

$$T_{12} = e^{-\tau_2/\mu}D + T_2e^{-\tau_1/\mu_0} + T_2D. \tag{3.2.44f}$$

The direct transmission function for the combined layer is given by $\exp[-(\tau_1 + \tau_2)/\mu_0]$. In these equations, the product of two functions implies an integration over the appropriate solid angle so that all possible multiple-scattering contributions are accounted for, as in the following example:

$$R_1^*R_2 = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R_1^*(\mu, \phi; \mu', \phi')R_2(\mu', \phi'; \mu_0, \phi_0)\mu'\,d\mu'\,d\phi'. \tag{3.2.45}$$

In the numerical computations, we may set $\tau_1 = \tau_2$. This is referred to as the *doubling method.* We start with an optical depth $\Delta\tau$ on the order of 10^{-8} and use Eqs. (3.2.33) and (3.2.34) to compute the reflection and transmission functions. Equations (3.2.44a–f) are subsequently employed to compute the reflection and transmission functions for an optical depth of $2\Delta\tau$. For the initial layers, $R_{1,2}^* = R_{1,2}$ and $T_{1,2}^* = T_{1,2}$. Using the adding equations, the computations may be repeated until a desirable optical depth is achieved.

For radiation emergent from below, R_{12}^* and T_{12}^* may be computed from a scheme analogous to Eq. (3.2.44). Let the incident direction be μ'; then the adding equations are as follows:

$$Q = R_2R_1^*, \tag{3.2.46a}$$

$$S = Q(1 - Q)^{-1}, \tag{3.2.46b}$$

$$U = T_2^* + ST_2^* + Se^{-\tau_2/\mu'}, \tag{3.2.46c}$$

$$D = R_1^*U + R_1^*e^{-\tau_2/\mu'}, \tag{3.2.46d}$$

$$R_{12}^* = R_2^* + T_2D + e^{-\tau_2/\mu}D, \tag{3.2.46e}$$

$$T_{12}^* = T_1^*U + e^{-\tau_1/\mu}U + T_1^*e^{-\tau_2/\mu'}. \tag{3.2.46f}$$

When polarization and azimuth dependence are neglected, the transmission function is the same regardless of whether radiation is from above or below; that is, $T^*(\mu, \mu') = T(\mu', \mu)$. This relation can be derived based on the Helmholtz principle of reciprocity in which the light beam may reverse its direction (Hovenier, 1969).

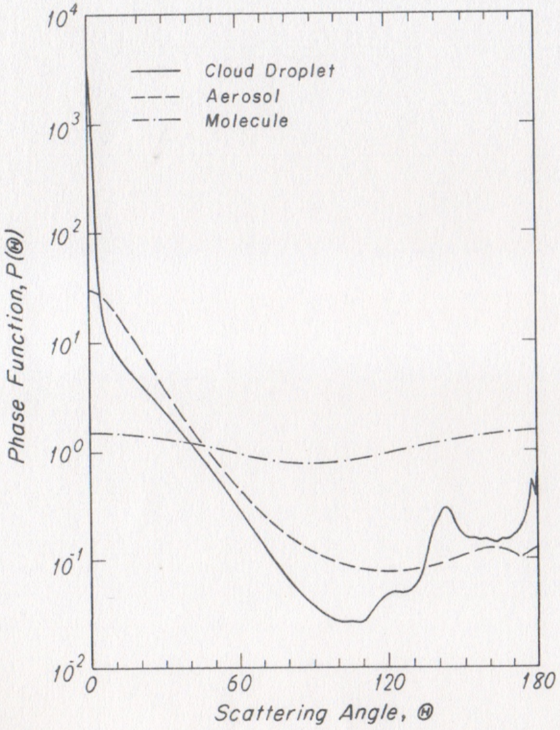

Fig. 3.5 Normalized phase functions for cloud droplets, aerosols, and molecules illuminated by a visible wavelength of 0.5 μm.

For practical applications, we begin with the computations of the reflection and transmission functions given in Eqs. (3.2.33) and (3.2.34). The phase function must be expressed as a function of the incoming and outgoing directions via Eq. (3.1.11a) in the form

$$P(\mu, \phi; \mu', \phi') = P^0(\mu, \mu') + 2 \sum_{\ell=1}^{N} P^m(\mu, \mu') \cos m(\phi' - \phi), \qquad (3.2.47)$$

where $P^m(\mu, \mu')$ $(m = 0, 1, \ldots, N)$ denotes the Fourier expansion coefficients. The number of terms required in the expansion depends on the sharpness of the forward diffraction peak in phase function (see Fig. 3.5).

The preceding adding equations for radiative transfer have been written in scalar forms involving diffuse intensity. However, these equations can be applied to the case that takes into account polarization in which the light beam is characterized by the Stokes parameters and the phase function is replaced by the phase matrix (see Subsection 5.1.2 for further discussion). The phase matrix must be expressed with respect to the local meridian plane in a manner defined in Eq. (5.5.6).

3.3 Two-stream and Eddington's approximations for radiative transfer

3.3.1 *Two-stream approximation*

On the basis of the analysis presented in Subsection 3.2.1 of the discrete-ordinates method for radiative transfer, Eq. (3.2.9) may be written in the form

$$\mu_i \frac{dI_i}{d\tau} = I_i - \sum_{j=-n}^{n} c_{i,j} I_j - c_{i,-0} I_\odot, \qquad i = -n, \ldots, n, \qquad (3.3.1)$$

where $c_{i,j}$ and $I_\odot$ have been defined in Eqs. (3.2.10) and (3.2.11), respectively. The simplest way of solving Eq. (3.3.1) is to take $n = 1$ and let the expansion term in the Legendre polynomials be $N = 1$. After rearranging terms and denoting $I^+ = I_1 = I(\tau, \mu_1)$ and $I^- = I_{-1} = I(\tau, -\mu_1)$, two simultaneous equations may be written in the forms

$$\mu_1 \frac{dI^+}{d\tau} = I^+ - \tilde{\omega}(1-b)I^+ - \tilde{\omega} b I^- - s^- e^{-\tau/\mu_0}, \qquad (3.3.2a)$$

$$-\mu_1 \frac{dI^-}{d\tau} = I^- - \tilde{\omega}(1-b)I^- - \tilde{\omega} b I^+ - s^+ e^{-\tau/\mu_0}, \qquad (3.3.2b)$$

where $\mu_1 = 1/\sqrt{3}$, $a_1 = a_{-1} = 1$ from the Gauss formula, and

$$s^\pm = F_\odot \tilde{\omega}(1 \pm 3g\mu_1\mu_0)/4\pi, \qquad (3.3.2c)$$

$$b = (1-g)/2. \qquad (3.3.2d)$$

The asymmetry factor g is defined by

$$g = \frac{\tilde{\omega}_1}{3} = \frac{1}{2} \int_{-1}^{1} P(\cos\Theta) \cos\Theta \, d\cos\Theta. \qquad (3.3.3)$$

The asymmetry factor is the first moment of the phase function. It is derived from Eq. (3.1.4) by using the orthogonal property of the Legendre polynomials. Note that the zero moment of the phase function is equal to $\tilde{\omega}_0 (= 1)$. For isotropic scattering, g is zero, as it is for Rayleigh scattering. The asymmetry factor increases as the diffraction peak of the phase function sharpens. Conceivably, the asymmetry factor may be negative if the phase function peaks in backward directions (90–180°). For Mie particles, whose phase function has a generally sharp peak at the 0° scattering angle, the asymmetry factor denotes the relative strength of forward scattering. Parameters b and $(1-b)$ can be interpreted as the integrated fraction of the energy that is backscattered and forward scattered, respectively. Thus it is apparent in Eq. (3.3.2) that the multiple-scattering contribution in the two-stream approximation is represented by upward and downward intensities weighted by appropriate fractions of the forward or backward phase function. The upward intensity is strengthened by its coupling with the forward fraction (0–90°) of the phase function plus the downward intensity, which appears in the backward fraction (90–180°) of the phase function. A similar argument is valid for the downward intensity.

The form of Eq. (3.3.2), without the direct solar source term, was first presented by Schuster (1905). Schuster's formulations have been used by Neiburger (1949) for solar reflection, absorption, and transmission measurements from California stratus clouds, and have been further discussed by Herman and Abraham (1960). An application of the two-stream approximation to planetary atmospheres has been presented by Sagan and Pollack (1967) for interpreting observed visual and near-ir reflectivity from the clouds of Venus.

The solutions of two first-order nonhomogeneous differential equations [Eqs. (3.3.2a) and (3.3.2b)] can be derived by straightforward analysis. Let $\tilde{\omega} \neq 1$; then we obtain

$$I^+ = I(\tau, \mu_1) = Kve^{k\tau} + Hue^{-k\tau} + \epsilon e^{-\tau/\mu_0}, \tag{3.3.4a}$$
$$I^- = I(\tau, -\mu_1) = Kue^{k\tau} + Hve^{-k\tau} + \gamma e^{-\tau/\mu_0}, \tag{3.3.4b}$$

where

$$k^2 = (1-\tilde{\omega})(1-\tilde{\omega}g)/\mu_1^2, \tag{3.3.4c}$$
$$v = (1+a)/2, \qquad u = (1-a)/2, \tag{3.3.4d}$$
$$a^2 = (1-\tilde{\omega})/(1-\tilde{\omega}g), \tag{3.3.4e}$$
$$\epsilon = (\alpha+\beta)/2, \qquad \gamma = (\alpha-\beta)/2, \tag{3.3.4f}$$
$$\alpha = Z_1\mu_0^2/(1-\mu_0^2k^2), \qquad \beta = Z_2\mu_0^2/(1-\mu_0^2k^2), \tag{3.3.4g}$$
$$Z_1 = -\frac{(1-\tilde{\omega}g)(s^- + s^+)}{\mu_1^2} + \frac{s^- - s^+}{\mu_1\mu_0}, \tag{3.3.4h}$$
$$Z_2 = -\frac{(1-\tilde{\omega})(s^- - s^+)}{\mu_1^2} + \frac{s^- + s^+}{\mu_1\mu_0}. \tag{3.3.4i}$$

The terms $\pm k$, in Eq. (3.3.4c) are the eigenvalues for the solution of the differential equations, and u and v represent the eigenfunctions, which are defined by the similarity parameter a in Eq. (3.3.4e) (see Section 3.4 for discussion on the similarity principle in radiative transfer). For conservative scattering, $\tilde{\omega} = 1$. Simpler solutions can be derived from Eqs. (3.3.2a) and (3.3.2b) with one of the eigenvalues, $k = 0$. In practice, however, we may set $\tilde{\omega} = 0.99999$ in Eqs. (3.3.4a–i) and obtain the results for conservative scattering. With two proper boundary conditons, the two integration constants, K and H, may be determined. The upward and downard fluxes in the context of the two-stream approximation are

$$F^+(\tau) = 2\pi\mu_1 I(\tau, \mu_1), \qquad F^-(\tau) = 2\pi\mu_1 I(\tau, -\mu_1). \tag{3.3.5}$$

3.3.2 *Eddington's approximation*

We begin with the general approach of decomposing the equation of radiative transfer using the property of Legendre polynomials. In line with the Legendre polynomial expansion for the phase function denoted in Eq. (3.1.15), the scattered intensity may be expanded in terms of Legendre polynomials such that

$$I(\tau, \mu) = \sum_{\ell=0}^{N} I_\ell(\tau)P_\ell(\mu). \tag{3.3.6}$$

Using the orthogonal and recurrence properties of Legendre polynomials, Eq. (3.1.16) may be decomposed in N harmonics in the form

$$\frac{\ell}{2\ell-1}\frac{dI_{\ell-1}}{d\tau} + \frac{\ell+1}{2\ell+3}\frac{dI_{\ell+1}}{d\tau} = I_\ell\left(1 - \frac{\tilde{\omega}\tilde{\omega}_\ell}{2\ell+1}\right) - \frac{\tilde{\omega}}{4\pi}\tilde{\omega}_\ell P_\ell(-\mu_0)F_\odot e^{-\tau/\mu_0}, \quad \ell = 0, 1, 2, \ldots, N. \tag{3.3.7}$$

The method of solving the basic radiative transfer equation using the aforementioned procedure is referred to as the *spherical harmonic method* (Kourganoff, 1952). Numerical solutions to a set of differential equations may be carried out in the same way as they are in the discrete-ordinates method (Dave, 1975).

