7.6 Repeat Problem 6.10 for the case that Surface 1 is coated with a material described in Problem 7.5.

7.7 Repeat Problem 6.12 for the case that the corner is coated with a diffusely emitting, specularly reflecting layer whose spectral behavior may be approximated by

$$\epsilon_\lambda = \begin{cases} 0.8, & 0 \le \lambda < 3\,\mu\text{m}, \\ 0.2, & 3\,\mu\text{m} < \lambda < \infty. \end{cases}$$

The line source consists of a long filament at 2500 K inside a quartz tube, i.e., the source behaves like a gray body for $\lambda < 2.5\,\mu$m but has no emission beyond 2.5 μm.

7.8 Repeat Problem 6.12 for the case that the corner is cold (i.e., has negligible emission), and that the surface is gray and specularly reflecting with $\epsilon = \rho^s = 0.5$, but has a directional emissivity/absorptivity of

$$\epsilon'(\theta) = \epsilon_n \cos\theta.$$

Determine local and total absorbed radiative heat fluxes.

7.9 Consider two infinitely long, parallel plates of width $w = 1$ m, spaced a distance $h = 0.5$ m apart (see Configuration 32 in Appendix D). Both plates are isothermal at 1000 K and are coated with a gray material with a directional emissivity of

$$\epsilon'(\theta_i) = \alpha'(\theta_i) = 1 - \rho'(\theta_i) = \epsilon_n \cos\theta_i,$$

and a hemispherical emissivity of $\epsilon = 0.5$. Reflection is neither diffuse nor specular, but the bidirectional reflection function of the material is

$$\rho''(\theta_i, \theta_r) = \frac{3}{2\pi}\rho'(\theta_i)\cos\theta_r.$$

Write a small computer program to determine the total heat lost (per unit length) from each plate. Compare with the case for a diffusely emitting/reflecting surface.

CHAPTER 8

THE EQUATION OF RADIATIVE TRANSFER IN PARTICIPATING MEDIA

8.1 INTRODUCTION

In previous chapters we have looked at radiative transfer between surfaces that were separated by vacuum or by a transparent ("radiatively nonparticipating") medium. However, in many engineering applications the interaction of thermal radiation with an absorbing, emitting, and scattering ("radiatively participating") medium must be accounted for. Examples in the heat transfer area are the burning of any fuel (be it gaseous, liquid or solid; be it for power production, within fires, within explosions, etc.), rocket propulsion, hypersonic shock layers, ablation systems on reentry vehicles, nuclear explosions, plasmas in fusion reactors, and many more.

In the present chapter we shall develop the general relationships that govern the behavior of radiative heat transfer in the presence of an absorbing, emitting, and/or scattering medium. We shall begin by making a radiative energy balance, known as the *Equation of Radiative Transfer*, which describes the radiative intensity field within the enclosure as a function of location (fixed by location vector **r**), direction (fixed by unit direction vector $\hat{\mathbf{s}}$) and spectral variable (wavenumber η).[1] To obtain the net radiative heat flux crossing a surface element, we must sum the contributions of radiative energy irradiating the surface from all possible directions and for all possible wavenumbers. Therefore, integrating the equation of transfer over all directions and wavenumbers leads to a *conservation of radiative energy* statement applied to an infinitesimal volume. Finally, this will be combined with a balance for all types of energy (including conduction and convection), leading to the *Overall Conservation of Energy* equation.

In the following three chapters we shall deal with the radiation properties of participating media, i.e., with how a substance can absorb, emit, and scatter thermal radiation. In Chapter 9 we discuss how a molecular gas can absorb and emit photons by changing its energy states, how to predict the radiation properties, and how to measure them experimentally. Chapter 10 is concerned with how small particles interact with electromagnetic waves—how they absorb, emit, and scatter radiative energy. Again, theoretical as well as experimental methods are covered. Finally, in Chapter 11 a very brief account is given of the radiation properties of solids and liquids that allow electromagnetic waves of certain wavelengths to penetrate into them for appreciable distances, known as semitransparent media.