Eddington's approximation uses an approach similar to that of the two-stream approximation, and was originally used for studies of radiative equilibrium in stellar atmospheres (Eddington, 1916). Letting $N = 1$, the phase function and intensity expressions may be written as follows:

$$P(\mu, \mu') = 1 + 3g\mu\mu', \qquad I(\tau, \mu) = I_0(\tau) + I_1(\tau)\mu, \quad -1 \le \mu \le 1. \tag{3.3.8}$$

Subsequently, Eq. (3.3.7) reduces to a set of two simultaneous equations in the forms

$$\frac{dI_1}{d\tau} = 3(1-\tilde{\omega})I_0 - \frac{3\tilde{\omega}}{4\pi}F_\odot e^{-\tau/\mu_0}, \tag{3.3.9a}$$
$$\frac{dI_0}{d\tau} = (1-\tilde{\omega}g)I_1 + \frac{3\tilde{\omega}}{4\pi}g\mu_0 F_\odot e^{-\tau/\mu_0}. \tag{3.3.9b}$$

Differentiating Eq. (3.3.9b) with respect to τ and substituting the expression for $dI_1/d\tau$ from Eq. (3.3.9a) leads to

$$\frac{d^2I_0}{d\tau^2} = k^2 I_0 - \chi e^{-\tau/\mu_0}, \tag{3.3.10}$$

where $\chi = 3\tilde{\omega}F_{\odot}(1 + g - \tilde{\omega}g)/4\pi$ and the eigenvalue is

$$k^2 = 3(1 - \tilde{\omega})(1 - \tilde{\omega}g). \tag{3.3.11}$$

Here, the eigenvalues are exactly the same as they are for the two-stream approximation depicted in Eq. (3.3.4c). Equation (3.3.10) represents a well-known diffusion equation for radiative transfer. The diffusion approximation is particularly applicable for the radiation field in the deep domain of an optically thick layer.

Straightforward analyses yield the following solutions for the diffusion equation:

$$I_0 = Ke^{k\tau} + He^{-k\tau} + \psi e^{-\tau/\mu_0}, \tag{3.3.12a}$$

where

$$\psi = \frac{3\tilde{\omega}}{4\pi}F_{\odot}[1 + g(1 - \tilde{\omega})]/(k^2 - 1/\mu_0^2).$$

Following a similar procedure, the solution for the second harmonic, I_1, is given by

$$I_1 = aKe^{k\tau} - aHe^{-k\tau} - \xi e^{-\tau/\mu_0}, \tag{3.3.12b}$$

where $a^2 = 3(1 - \tilde{\omega})/(1 - \tilde{\omega}g)$, defined in the two-stream approximation [Eq. (3.3.4e)], and

$$\xi = \frac{3\tilde{\omega}}{4\pi}\frac{F_{\odot}}{\mu_0}\left[1 + 3g(1 - \tilde{\omega})\mu_0^2\right]/(k^2 - 1/\mu_0^2).$$

The integration constants, K and H, are to be determined from proper boundary conditions. Finally, the upward and downward fluxes are given by

$$\left.\begin{matrix} F^+(\tau) \\ F^-(\tau) \end{matrix}\right\} = 2\pi\int_0^{\pm 1}(I_0 + \mu I_1)\mu\, d\mu = \left\{\begin{matrix} \pi\left(I_0 + \frac{2}{3}I_1\right) \\ \pi\left(I_0 - \frac{2}{3}I_1\right). \end{matrix}\right. \tag{3.3.13}$$

3.3.3 *Generalized two-stream equation*

Using the radiative transfer equation denoted in Eq. (3.1.16) and the upward and downward diffuse fluxes defined in Eq. (3.1.19b), we may write

$$\frac{1}{2\pi}\frac{dF^+(\tau)}{d\tau} = \int_0^1 I(\tau,\mu)\,d\mu - \frac{\tilde{\omega}}{2}\int_0^1\int_{-1}^1 I(\tau,\mu)P(\mu,\mu')\,d\mu'\,d\mu$$
$$- \frac{\tilde{\omega}}{4\pi}F_{\odot}e^{-\tau/\mu_0}\int_0^1 P(\mu,-\mu_0)\,d\mu, \tag{3.3.14a}$$

$$\frac{1}{2\pi}\frac{dF^-(\tau)}{d\tau} = -\int_0^1 I(\tau,-\mu)\,d\mu + \frac{\tilde{\omega}}{2}\int_0^1\int_{-1}^1 I(\tau,\mu')P(-\mu,\mu')\,d\mu'\,d\mu$$
$$+ \frac{\tilde{\omega}}{4\pi}F_{\odot}e^{-\tau/\mu_0}\int_0^1 P(-\mu,-\mu_0)\,d\mu. \tag{3.3.14b}$$

Table 3.1 Coefficients in two-stream approximations

Method	γ_1	γ_2	γ_3
Two-stream	$[1 - \tilde{\omega}(1 + g)/2]/\mu_1$	$\tilde{\omega}(1 - g)/2\mu_1$	$(1 - 3g\mu_1\mu_0)/2$
Eddington's	$[7 - (4 + 3g)\tilde{\omega}]/4$	$-[1 - (4 - 3g)\tilde{\omega}]/4$	$(2 - 3g\mu_0)/4$

The generalized two-stream approximation may be expressed by

$$\frac{dF^+(\tau)}{d\tau} = \gamma_1 F^+(\tau) - \gamma_2 F^-(\tau) - \gamma_3\tilde{\omega}F_{\odot}e^{-\tau/\mu_0}, \tag{3.3.15a}$$

$$\frac{dF^-(\tau)}{d\tau} = \gamma_2 F^+(\tau) - \gamma_1 F^-(\tau) + (1 - \gamma_3)\tilde{\omega}F_{\odot}e^{-\tau/\mu_0}. \tag{3.3.15b}$$

The differential changes in upward and downward diffuse fluxes are directly related to the upward and downward diffuse fluxes, as well as the downward direct flux. The coefficients γ_i $(i = 1, 2, 3)$ depend on the manner in which the intensity and phase function are approximated in Eq. (3.3.14). In the two-stream approximation, there are only upward and downward intensities in the directions μ_1 and $-\mu_1$ given by the Gauss quadrature formula, while the phase function is expanded in two terms in Legendre polynomials. In Eddington's approximation, both intensity and phase function are expanded in two polynomial terms. The coefficients γ_i can be directly derived from Eqs. (3.3.2a, b) and (3.3.9a, b), and are given in Table 3.1.

In Eq. (3.3.14), we let the last integral involving the phase function be

$$q = \frac{1}{2}\int_0^1 P(\mu, -\mu_0)\,d\mu. \tag{3.3.16a}$$

Since the phase function is normalized to unity, we have

$$\frac{1}{2}\int_0^1 P(-\mu, -\mu_0)\,d\mu = 1 - q. \tag{3.3.16b}$$

Equations (3.3.16a, b) can be evaluated exactly by numerical means. We may take $\gamma_3 = q$ in the two-stream approximation. This constitutes the modified two-stream approximation proposed by Liou (1973b) and Meador and Weaver (1980). The two-stream approximation yields negative albedo values for a thin atmosphere when $\gamma_3 < 0$ (i.e., $g > \mu_1/\mu_0$). This also occurs in Eddington's approximation when $g > 0.75/\mu_0$. These negative albedo values can be avoided by using q, the full phase function integration for the direct solar beam, denoted in Eq. (3.3.16b). The accuracy of the two-stream approximation has been discussed in Liou (1973a). The overall accuracy of the two-stream and Eddington's approximations can be improved by incorporating the δ-function adjustment for forward scattering. We will discuss forward scattering in Subsection 3.4. There are other two-stream approximations, such as those discussed by Zdunkowski et al. (1974) and Coakley and Chýlek (1975), who used hemispheric average intensities for the upward and downward diffuse fluxes, and by Meador and Weaver (1980), who used a combination of Eddington and δ-function methods to get the two-stream approximations. The accuracy of these methods is generally on the same order of magnitude as that of the δ–two-stream or δ-Eddington approximations (see Section 3.5).

The solutions for the equations of the generalized two-stream approximation expressed in Eq. (3.3.15a, b) are as follows:

$$F^+ = vKe^{k\tau} + uHe^{-k\tau} + \epsilon e^{-\tau/\mu_0}, \quad (3.3.17a)$$
$$F^- = uKe^{k\tau} + vHe^{-k\tau} + \gamma e^{-\tau/\mu_0}, \quad (3.3.17b)$$

where

$$k^2 = \gamma_1^2 - \gamma_2^2, \quad (3.3.18a)$$
$$v = \frac{1}{2}[1 + (\gamma_1 - \gamma_2)/k], \qquad u = \frac{1}{2}[1 - (\gamma_1 - \gamma_2)/k], \quad (3.3.18b)$$
$$\epsilon = [\gamma_3(1/\mu_0 - \gamma_1) - \gamma_2(1 - \gamma_3)]\mu_0^2\tilde{\omega}F_\odot, \quad (3.3.18c)$$
$$\gamma = -[(1 - \gamma_3)(1/\mu_0 + \gamma_1) + \gamma_2\gamma_3]\mu_0^2\tilde{\omega}F_\odot, \quad (3.3.18d)$$

and where K and H are unknown coefficients to be determined from the boundary conditions.

3.4 Delta-function adjustment and similarity principle

The two-stream and Eddington methods for radiative transfer are good approximations for optically thick layers, but they produce inaccurate results for thin layers and when significant absorption is involved. The basic problem is that scattering by atmospheric particulates is highly peaked in the forward directions. Figure 3.5 illustrates the phase functions for cloud water droplets, aerosols, and molecules. The phase functions for cloud and aerosol particles are highly peaked in the forward direction. This is especially evident for cloud particles, for which the forward-scattered energy within $\sim 5^\circ$ scattering angles produced by diffraction is four to five orders of magnitude greater than it is in the side and backward directions. The highly peaked diffraction pattern is typical for atmospheric particulates. It is clear that a two-term expansion in the phase function is far from adequate.

To incorporate the forward peak contribution in multiple scattering, we may consider an adjusted absorption and scattering atmosphere, such that the fraction of scattered energy residing in the forward peak, f, is removed from the scattering parameters: optical depth, τ; single-scattering albedo, $\tilde{\omega}$; and asymmetry factor, g. We use primes to represent the adjusted radiative parameters, as shown in Fig. 3.6. The optical (extinction) depth is the sum of the scattering (τ_s) and absorption (τ_a) optical depths. The forward peak is produced by diffraction without the contribution of absorption. Thus the adjusted scattering and absorption optical depths must be

$$\tau_s' = (1 - f)\tau_s, \quad (3.4.1)$$
$$\tau_a' = \tau_a. \quad (3.4.2)$$

The total adjusted optical depth is

$$\tau' = \tau_s' + \tau_a' = (1 - f)\tau_s + \tau_a = \tau(1 - f\tilde{\omega}). \quad (3.4.3)$$

The adjusted single-scattering albedo is

$$\tilde{\omega}' = \frac{\tau_s'}{\tau'} = \frac{(1 - f)\tau_s}{(1 - f\tilde{\omega})\tau} = \frac{(1 - f)\tilde{\omega}}{1 - f\tilde{\omega}}. \quad (3.4.4)$$

Moreover, we multiply the asymmetry factor by the scattering optical depth to get the similarity equation

$$\tau_s'g' = \tau_s g - \tau_s f, \quad \text{that is,} \ g' = \frac{g - f}{1 - f}, \quad (3.4.5)$$

where we note that the asymmetry factor in the forward peak is equal to unity. In the diffusion domain, the solution of the diffuse intensity is given by exponential functions with eigenvalues defined in Eq. (3.3.11). We may set the intensity solution in the adjusted atmosphere so that it is equivalent to that in the real atmosphere, in the form

$$k\tau = k'\tau'. \quad (3.4.6)$$

From Eqs. (3.4.3)–(3.4.6), the similarity relations for radiative transfer can be expressed in the forms

$$\frac{\tau}{\tau'} = \frac{k'}{k} = \frac{1 - \tilde{\omega}'}{1 - \tilde{\omega}} = \frac{\tilde{\omega}'(1 - g')}{\tilde{\omega}(1 - g)}. \quad (3.4.7a)$$

Using the expression for the eigenvalue defined in Eq. (3.3.4c), we also find the relation for the similarity parameter defined in Eq. (3.3.4e), as follows:

$$a = \left(\frac{1 - \tilde{\omega}}{1 - \omega g}\right)^{1/2} = \left(\frac{1 - \tilde{\omega}'}{1 - \tilde{\omega}'g'}\right)^{1/2}. \quad (3.4.7b)$$

The similarity principle can also be derived from the basic radiative transfer equation. We may begin with this equation in the form

$$\mu\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}}{2}\int_{-1}^{1} I(\tau, \mu')P(\mu, \mu')\, d\mu'. \quad (3.4.8)$$

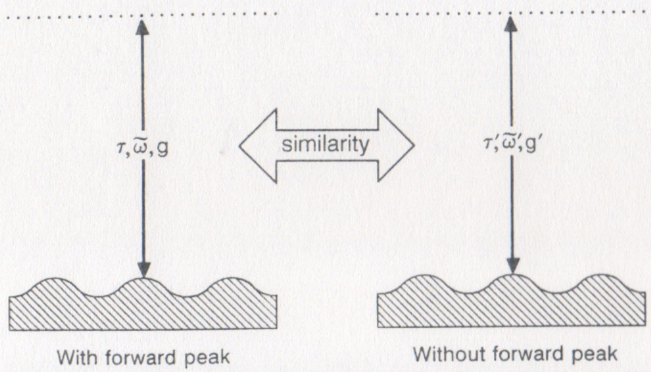

FIG. 3.6 Similarity principle for radiative transfer. The prime system represents adjustment radiative parameters such that the forward diffraction peak in scattering processes is removed.

From Eq. (3.3.8), the phase function in the limit of the two-stream and Eddington's approximations is given by $P(\mu, \mu') = 1 + 3g\mu\mu'$. However, the phase functions involving cloud and aerosol particles are highly peaked in the forward direction, and two-term expansions do not adequately account for the strong forward scattering. Let the fraction of the energy scattered in the forward direction ($\Theta = 0°$) be denoted by f. The normalized phase function may be expressed in terms of this value, as follows:

$$P(\mu, \mu') = 2f\delta(\mu - \mu') + (1 - f)(1 + 3g'\mu\mu'), \qquad (3.4.9)$$

where $\mu = \mu'$ when $\Theta = 0$, δ is the δ function, and g' denotes the scaled asymmetry factor. The phase function so defined is normalized to unity, and the asymmetry factor defined in Eq. (3.3.3) is given by

$$g = f + (1 - f)g'. \qquad (3.6.10)$$

The second moment of the phase function expansion is

$$\tilde{\omega}_2/5 = f. \qquad (3.6.11)$$

Thus the scaled asymmetry factor can be expressed by

$$g' = \frac{g - \tilde{\omega}_2/5}{1 - \tilde{\omega}_2/5}. \qquad (3.4.12)$$

Now, substituting Eq. (3.4.9) into Eq. (3.4.8), we obtain

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu)(1 - \tilde{\omega} f) - \frac{\tilde{\omega}(1 - f)}{2} \times \int_{-1}^{1} (1 + 3g'\mu\mu') I(\tau, \mu')\, d\mu'. \qquad (3.4.13)$$

Consequently, if we redefine the optical depth, single-scattering albedo, and phase function such that

$$\tau' = (1 - \tilde{\omega} f)\tau, \qquad (3.4.14a)$$

$$\tilde{\omega}' = \frac{(1 - f)\tilde{\omega}}{1 - \tilde{\omega} f}, \qquad (3.4.14b)$$

$$P'(\mu, \mu') = 1 + 3g'\mu\mu', \qquad (3.4.14c)$$

Eq. (3.4.13) then becomes

$$\mu \frac{dI(\tau', \mu)}{d\tau'} = I(\tau', \mu) - \frac{\tilde{\omega}'}{2} \int_{-1}^{1} I(\tau', \mu') P'(\mu, \mu')\, d\mu'. \qquad (3.4.15)$$

Equation (3.4.15) is exactly the same as Eq. (3.4.8), except that g, τ, and $\tilde{\omega}$ have been replaced by g', τ', and $\tilde{\omega}'$. By redefining the asymmetry factor, optical depth, and single-scattering albedo, the forward-scattering nature of the phase function is approximately accounted for in the basic radiative transfer equation. In essence, we have incorporated the second moment of the phase function expansion in the formulation of the radiative transfer equation. The "equivalence" between Eqs. (3.4.8) and (3.4.15) is the similarity principle that has been stated previously.

The phase functions for aerosol and cloud particles require involved scattering calculations, as will be discussed in Section 5.1. For many applications to radiative transfer in planetary atmospheres, an analytic expression for the phase function in terms of the asymmetry factor has been proposed:

$$P_{HG}(\cos\Theta) = (1 - g^2)/(1 + g^2 - 2g\cos\Theta)^{3/2} = \sum_{\ell=0}^{N} (2\ell + 1) g^\ell P_\ell(\cos\Theta). \qquad (3.4.16)$$

This is referred to as the *Henyey–Greenstein phase function* (Henyey and Greenstein, 1941), which is adequate for scattering patterns that are not strongly peaked in the forward direction. Using this expression, the second moment for the phase function is given by

$$\tilde{\omega}_2/5 = f = g^2. \qquad (3.4.17)$$

Thus, in the limit of the Henyey–Greenstein approximation, the forward fraction of the scattered light is now expressed in terms of the asymmetry factor. Subsequently, the scaled asymmetry factor, optical depth, and single-scattering albedo can now be expressed by

$$g' = \frac{g}{1 + g}, \qquad \tau' = (1 - \tilde{\omega} g^2)\tau, \qquad \tilde{\omega}' = \frac{(1 - g^2)\tilde{\omega}}{1 - \tilde{\omega} g^2}. \qquad (3.4.18)$$

The similarity principle for radiative transfer was first stated by Sobolev (1975) for isotropic scattering. The general similarity relationships have been presented by van de Hulst (1980). The application of employing a Dirac δ function to approximate highly peaked forward scattering in radiative transfer has been discussed by a number of researchers, including Hansen (1969), Potter (1970), Joseph et al. (1976), and Wiscombe (1977).

The two-stream approximations are popular because they enable the analytic solutions for upward and downward fluxes to be derived, and the numerical computations for these fluxes to be efficiently performed. The incorporation of the delta-function adjustment to account for the strong forward scattering of large size parameters in the context of two-stream approximations has led to a significant improvement in the accuracy of radiative flux calculations. As pointed out previously, the δ adjustment provides a third term closure through the second moment of the phase function expansion. Schaller (1979) has illustrated that the δ–two-stream and δ–Eddington approximations have the same accuracy. King and Harshvardhan (1986) have undertaken a more comprehensive examination of the accuracy of various two-stream approximations. They have shown that relative errors of 15–20% could result for some values of optical depths, solar zenith angles and single-scattering albedos. In the next section, we will introduce the δ–four-stream approximation, which produces an accuracy within ~5% in the radiative flux calculations.

3.5 δ–four-stream approximation for radiative transfer parameterization

For atmospheric flux computations, Liou (1974) has proposed that the four-stream approximation could be of value. For this approximation, the solution for eigenvalues associated with the homogeneous part of the discretized equations can be derived analytically from the recurrence equation for eigenvalues. Thus, the computational time for the flux calculations does not significantly exceed that required for the two-stream approximation. Cuzzi et al. (1982) have carried out an examination of the four-stream approximation. Their findings indicate that the four-stream approximation, as well as the incorporation of the forward-peak adjustment in this approximation, does indeed have much to offer for flux calculations in terms of both accuracy and efficiency.

The four-stream approximation, as given in Liou (1974), is based on the general solution for the discrete-ordinates method for radiative transfer. In order to be able to understand the merit of the four-stream approximation, it is necessary to have some background in solving a set of differential equations based on Chandrasekhar's (1950) formulations. In particular, it is noted that the search for eigenvalues from the recurrence equation developed in the solution is both mathematically ambiguous and numerically troublesome, as pointed out in Subsection 3.2.1.

A systematic and independent development of the solution for this approximation has been presented by Liou et al. (1988). Specifically, this solution involves the computation of solar radiative fluxes using a relatively simple, convenient, and accurate method. Knowledge of the discrete-ordinates method for radiative transfer is desirable but not necessary. In addition, a wide range of accuracy checks for this approximation has been provided, including the δ adjustment to account for the forward diffraction peak based on the generalized similarity principle for radiative transfer.

Consider two radiative streams in the upper and lower hemispheres (i.e., let $n = 2$). At the same time, expand the scattering phase function into four terms (i.e., $N = 3$) in line with the four radiative streams. On the basis of Eq. (3.2.16), four first-order differential equations can then be written explicitly in matrix form:

$$\frac{d}{d\tau}\begin{bmatrix} I_{-2} \\ I_{-1} \\ I_1 \\ I_2 \end{bmatrix} = -\begin{bmatrix} -b_{2,2} & -b_{2,1} & -b_{2,-1} & -b_{2,-2} \\ -b_{1,2} & -b_{1,1} & -b_{1,-1} & -b_{1,-2} \\ b_{1,-2} & b_{1,-1} & b_{1,1} & b_{1,2} \\ b_{2,-2} & b_{2,-1} & b_{2,1} & b_{2,2} \end{bmatrix}\begin{bmatrix} I_{-2} \\ I_{-1} \\ I_1 \\ I_2 \end{bmatrix} - \begin{bmatrix} b_{-2,-0} \\ b_{-1,-0} \\ b_{1,-0} \\ b_{2,-0} \end{bmatrix} I_\odot, \tag{3.5.1}$$

where $I_\odot = I(\tau, -\mu_0)$ defined in Eq. (3.2.11). The 4×4 matrix represents the contribution of multiple scattering. Thus the derivative of the diffuse intensity at a specific quadrature angle is the weighted sum of the multiple-scattered intensity from all four quadrature angles. The last term represents the contribution of the unscattered component of the direct solar flux at position τ.