8.2 RADIATIVE INTENSITY IN VACUUM

Before we discuss how radiative intensity is affected by absorption, emission, and scattering, it is important to understand how intensity penetrates through a vacuum. When discussing surface radiation we noticed that the concept of intensity had one advantage over emissive power, namely, that the emitted intensity from a black surface did not vary with direction. Within a medium, the definition of an emissive power is not possible since there is no surface to which to relate it. The intensity, defined as radiative energy transferred per unit time, solid angle, spectral variable, and area normal to the pencil of rays, is the most appropriate variable to describe the radiative transfer within a medium.

Consider radiative intensity penetrating at normal angle through a (fictitious) infinitesimal area dA_1, at location s_1 and time t_1, as shown in the sketch of Fig. 8-1. Based on the definition of intensity, we see that the amount of energy passing through dA_1 over a duration dt and spectral range $d\eta$ that will—a little later—fall onto the infinitesimal surface dA_2, is

$$I_\eta(s_1, t_1)\, dt\, d\Omega_{1\to2}\, d\eta\, dA_1 = I_\eta(s_1, t_1)\, dt \frac{dA_2}{(s_2 - s_1)^2}\, d\eta\, dA_1,$$

[1] In our discussion of surface radiative transport we have used wavelength λ as the spectral variable throughout, largely to conform with the majority of other publications. However, for gases, frequency ν or wavenumber η are considerably more convenient to use. Again, to conform with the majority of the literature, we shall use wavenumber throughout this part.

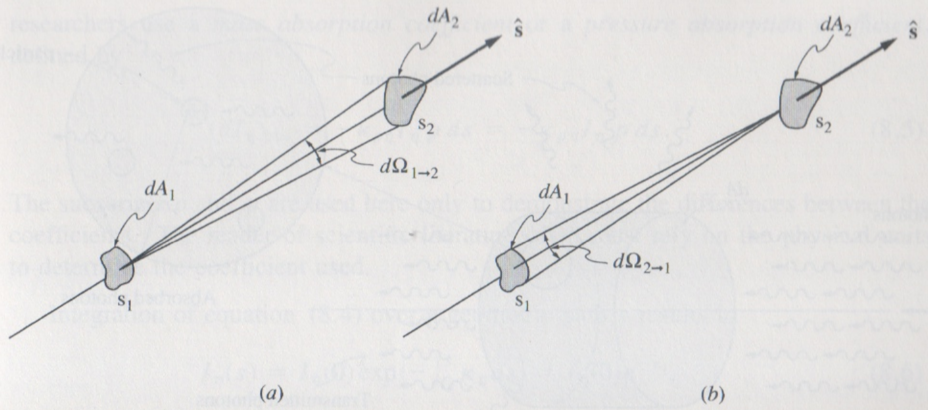

FIGURE 8-1
Radiative intensity in vacuum.

where $d\Omega_{1\to2}$ is the solid angle with which dA_2 is seen from an observer on dA_1. Since it takes the radiation until the time $t_2 = t_1 + (s_2 - s_1)/c$ to travel from s_1 to s_2, we can say that the energy going through dA_2 that is coming from dA_1 is

$$I_\eta(s_2, t_2)\, dt\, d\Omega_{2\to1}\, d\eta\, dA_2 = I_\eta(s_2, t_2)\, dt \frac{dA_1}{(s_2 - s_1)^2}\, d\eta\, dA_2.$$

Since both energies must be equal, we conclude that

$$I_\eta(s_2, t_1 + (s_2 - s_1)/c) = I_\eta(s_1, t_1). \tag{8.1}$$

Since the speed of light is so large in comparison with nearly every time scale in engineering problems, we may almost always assume that the radiative energy arrives "instantaneously" everywhere in the medium,[2] or

$$I_\eta(s_2) = I_\eta(s_1), \tag{8.2}$$

or

$$I_\eta(\hat{\mathbf{s}}) = \text{const.} \tag{8.3}$$

Therefore, within a radiatively nonparticipating medium, the radiative intensity in any given direction is constant along its path. This property of the intensity makes it a most suitable quantity for the description of absorption, emission and scattering of energy within a medium, because any changes in intensity along any given path must be due to one or more of these phenomena.