We proceed with a direct approach to find the eigenvalues and eigenvectors for Eq. (3.5.1). To do so, we define the sum and difference of the upward and downward intensities in the form

$$M_{1,2}^{\pm} = I_{1,2} \pm I_{-1,-2}. \tag{3.5.2}$$

From Eq. (3.5.1), we obtain the following four equations:

$$-\frac{dM_2^+}{d\tau} = b_{22}^- M_2^- + b_{21}^- M_1^- + b_2^+ I_\odot, \tag{3.5.3a}$$

$$-\frac{dM_2^-}{d\tau} = b_{22}^+ M_2^+ + b_{21}^+ M_1^+ + b_2^- I_\odot, \tag{3.5.3b}$$

$$-\frac{dM_1^+}{d\tau} = b_{12}^- M_2^- + b_{11}^- M_1^- + b_1^+ I_\odot, \tag{3.5.3c}$$

$$-\frac{dM_1^-}{d\tau} = b_{12}^+ M_2^+ + b_{11}^+ M_1^+ + b_1^- I_\odot, \tag{3.5.3d}$$

where the coefficients are defined by

$$\begin{aligned} b_{22}^{\pm} &= b_{2,2} \pm b_{2,-2}, & b_{21}^{\pm} &= b_{2,1} \pm b_{2,-1}, \\ b_{12}^{\pm} &= b_{1,2} \pm b_{1,-2}, & b_{11}^{\pm} &= b_{1,1} \pm b_{1,-1}, \\ b_2^{\pm} &= b_{2,-0} \pm b_{-2,-0}, & b_1^{\pm} &= b_{1,-0} \pm b_{-1,-0}, \end{aligned} \tag{3.5.4}$$

and $b_{i,j}$ have been defined in Eq. (3.2.13). Equations (3.5.3a–d) can be combined to yield

$$\frac{d^2}{d\tau^2}\begin{bmatrix} M_2^+ \\ M_1^+ \end{bmatrix} = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix}\begin{bmatrix} M_2^+ \\ M_1^+ \end{bmatrix} + \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} I_\odot, \tag{3.5.5}$$

where

$$\begin{aligned} a_{22} &= b_{22}^+ b_{22}^- + b_{12}^+ b_{21}^-, \qquad a_{21} = b_{22}^- b_{21}^+ + b_{21}^- b_{11}^+, \\ a_{12} &= b_{12}^- b_{22}^+ + b_{11}^- b_{12}^+, \qquad a_{11} = b_{12}^- b_{21}^+ + b_{11}^- b_{11}^+, \\ d_2 &= b_{22}^- b_2^- + b_{21}^- b_1^- + b_2^+/\mu_0, \\ d_1 &= b_{12}^- b_2^- + b_{11}^- b_1^- + b_1^+/\mu_0. \end{aligned} \tag{3.5.6}$$

Performing differential operations on Eq. (3.5.5) leads to

$$\frac{d^4 M_2^+}{d\tau^4} = b\,\frac{d^2 M_2^+}{d\tau^2} + cM_2^+ + \left(\frac{d_2}{\mu_0^2} + a_{21}d_1 - a_{11}d_2\right) I_\odot, \tag{3.5.7a}$$

$$\frac{d^4 M_1^+}{d\tau^4} = b\,\frac{d^2 M_1^+}{d\tau^2} + cM_1^+ + \left(\frac{d_1}{\mu_0^2} + a_{12}d_2 - a_{22}d_1\right) I_\odot, \tag{3.5.7b}$$

where the terms $b = a_{22} + a_{11}$ and $c = a_{21}a_{12} - a_{11}a_{22}$. The complete solution for M_2^+ (or M_1^+) is the sum of the solution for the homogeneous part of the fourth-order differential equation plus a particular solution. Thus,

$$\begin{bmatrix} M_2^+ \\ M_1^+ \end{bmatrix} = \sum_{j=-2}^{2} \begin{bmatrix} G_j \\ H_j \end{bmatrix} e^{-k_j\tau} + \begin{bmatrix} \eta_2 \\ \eta_1 \end{bmatrix} e^{-\tau/\mu_0}, \tag{3.5.8}$$

where G_j and H_j are associated with eigenvectors, and η_2 and η_1 are results for the particular solutions. Considering the homogeneous part in Eq. (3.5.7a) and substituting the homogeneous solution for M_2^+ into this equation, we find

$$\sum_{j=-2}^{2} \left(k_j^4 - bk_j^2 - c\right) G_j e^{-k_j\tau} = 0. \tag{3.5.9}$$

In order to have a nontrivial solution for M_2^+ (or M_1^+), we must have

$$f(k) = k^4 - bk^2 - c = 0. \tag{3.5.10}$$

It follows that the eigenvalues are given by

$$k^2 = \left[b \pm (b^2 + 4c)^{1/2}\right]/2. \tag{3.5.11}$$

From the definitions of b and c, we have $b^2 + 4c = (a_{11} - a_{22})^2 + 4a_{21}a_{12}$. The terms a_{21} and a_{12} can be expressed in terms of $c_{i,j}$ defined in Eq. (3.2.10).

Under the conditions that $0 < g < 1$ and $0 < \tilde{\omega} < 1$, we find that $a_{21} < 0$ and $a_{12} < 0$. This implies that $b^2 + 4c > 0$. Also, it can be shown that the term $c = -\prod_{\ell=0}^{3}(1 - \tilde{\omega}g^\ell)/\mu_1^2\mu_2^2$. This term is less than zero, so that the eigenvalues are all real numbers. In the case of conservative scattering, $c = 0$. As a result, the two eigenvalues are zero. By substituting the particular solution for $M_{1,2}^+$ into Eqs. (3.5.7a, b), we obtain

$$\eta_2 = \frac{d_2/\mu_0^2 + a_{21}d_1 - a_{11}d_2}{f(1/\mu_0)}\,\frac{F_\odot}{2\pi}, \tag{3.5.12a}$$

$$\eta_1 = \frac{d_1/\mu_0^2 + a_{12}d_2 - a_{22}d_1}{f(1/\mu_0)}\,\frac{F_\odot}{2\pi} \tag{3.5.12b}$$

The function f in this equation has been defined in Eq. (3.5.10). Because G_j and H_j in Eq. (3.5.8) are defined after high-order differentiations, they are not mutually independent. We may determine their relationship from the homogeneous part of Eq. (3.5.5). A straightforward substitution yields

$$H_1e^{-k_1\tau} + H_{-1}e^{k_1\tau} = A_1(G_1e^{-k_1\tau} + G_{-1}e^{k_1\tau}), \tag{3.5.13a}$$

$$H_2e^{-k_2\tau} + H_{-2}e^{k_2\tau} = A_2(G_2e^{-k_2\tau} + G_{-2}e^{k_2\tau}), \tag{3.5.13b}$$

where $A_{1,2} = (k_{1,2}^2 - a_{22})/a_{21}$, and k_1 and k_2 are eigenvalues from Eq. (3.5.11).

Following the preceding procedures and analogous to Eq. (3.5.7), we may obtain expressions for $M_{1,2}$ in the form

$$\frac{d^4 M_2^-}{d\tau^4} = b'\,\frac{d^2 M_2^-}{d\tau^2} + c'M_2^- + \left(\frac{d_2'}{\mu_0^2} + a_{21}'d_1' - a_{11}'d_2'\right) I_\odot, \tag{3.5.14a}$$

$$\frac{d^4 M_1^-}{d\tau^4} = b'\,\frac{d^2 M_1^-}{d\tau^2} + c'M_1^- + \left(\frac{d_1'}{\mu_0^2} + a_{12}'d_2' - a_{22}'d_1'\right) I_\odot, \tag{3.5.14b}$$

where the primed coefficients can be obtained by replacing the superscripts + and − in Eq. (3.5.6) with − and +, respectively. Also, we note that $b' = a_{22}' + a_{11}' = b$, and $c' = a_{21}'a_{12}' - a_{11}'a_{22}' = c$. The particular solutions for $M_{2,1}^-$ are

$$M_{2,1}^- = \eta_{2,1}' e^{-\tau/\mu_0}, \tag{3.5.15}$$

with

$$\eta_2' = \frac{d_2'/\mu_0^2 + a_{21}'d_1' - a_{11}'d_2'}{f(1/\mu_0)}\,\frac{F_\odot}{2\pi}, \tag{3.5.16a}$$

$$\eta_1' = \frac{d_1'/\mu_0^2 + a_{12}'d_2' - a_{22}'d_1'}{f(1/\mu_0)}\,\frac{F_\odot}{2\pi}. \tag{3.5.16b}$$

From Eqs. (3.5.3b, d), the homogeneous solutions for $M_{2,1}$ are given by

$$M_2^- = \frac{A_1 b_{21}^- - b_{11}^-}{a^-} k_1(-G_1 e^{-k_1\tau} + G_{-1} e^{k_1\tau})$$
$$+ \frac{A_2 b_{21}^- - b_{11}^-}{a^-} k_2(-G_2 e^{-k_2\tau} + G_{-2} e^{k_2\tau}), \quad (3.5.17a)$$

$$M_1^- = \frac{b_{12}^- - A_1 b_{22}^-}{a^-} k_1(-G_1 e^{-k_1\tau} + G_{-1} e^{k_1\tau})$$
$$+ \frac{b_{12}^- - A_2 b_{22}^-}{a^-} k_2(-G_2 e^{-k_2\tau} + G_{-2} e^{k_2\tau}), \quad (3.5.17b)$$

where $a^- = b_{22}^- b_{11}^- - b_{12}^- b_{21}^-$.

Finally, combining Eqs. (3.5.8), (3.5.13), and (3.5.17), the complete solutions for I_i $(i = -2, -1, 1, 2)$ are given by

$$\begin{bmatrix} I_1 \\ I_{-1} \\ I_2 \\ I_{-2} \end{bmatrix} = \begin{bmatrix} \Phi_1^+ e_1^- & \Phi_1^- e_1^+ & \Phi_2^+ e_2^- & \Phi_2^- e_2^+ \\ \Phi_1^- e_1^- & \Phi_1^+ e_1^+ & \Phi_2^- e_2^- & \Phi_2^+ e_2^+ \\ \phi_1^+ e_1^- & \phi_1^- e_1^+ & \phi_2^+ e_2^- & \phi_2^- e_2^+ \\ \phi_1^- e_1^- & \phi_1^+ e_1^+ & \phi_2^- e_2^- & \phi_2^+ e_2^+ \end{bmatrix} \begin{bmatrix} G_1 \\ G_{-1} \\ G_2 \\ G_{-2} \end{bmatrix} + \begin{bmatrix} Z_1^+ \\ Z_1^- \\ Z_2^+ \\ Z_2^- \end{bmatrix} e^{-\tau/\mu_0}, \quad (3.5.18)$$

where the elements $e_1^- = e^{-k_1\tau}$, $e_1^+ = e^{k_1\tau}$, $e_2^- = e^{-k_2\tau}$, and $e_2^+ = e^{k_2\tau}$, and the eigenvectors are:

$$\phi_{1,2}^{\pm} = \frac{1}{2}\left(1 \pm \frac{b_{11}^- - A_{1,2} b_{21}^-}{a^-} k_{1,2}\right), \quad (3.5.19a)$$

$$\Phi_{1,2}^{\pm} = \frac{1}{2}\left(A_{1,2} \pm \frac{A_{1,2} b_{22}^- - b_{12}^-}{a^-} k_{1,2}\right). \quad (3.5.19b)$$

In Eqs. (3.5.19a, b), $b_{ij}^{\pm}$ is defined by Eq. (3.5.4), with $b_{i,j}$ given in Eq. (3.2.13), and $k_{1,2}$ by eigenvalues of Eq. (3.5.11) with b and c defined below Eq. (3.5.7b). a^- and $A_{1,2}$ are defined by expressions below Eqs. (3.5.17b) and (3.5.13b), respectively, with a_{ij} given in Eq. (3.5.6). The Z functions are defined by

$$Z_{1,2}^{\pm} = \frac{1}{2}(\eta_{1,2} \pm \eta_{1,2}'), \quad (3.5.19c)$$

where $\eta_{1,2}$ and $\eta_{1,2}'$ are defined by Eqs. (3.5.12) and (3.5.16). d_i and a_{ij} are given in Eq. (3.5.6), and $f(1/\mu_0)$ has the same expression as that in Eq. (3.5.10), except k is replaced by $1/\mu_0$. d_i' and a_{ij}' have the same expressions as those in Eq. (3.5.6) except that the superscripts + and − are replaced by − and +, respectively. The coefficients G_j $(j = 1, 2, -1, -2)$ are to be determined from radiation boundary conditions.

Consider a homogeneous layer characterized by an optical depth τ_* and assume that there is no diffuse radiation from the top and bottom of this layer; then the boundary conditions are

$$I_{-1,-2}(\tau = 0) = 0$$
$$I_{1,2}(\tau = \tau_*) = 0. \quad (3.5.20)$$

The lower boundary condition can be modified to include surface albedo effects. Using these boundary conditions, G_j can be obtained by an inversion of the 4×4 matrix given in Eq. (3.5.18). The upward and total (diffuse plus direct) downward fluxes at a given level τ are given by

$$F^+(\tau) = 2\pi(a_1\mu_1 I_1 + a_2\mu_2 I_2), \quad (3.5.21a)$$
$$F^-(\tau) = 2\pi(a_1\mu_1 I_{-1} + a_2\mu_2 I_{-2}) + \mu_0 F_\odot e^{-\tau/\mu_0}. \quad (3.5.21b)$$

We may also apply the four-stream solutions to nonhomogeneous atmospheres in the manner presented in Section 3.7.

The regular Gauss quadratures and weights in the four-stream approximation are $\mu_1 = 0.3399810$, $\mu_2 = 0.8611363$, $a_1 = 0.6521452$, and $a_2 = 0.3478548$. When the isotropic surface reflection is included in this approximation or when it is applied to the thermal infrared radiative transfer involving isothermal emission, double Gauss quadratures and weights ($\mu_1 = 0.2113248$, $\mu_2 = 0.7886752$, and $a_1 = a_2 = 0.5$) offer some advantage in flux calculations because $\sum_i a_i\mu_i = 1/2$ in this case. In the case of conservative scattering, $\tilde{\omega} = 1$, $\phi_2^{\pm} = \Phi_2^{\pm} = 0.5$, the 4×4 matrix becomes 0 in Eq. (3.5.18). The solution for this equation does not exist. Direct formulation and solution from Eq. (3.5.1) are required by setting $\tilde{\omega} = 1$. However, we may use $\tilde{\omega} = 0.999999$ in numerical calculations and obtain the results for conservative scattering. In the case $\tilde{\omega} = 0$, the multiple-scattering term vanishes.

It is possible to incorporate a δ-function adjustment to account for the forward diffraction peak in the context of the four-stream approximation. In reference to Eq. (3.1.8), we may express the normalized phase function expansion by incorporating the δ-forward adjustment in the form

$$P_\delta(\cos\Theta) = 2f\delta(\cos\Theta - 1) + (1 - f)\sum_{\ell=0}^{N} \tilde{\omega}_\ell' P_\ell(\cos\Theta), \quad (3.5.22)$$

where $\tilde{\omega}_\ell'$ is the adjusted coefficient in the phase function expansion. The forward peak coefficient f in the four-stream approximation can be evaluated by demanding that the next-highest-order coefficient in the prime expansion, $\tilde{\omega}_4'$, vanish. Setting $P(\cos\Theta) = P_\delta(\cos\Theta)$ and utilizing the orthogonal property of Legendre polynomials, we find

$$\tilde{\omega}_\ell' = [\tilde{\omega}_\ell - f(2\ell + 1)]/(1 - f). \quad (3.5.23)$$

Letting $\tilde{\omega}_4' = 0$, we obtain $f = \tilde{\omega}_4/9$. Based on Eq. (3.5.23), $\tilde{\omega}_\ell'$ ($\ell = 0, 1, 2, 3$) can be evaluated from the expansion coefficients of the phase function, $\tilde{\omega}_\ell$ ($\ell = 0, 1, 2, 3, 4$).

The adjusted phase function from Eq. (3.5.22) is given by

$$P'(\cos\Theta) = \sum_{\ell=0}^{N} \tilde{\omega}_\ell' P_\ell(\cos\Theta). \quad (3.5.24)$$

This equation, together with Eqs. (3.4.14a, b), constitutes the generalized similarity principle for radiative transfer. That is, the removal of the forward diffraction peak in scattering processes using adjusted single- scattering parameters is "equivalent" to actual scattering processes.

The reflectance r and total transmittance t of the solar flux $\mu_0 F_\odot$ are defined in the forms

$$r(\mu_0) = F^+(0)/\mu_0 F_\odot, \quad (3.5.25a)$$

$$t(\mu_0) = F^-(\tau_*)/\mu_0 F_\odot\,. \quad (3.5.25b)$$

The accuracy of the δ–two-stream and δ–four-stream approximations is examined by comparing the approximate results with the "exact" values computed from the adding method for radiative transfer. Let the reflectance computed from the approximate and "exact" methods be denoted by $\hat{r}$ and r, respectively. Then the relative accuracy is defined by $(\Delta r/r)100\% = [(\hat{r} - r)/r]100\%$. Likewise, the relative accuracy of the total transmittance is defined by $(\Delta t/t)100\%$. The analytic Henyey–Greenstein phase function expanded in the asymmetry factor g was used in the computation [Eq. (3.4.16)].

Numerous asymmetry factors, single-scattering albedos, optical depths, and solar zenith angles were used in the computations. For presentation purposes, however, we select two single-scattering albedos of 1 and 0.8, optical depths from 0.1 to 50 (intervals of 0.1 from 0.1 to 1, 1 from 1 to 10, and 5 from 10 to 50), and cosines of the solar zenith angle from 0 (0.01) to 1 (intervals of 0.1). The asymmetry factor chosen for the graphic presentation is 0.75. To highlight the relative accuracy of the presentation, heavy shading is used for accuracy within 5%, while accuracy within 5–10% is denoted by light shading. White regions show errors greater than 10%.

Figure 3.7 shows the relative accuracy of the δ–two-stream (top graphs) and δ–four-stream (bottom graphs) approximations displayed in intervals of 0, 1, 2, 5, 10%, etc. The accuracy of the δ–two-stream approximation is comparable to that of the δ-Eddington approximation presented by King and Harshvardhan (1986). For conservative scattering, the reflection values produced by both approximations have low accuracy, on the order of 10 to 30% for $\mu_0 < 0.5$ and $\mu_0 > 0.9$ for $\tau < 1$. Errors greater than 10% occur for the total transmittance when $\mu_0 < 0.2$.

In general, reflectance and total transmittance values computed from the δ–four-stream approximation are accurate within about 5%, except for three small regions. For reflectance, 5–10% errors occur for $\mu_0 < 0.3$ and $0.6 < \mu_0 < 1$ when $\tau < 1$. For total transmittance, errors greater than 5% are produced for very high solar zenith angles ($\mu_0 < 0.1$). It is noted that these regions are associated with very small values. Thus, absolute errors are extremely small ($< 1\%$). In the case of $\tilde{\omega} = 0.8$, significant absorption could be built up for large optical depths and/or small solar zenith angles. The δ–two-stream (or δ-Eddington) approximation generally produces errors greater than 5–10%, as is evident from the graphic presentation. In particular, due to small transmittance values, errors of more than 50% may result in the case of large optical depths. The δ–four-stream approximation, on the other hand, has a relative accuracy (i.e., within about 5%) that is comparable to the case of conservative scattering. Errors of 5–10% occur only for very low solar zenith angles ($\mu_0 < 0.2$). Tables 3.2 and 3.3 present numerical results of reflectance and total transmittance computed from δ–two-stream, δ–four-stream, and doubling methods for $\tilde{\omega} = 1$ and 0.8.

In addition to the aforementioned results, computations have also been carried out using asymmetry factors of 0.7, 0.8, and 0.85 for the analytic Henyey–Greenstein phase function. The actual phase functions for cloud droplets were employed in accuracy checks, as were the surface albedos. The accuracy of the δ–four-stream approximation and, for that matter, the δ–two-stream or δ-Eddington approximation is not sensitive to small variations in the asymmetry factor and the detailed structure of the phase function. Also, variations in the surface albedo do not significantly alter the accuracy of the approximations.

Lastly, we have examined the accuracy of the δ–two-stream and δ–four-stream approximations in the case of Rayleigh scattering. Since $g = 0$ for Rayleigh atmospheres, there is no δ adjustment, and use of the two-stream method is equivalent to the isotropic scattering approximation. The four-stream approximation for flux calculations in Rayleigh atmospheres has an accuracy within about 3%.

For applications to the solar absorption bands, in which gaseous absorption in scattering atmospheres must be accounted for, the single-scattering albedo could be small. For this reason, we investigated the accuracy of the δ–two-stream and δ–four stream approximations using single-scattering albedos of 0.5 and 0.3 and keeping the other parameters the same as in Fig. 3.7. For cases involving large absorption, the reflectance values are generally very small. Thus we have presented the percentage of relative accuracy for absorptance, $(\Delta A/A)100\%$, where $A = 1 - r - t$, and total transmittance. Fig. 3.8 shows that the δ–two-stream approximation for absorption calculations produces adequate accuracy, which increases as $\tilde{\omega}$ decreases (i.e., absorptance increases). The δ–four-stream approximation has better accuracy than the δ–two-stream approximation, with errors for absorptance generally less than 2%. It is noted that as $\tilde{\omega}$ decreases, the effects of multiple-scattering on the flux calculations become less important. For total transmittance, errors from the

Table 3.2 Comparison of reflectance and total transmittance as computed by the δ–two-stream (a), δ–four-stream (b), and doubling (c) methods for conservative scattering; $\tilde{\omega} = 1$.

		Reflectance					Transmittance				
τ \ μ_0		0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.25	a	0.31216	0.12096	0.06384	0.03676	0.02101	0.68784	0.87904	0.93617	0.96324	0.97899
	b	0.38544	0.15451	0.07565	0.03972	0.02193	0.61457	0.84549	0.92436	0.96028	0.97807
	c	0.41610	0.15795	0.07179	0.03801	0.02249	0.58390	0.84205	0.92821	0.96200	0.97751
1.0	a	0.51352	0.32948	0.21009	0.13740	0.08956	0.48647	0.67050	0.78993	0.86260	0.91046
	b	0.58329	0.39015	0.24725	0.15534	0.09707	0.41670	0.60986	0.75277	0.84466	0.90293
	c	0.58148	0.38570	0.24048	0.15019	0.09672	0.41852	0.61430	0.75952	0.84981	0.90328
4.0	a	0.68564	0.59244	0.49891	0.41185	0.33578	0.31436	0.40756	0.50109	0.58814	0.66422
	b	0.72859	0.61975	0.52116	0.43006	0.34888	0.27140	0.38024	0.47883	0.56994	0.65111
	c	0.73254	0.61732	0.51932	0.42945	0.34822	0.26746	0.38268	0.48068	0.57055	0.65178
16.0	a	0.86858	0.82978	0.79098	0.75218	0.71340	0.13140	0.17020	0.20900	0.24780	0.28657
	b	0.87853	0.82988	0.78613	0.74539	0.70718	0.12145	0.17009	0.21384	0.25458	0.29278
	c	0.88103	0.82995	0.78658	0.74618	0.70721	0.11897	0.17005	0.21342	0.25382	0.29279
32.0	a	0.92599	0.90413	0.88228	0.86043	0.83858	0.07398	0.09582	0.11766	0.13951	0.16135
	b	0.93004	0.90202	0.87682	0.85335	0.83135	0.06993	0.09793	0.12313	0.14658	0.16858
	c	0.93151	0.90211	0.87714	0.85388	0.83145	0.06849	0.09789	0.12286	0.14612	0.16855

Table 3.3 Same as Table 3.2 except for the single-scattering albedo $\tilde{\omega}$ of 0.8.

		Reflectance					Transmittance				
τ \ μ_0		0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.25	a	0.21542	0.08867	0.04736	0.02733	0.01553	0.49230	0.76749	0.85858	0.90290	0.92904
	b	0.26907	0.10881	0.05290	0.02759	0.01540	0.45040	0.73848	0.84608	0.89840	0.92740
	c	0.28961	0.10686	0.04855	0.02590	0.01547	0.43017	0.73172	0.84756	0.89938	0.92669
1.0	a	0.29926	0.19772	0.12788	0.08327	0.05310	0.23476	0.40572	0.55685	0.65946	0.73071
	b	0.34599	0.21293	0.12880	0.07856	0.04963	0.19685	0.35288	0.51237	0.63149	0.71728
	c	0.35487	0.20714	0.12342	0.07622	0.04929	0.20556	0.35621	0.51606	0.63580	0.71772
4.0	a	0.32304	0.24597	0.18736	0.14125	0.10427	0.05288	0.07730	0.11690	0.17210	0.23407
	b	0.36061	0.24348	0.16937	0.12035	0.08906	0.04775	0.07364	0.10892	0.15813	0.21859
	c	0.37148	0.23862	0.16615	0.12036	0.08925	0.04539	0.07179	0.10718	0.15697	0.21953
16.0	a	0.32432	0.24785	0.19022	0.14525	0.10916	0.00015	0.00022	0.00032	0.00052	0.00098
	b	0.36145	0.24478	0.17131	0.12316	0.09281	0.00026	0.00040	0.00059	0.00086	0.00126
	c	0.37229	0.23990	0.16808	0.12315	0.09296	0.00027	0.00042	0.00062	0.00092	0.00139
32.0	a	0.32432	0.24785	0.19022	0.14525	0.10916	0.00000	0.00000	0.00000	0.00000	0.00000
	b	0.36145	0.24478	0.17131	0.12316	0.09281	0.00000	0.00000	0.00000	0.00000	0.00000
	c	0.37229	0.23990	0.16808	0.12315	0.09296	0.00000	0.00000	0.00000	0.00000	0.00000

δ–four-stream approximation are again within about 5%. Large relative errors can be produced by the δ–two-stream approximation when the transmittance values are small.

We have presented a simple and systematic formulation of the δ–four-stream approximation for solar flux calculations. While all approximate methods for radiative flux transfer have advantages and shortcomings in terms of their computational accuracy for different $\tilde{\omega}$, τ, and μ_0, this approximation can achieve relative accuracy within about 5% for all reasonable ranges of the single-scattering parameters at a given wavelength. For computations of solar fluxes covering the entire solar spectrum, the averaged accuracy should also be within about 5%. By virtue of the two intensity streams in the upper hemisphere and the two in the lower hemisphere, the δ–four-stream approximation has all the radiative characteristics inherent in the δ–two-stream approximation. The solution of this approximation, like various two-stream methods, is in analytic form so that the computer time involved is minimal. The method can be easily applied to nonhomogeneous atmospheres, as described in Section 3.7. For radiative transfer parameterizations in numerical models in which a single radiative transfer approximation is required, the δ–four-stream approximation would be an excellent method.

3.6 Principles of invariance and radiative flux transfer

The transfer of light beams in planetary atmospheres depends on the incoming and outgoing directions. If computations of flux are required, numerical integrations over the outgoing directions must be performed by virtue of the definition of flux. Numerical integrations require considerable computational effort. Hence, it is desirable to seek simplified and approximate expressions for the representation of a flux field. In this section, we wish to introduce the transfer of radiative flux based on the principles of invariance. It suffices to consider azimuthal-independent radiative transfer. Consider an atmosphere with an optical depth τ_*. The reflection function R and transmission function T are defined by

$$R(\mu, \mu_0) = \frac{\pi I_r(0, \mu)}{\mu_0 F_\odot}, \tag{3.6.1a}$$

$$T(\mu, \mu_0) = \frac{\pi I_t(\tau_*, -\mu)}{\mu_0 F_\odot}, \tag{3.6.1b}$$

$$T^{\text{dir}}(\mu_0) = \frac{\pi I_t^{\text{dir}}(\tau_*, -\mu_0)}{\mu_0 F_\odot} = e^{-\tau_*/\mu_0}, \tag{3.6.1c}$$

where I_r and I_t represent the reflected and transmitted intensities at the top and bottom of the atmosphere, respectively. The minus sign associated with μ indicates that the direction of the light beam is downward. The diffuse and direct (dir) components of the transmission function are separated in the definitions.

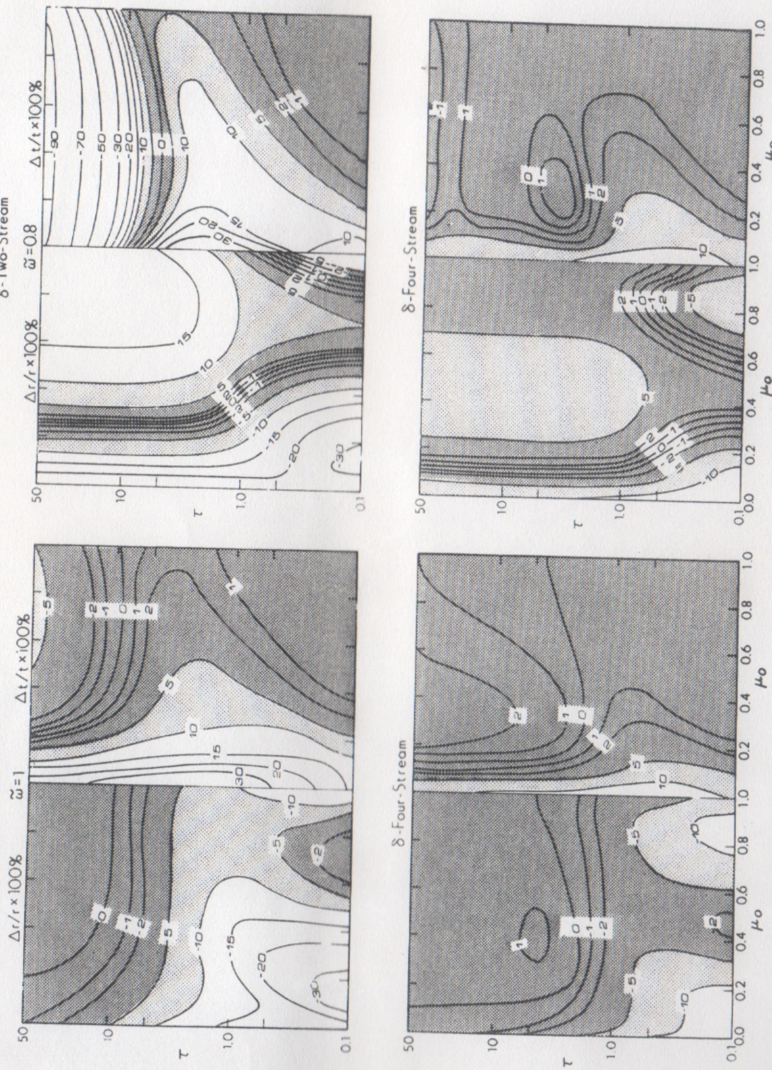

Fig. 3.7 Relative accuracy of the reflectance ($\hat{r}$) and total transmittance ($\hat{t}$) computed from the δ–two-stream (upper graphs) and δ–four-stream (lower graphs) approximations with respect to r and t derived from the adding method for radiative transfer. The relative accuracy is defined by $\Delta r/r = (\hat{r} - r)/r$ for reflectance and $\Delta t/t = (\hat{t} - t)/t$ for total transmittance. The results are shown in the domain of the optical depth τ and the cosine of the solar zenith angle μ_0, and expressed in terms of percentage. The heavy and light shadings denote errors within 5% and within 5–10%, respectively, while the white area represents errors greater than 10%. The left and right graphs are, respectively, for $\tilde{\omega} = 1$ (conservative scattering) and $\tilde{\omega} = 0.8$ (after Liou et al., 1988).

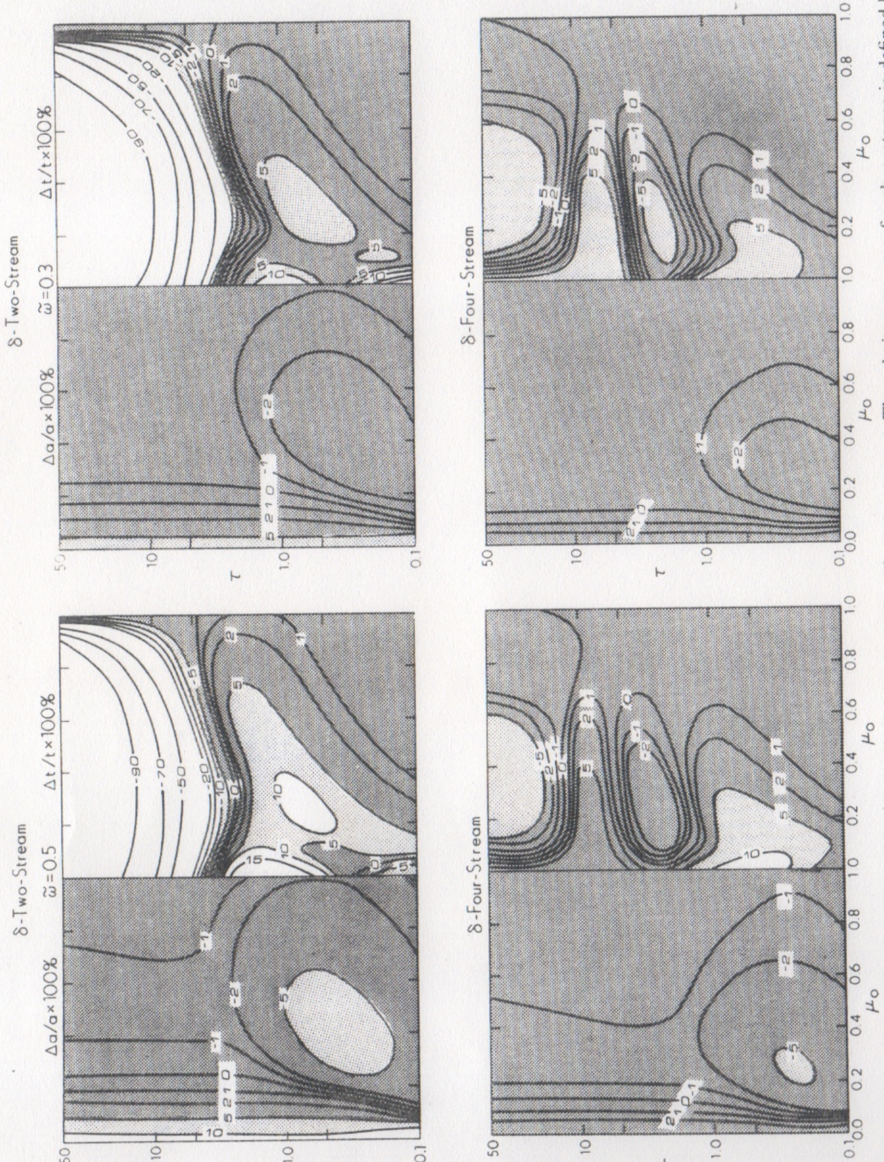

FIG. 3.8 Same as Fig. 3.7, except for absorptance A and total transmittance t, where $A = 1 - r - t$. The relative accuracy for absorptance is defined by $\Delta A/A = (\hat{A} - A)/A$. The left and right graphs are, respectively, for $\tilde{\omega} = 0.5$ and 0.3 (after Liou et al., 1988).

Next, we introduce the corresponding nondimensional parameters in flux forms. *Reflectance* (also referred to as *reflection* or *local* or *planetary albedo*) and *transmittance* (also referred to as *transmission*) are defined by

$$r(\mu_0) = \frac{F^+(0)}{\mu_0 F_\odot} = 2\int_0^1 R(\mu, \mu_0)\mu \, d\mu, \tag{3.6.2a}$$

$$t(\mu_0) = \frac{F^-(\tau_*)}{\mu_0 F_\odot} = 2\int_0^1 T(\mu, \mu_0)\mu \, d\mu, \tag{3.6.2b}$$

$$t^{\text{dir}}(\mu_0) = 2\int_0^1 e^{-\tau_*/\mu_0}\mu \, d\mu = e^{-\tau_*/\mu_0}, \tag{3.6.2c}$$

where F^+ and F^- represent the diffuse upward and downward fluxes, respectively. These are obtained by integrating the upward and downward intensities over the upper and lower hemispheres. The total transmission $t(\mu_0)$ is therefore the sum of t and t^{dir}.

Finally, the global reflectance (or global albedo) $\bar{r}$ and global transmittance $\bar{t}$ may be defined in the forms

$$\bar{r} = \frac{f^+(0)}{\pi a_e^2 F_\odot} = 2\int_0^1 r(\mu_0)\mu_0 \, d\mu_0, \tag{3.6.3a}$$

$$\bar{t} = \frac{f^-(\tau_*)}{\pi a_e^2 F_\odot} = 2\int_0^1 t(\mu_0)\mu_0 \, d\mu_0, \tag{3.6.3b}$$

$$\bar{t}^{\text{dir}} = 2\int_0^1 e^{-\tau_*/\mu_0}\mu_0 \, d\mu_0, \tag{3.6.3c}$$

where f^+ and f^- represent the total outgoing flux at the top and bottom of the atmosphere, respectively, a_e is the radius of the planet, and $\pi a_e^2 F_\odot$ represents the total incoming solar flux at TOA.

For a semi-infinite, plane-parallel atmosphere, the diffuse reflected intensity cannot be changed if a layer of finite optical depth, having the same optical properties as those of the original layer, is added (Ambartzumian, 1942). Based on this invariant principle, the reflection function at the top of a plane-parallel atmosphere can be expressed in terms of a known mathematical function, the so-called H function. More general principles of invariance for a finite, plane-parallel atmosphere have been developed by Chandrasekhar (1950), who used scattering and transmission functions in defining the four principles governing the reflection and transmission of a light beam in two layers. Liou (1980) has developed these four principles in terms of the conventional reflection and transmission functions defined in Subsection 3.2.2.

The four principles of invariance governing the reflection and transmission of light beam may be stated as follows:

1. The reflected intensity at any given optical depth level τ results from the reflection of (a) the attenuated solar flux and (b) the downward diffuse intensity at that level by the optical depth $\tau_* - \tau$.
2. The diffusely transmitted intensity at τ results from (a) the transmission of solar flux and (b) the reflection of the upward diffuse intensity above the level τ.
3. The reflected intensity at the top of the finite atmosphere ($\tau = 0$) is equivalent to (a) the reflection of solar flux plus (b) the direct and diffuse transmission of the upward diffuse intensity above the level τ.
4. The diffusely transmitted intensity at the bottom of a finite atmosphere ($\tau = \tau_*$) is equivalent to (a) the transmission of the attenuated solar flux at level τ plus (b) the direct and diffuse transmission of the downward diffuse intensity at the level τ by the optical depth $\tau_* - \tau$.

Using the definitions of the reflection and transmission functions in Eqs. (3.6.1a,b), letting $\tau = \tau_1$, and $\tau_* - \tau = \tau_2$, and defining the dimensionless upward and downward internal intensities by

$$U(\mu, \mu_0) = \frac{\pi I(\tau_1, \mu)}{\mu_0 F_\odot}, \tag{3.6.4a}$$

$$D(\mu, \mu_0) = \frac{\pi I(\tau_1, -\mu)}{\mu_0 F_\odot}, \tag{3.6.4b}$$

the four principles of invariance may be expressed in terms of the reflection and transmission functions as follows:

$$U(\mu, \mu_0) = R_2(\mu, \mu_0)e^{-\tau_1/\mu_0} + 2\int_0^1 R_2(\mu, \mu')D(\mu', \mu_0)\mu'\, d\mu', \tag{3.6.4}$$

$$D(\mu, \mu_0) = T_1(\mu, \mu_0) + 2\int_0^1 R_1(\mu, \mu'')U(\mu'', \mu_0)\mu''\, d\mu'', \tag{3.6.5}$$

$$R_{12}(\mu, \mu_0) = R_1(\mu, \mu_0) + e^{-\tau_1/\mu_0}U(\mu, \mu_0) + 2\int_0^1 T_1(\mu, \mu')U(\mu', \mu_0)\mu'\, d\mu', \tag{3.6.6}$$

$$T_{12}(\mu, \mu_0) = T_2(\mu, \mu_0)e^{-\tau_1/\mu_0} + e^{-\tau_2/\mu}D(\mu, \mu_0) + 2\int_0^1 T_2(\mu, \mu')D(\mu', \mu_0)\mu'\, d\mu'. \tag{3.6.7}$$

The geometric configuration involving the basic variables is illustrated in Fig. 3.9. Although the preceding equations are written for azimuthal-independent cases, these equations may be modified for applications to general radiative transfer involving azimuthal terms and polarization effects by replacing μ with (μ, ϕ)

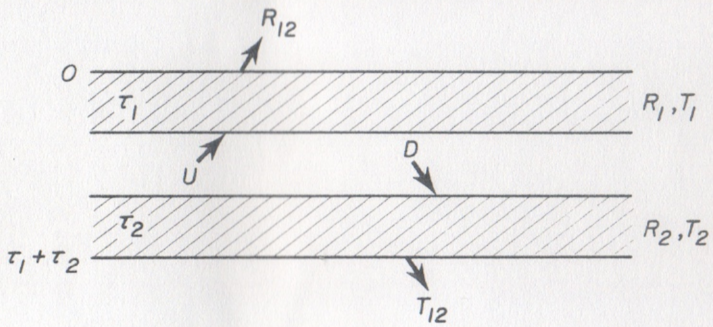

FIG. 3.9 Geometric configuration for the reflection and transmission functions defined in Eqs. (3.6.4)–(3.6.7) based on the principles of invariance for a finite atmosphere. For illustration purposes, we have defined $\tau = \tau_1$ and $\tau_* - \tau = \tau_2$ in the text.

and the diffuse intensity with the Stokes parameters defined in Subsection 5.1.2. Substituting Eq. (3.6.4) into Eq. (3.6.5) leads to

$$D(\mu, \mu_0) = T_1(\mu, \mu_0) + S_{12}(\mu, \mu_0)e^{-\tau_1/\mu_0} + 2\int_0^1 S_{12}(\mu, \mu'')D(\mu'', \mu_0)\mu''\, d\mu'', \tag{3.6.8}$$

where

$$S_{12}(\mu, \mu'') = 2\int_0^1 R_1(\mu, \mu')R_2(\mu', \mu'')\mu'\, d\mu'. \tag{3.6.9}$$

Equations (3.6.4) and (3.6.6)–(3.6.9), which are postulated from the principles of invariance, are "equivalent" to the adding equations presented in Eqs. (3.2.44a–f) for the case involving radiation from above. The principles of invariance can also be formulated for the case involving radiation from below, and the resulting equations would be "equivalent" to the adding equations presented in Eqs. (3.2.46a–f).

We wish to derive a set of equations for the computation of fluxes. Analogous to the derivation of Eq. (3.6.8), substituting Eq. (3.6.5) into Eq. (3.6.4) yields

$$U(\mu, \mu_0) = R_2(\mu, \mu_0)e^{-\tau_1/\mu_0} + 2\int_0^1 R_2(\mu, \mu')T_1(\mu', \mu_0)\mu'\, d\mu' + 2\int_0^1 S_{12}(\mu, \mu'')U(\mu'', \mu_0)\mu''\, d\mu''. \tag{3.6.9a}$$

In order to obtain flux forms based on the principles of invariance, we define

$$y(\mu_0) = 2\int_0^1 Y(\mu, \mu_0)\mu\, d\mu, \tag{3.6.10}$$

where the notations $y(Y)$ can be $u(U)$, $d(D)$, $r_{12}(R_{12})$, $t_{12}(T_{12})$, or $s_{12}(S_{12})$. Carrying out solid-angle integrations on Eqs. (3.6.9a) and (3.6.5)–(3.6.7), we obtain

$$u(\mu_0) = r_2(\mu_0)e^{-\tau_1/\mu_0} + 2\int_0^1 r_2(\mu')T_1(\mu', \mu_0)\mu'\, d\mu' + 2\int_0^1 s_{12}(\mu'')U(\mu'', \mu_0)\mu''\, d\mu'' , \tag{3.6.11a}$$

$$d(\mu_0) = t_1(\mu_0) + 2\int_0^1 r_1(\mu')U(\mu', \mu_0)\mu'\, d\mu' , \tag{3.6.11b}$$

$$r_{12}(\mu_0) = r_1(\mu_0) + 2\int_0^1 e^{-\tau_1/\mu}U(\mu, \mu_0)\mu\, d\mu + 2\int_0^1 t_1(\mu')U(\mu', \mu_0)\mu'\, d\mu' , \tag{3.6.11c}$$

$$t_{12}(\mu_0) = t_2(\mu_0)e^{-\tau_1/\mu_0} + 2\int_0^1 e^{-\tau_2/\mu}D(\mu, \mu_0)\mu\, d\mu + 2\int_0^1 t_2(\mu')D(\mu', \mu_0)\mu'\, d\mu' . \tag{3.6.11d}$$

Moreover, by using global values for reflectance and for direct and diffuse transmittance in these equations, viz.,

$$2\int_0^1 r_2(\mu')T_1(\mu', \mu_0)\mu'\, d\mu' \cong \bar{r}_2\, 2\int_0^1 T_1(\mu', \mu_0)\mu'\, d\mu' = \bar{r}_2 t_1(\mu_0), \tag{3.6.12a}$$

$$2\int_0^1 e^{-\tau_1/\mu}U(\mu, \mu_0)\mu\, d\mu \cong e^{-\tau_1/\bar{\mu}}\, 2\int_0^1 U(\mu, \mu_0)\mu\, d\mu = e^{-\tau_1/\bar{\mu}}u(\mu_0), \tag{3.6.12b}$$

where $1/\bar{\mu}$ denotes the diffusivity factor to be determined numerically, we have

$$u(\mu_0) \cong r_2(\mu_0)e^{-\tau_1/\mu_0} + \bar{r}_2 t_1(\mu_0) + \bar{s}_{12}u(\mu_0), \tag{3.6.13a}$$

$$d(\mu_0) \cong t_1(\mu_0) + \bar{r}_1 \mu(\mu_0), \tag{3.6.13b}$$

$$r_{12}(\mu_0) \cong r_1(\mu_0) + e^{-\tau_1/\bar{\mu}}u(\mu_0) + \bar{t}_1 u(\mu_0), \tag{3.6.13c}$$

$$t_{12}(\mu_0) \cong t_2(\mu_0)e^{-\tau_1/\mu_0} + e^{-\tau_2/\bar{\mu}}d(\mu_0) + \bar{t}_2 d(\mu_0), \tag{3.6.13d}$$

where

$$\bar{s}_{12} = 2\int_0^1 r_1(\mu')r_2(\mu')\mu'\, d\mu' \cong \bar{r}_1\bar{r}_2 . \tag{3.6.14}$$

To finalize the iterative equations for reflectance and transmittance for a combined layer with an optical depth $(\tau_1 + \tau_2)$, we introduce a parameter referred to as the *upward generation function*, based on Eq. (3.6.13a), in the form

$$u(\mu_0) = \left[r_2(\mu_0)e^{-\tau_1/\mu_0} + \bar{r}_2 t_1(\mu_0)\right](1 - \bar{s}_{12})^{-1} . \tag{3.6.15}$$

Further, we may define a number of total global transmittances as follows:

$$\bar{\tilde{t}}_{12} = \bar{t}_{12} + \exp(-\tau_{1,2}/\bar{\mu}), \tag{3.6.16a}$$

$$\tilde{t}_{1,2} = t_{1,2}(\mu_0) + \exp(-\tau_{1,2}/\mu_0). \tag{3.6.16b}$$

By adding the direct transmittance, $\exp[-(\tau_1 + \tau_2)/\mu_0]$, to the diffuse transmittance in Eq. (3.6.13d), we obtain

$$r_{12}(\mu_0) = r_1(\mu_0) + \bar{\tilde{t}}_1 u(\mu_0), \tag{3.6.17}$$

$$\tilde{t}_{12}(\mu_0) = \tilde{t}_2(\mu_0)e^{-\tau_1/\mu_0} + \bar{\tilde{t}}_2[t_1(\mu_0) + \bar{r}_1 u(\mu_0)]. \tag{3.6.18}$$

Equations (3.6.15), (3.6.17), and (3.6.18) constitute a closed set of iterative equations for computing reflectance and transmittance for a combined layer. The physical meaning and configurations of these equations may be understood from Fig. 3.10. The reflectance of a combined layer is produced by (a) the reflectance of the first layer plus (b) the diffuse transmittance of the upward generation function, $u(\mu_0)$. The total transmittance of a combined layer is the result of (a) the transmittance of the direct transmittance component of the first layer and (b) the global diffuse transmittance of the diffuse transmittance plus the global diffuse reflectance of the upward generation function by the second layer. The upward generation function is the sum of (a) the reflectance of the direct transmittance of the first layer and (b) the global diffuse reflectance of the diffuse transmittance of the first layer by the second layer that undergoes multiple reflections.

In order to proceed with the computational procedures for flux, it is necessary to determine the reflectance and transmittance values as functions of the solar zenith angle for each layer. We may begin with a layer that is optically thin and use the single-scattering approximation given in Eqs. (3.2.31)–(3.2.34). Subsequently, we may perform zenith angle integrations to obtain reflectance, $r(\mu_0)$, and transmittance, $t(\mu_0)$. The global albedo $\bar{r}$ and global transmittance $\bar{t}$ may also be calculated. The forward diffraction peak may be incorporated in the computation through the δ-function adjustment discussed in Section 3.4.

The principles of invariance may be applied to a combination of a finite homogeneous layer with an optical depth of τ_* and a surface. Consider a Lambertian surface with an isotropic reflectance of r_s and zero transmittance. From Eqs. (3.6.17), the reflectance at the top of the layer is

$$r_*(\mu_0) = r_1(\mu_0) + \bar{\tilde{t}}_1 u(\mu_0), \tag{3.6.19}$$

where

$$u(\mu_0) = r_s\tilde{t}_1(\mu_0)(1 - r_s\bar{r}_1)^{-1} . \tag{3.6.20}$$

Based on Eq. (3.6.13b), the total transmission at the bottom of the layer (or at the surface) is given by

$$t_*(\mu_0) = \tilde{t}_1(\mu_0) + \bar{r}_1 u(\mu_0). \tag{3.6.21}$$

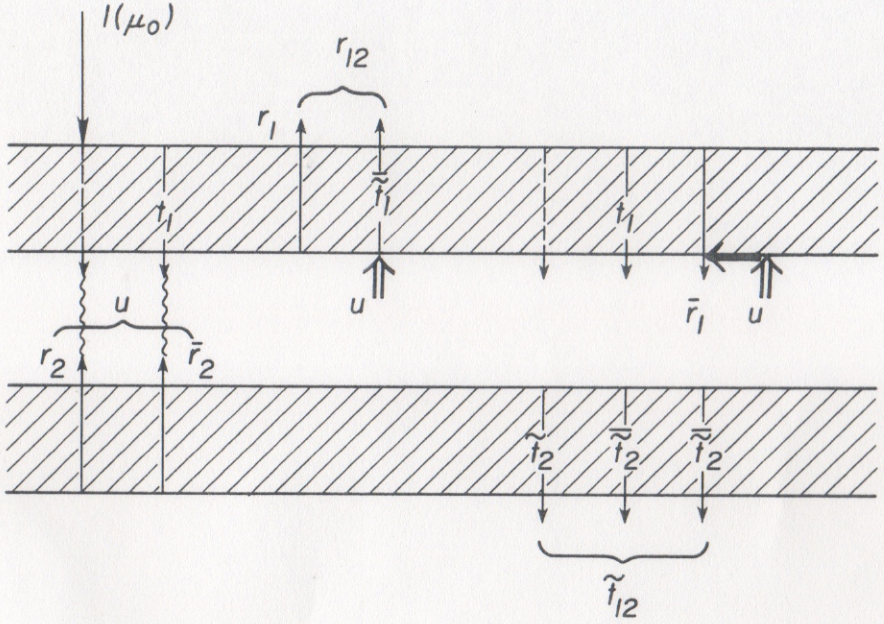

FIG. 3.10 Adding principles for reflectance, r_{12}, and total transmittance, $\tilde{t}_{12}$, according to the terms in Eqs. (3.6.17) and (3.6.18). The term u is defined in Eq. (3.6.15), $1(\mu_0)$ denotes unit solar flux, and the dashed lines in the diagram represent exponential attenuation. The wavy lines illustrate multiple reflections. The meanings of all other terms are explained in the text.

Equations (3.6.19)–(3.6.21) can also be derived from the adding principle for radiative transfer presented in Fig. 3.4.

3.7 Application of radiative transfer methods to nonhomogeneous atmospheres

One of the fundamental difficulties in radiative transfer involves accounting for the nonhomogeneous nature of the atmosphere. Figure 3.11 shows the profiles of molecular and aerosol number densities in the earth's atmosphere. Two typical aerosol concentrations are displayed. The clear condition has a visibility of ~25 km, whereas the hazy condition represents an extreme aerosol concentration in an urban environment. These profiles illustrate that the atmosphere, even without clouds, is nonhomogeneous and cannot be represented by a single single-scattering albedo $\tilde{\omega}$ and a phase function P. The radiative transfer equation for diffuse intensities must be modified to include variations in $\tilde{\omega}$ and P with optical depth. Using the basic radiative transfer equation denoted in Eq. (3.1.16), we may

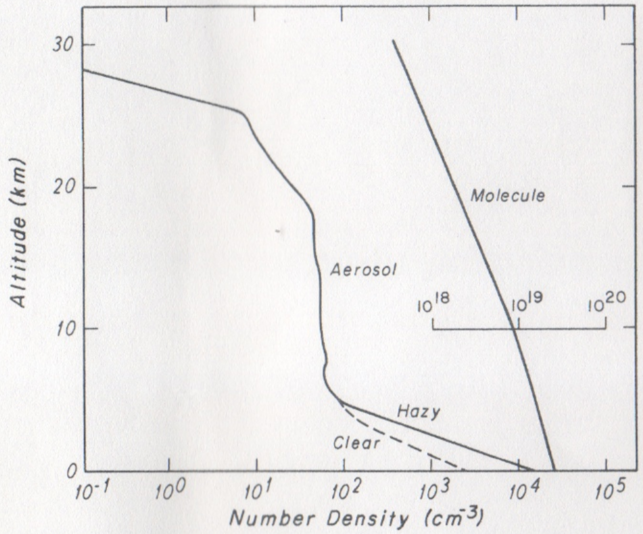

FIG. 3.11 The number density of aerosols and molecules as functions of altitude in a model atmosphere. Two aerosol concentrations, representing background (clear) and urban (hazy) conditions, are shown.

write

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\tilde{\omega}(\tau)}{2}\int_{-1}^{1} I(\tau,\mu')P(\tau;\mu,\mu')\,d\mu' - \frac{\tilde{\omega}(\tau)}{4\pi}P(\tau;\mu,-\mu_0)F_{\odot}e^{-\tau/\mu_0}. \tag{3.7.1}$$

Since $\tilde{\omega}$ and P are functions of optical depth, analytic solutions for this equation are generally not possible. We may, however, devise a numerical procedure to compute the diffuse intensities in nonhomogeneous atmospheres.

3.7.1 *Discrete-ordinates method*

The discrete-ordinates method for radiative transfer can be applied to nonhomogeneous atmospheres by numerical approximations (Liou, 1975). For the present analysis, consider the azimuth-independent component. As illustrated in Fig. 3.12, the atmosphere may be divided into N homogeneous layers, each of which is characterized by a single-scattering albedo, a phase function, and an extinction coefficient (or optical depth). The solution for the azimuthally independent diffuse intensity, as given in Eq. (3.2.7), may be written for each individual layer ℓ in the form

$$I^{(\ell)}(\tau,\mu_i) = \sum_j L_j^{(\ell)}\phi_j^{(\ell)}(\mu_i)e^{-k_j^{(\ell)}\tau} + Z^{(\ell)}(\mu_i)e^{-\tau/\mu_0}, \quad \ell = 1,2,\ldots,N. \tag{3.7.2}$$