[2] Using slightly different arguments this relation had already been established during the discussion on surface radiation between nonideal surfaces, equation (7.9).

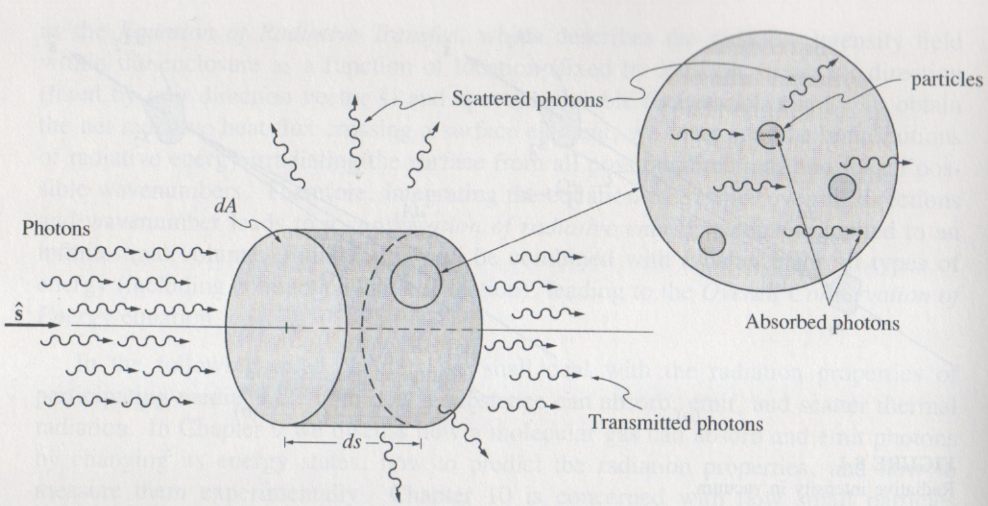

FIGURE 8-2
Attenuation of radiative intensity by absorption and scattering.

8.3 ATTENUATION BY ABSORPTION AND SCATTERING

If the medium through which radiative energy travels is "participating," then any incident beam will be attenuated by absorption and scattering while it travels through the medium, as schematically shown in Fig. 8-2. In the following we shall develop expressions for this attenuation for a light beam which travels within a pencil of rays into the direction $\hat{s}$. The present discussion will be limited to media with constant refractive index, i.e., media through which electromagnetic waves travel along straight lines [while a varying refractive index will bend the ray, as shown by Snell's law, equation (2.71), for an abrupt change].

Absorption

The absolute amount of absorption has been observed to be directly proportional to the magnitude of the incident energy as well as the distance the beam travels through the medium. Thus, we may write,

$$(dI_\eta)_{\text{abs}} = -\kappa_\eta I_\eta \, ds, \tag{8.4}$$

where the proportionality constant κ_η is known as the *(linear) absorption coefficient*, and the negative sign has been introduced since the intensity decreases. As will be discussed in the following chapter, the absorption of radiation in molecular gases depends also on the number of receptive molecules per unit volume, so that some

researchers use a *mass absorption coefficient* or a *pressure absorption coefficient*, defined by

$$(dI_\eta)_{\text{abs}} = -\kappa_{\rho\eta} I_\eta \, \rho \, ds = -\kappa_{p\eta} I_\eta \, p \, ds. \tag{8.5}$$

The subscripts ρ and p are used here only to demonstrate the differences between the coefficients. The reader of scientific literature often must rely on the physical units to determine the coefficient used.

Integration of equation (8.4) over a geometric path s results in

$$I_\eta(s) = I_\eta(0) \exp\left(-\int_0^s \kappa_\eta \, ds\right) = I_\eta(0) e^{-\tau_\eta}, \tag{8.6}$$

where

$$\tau_\eta = \int_0^s \kappa_\eta \, ds \tag{8.7}$$

is the optical thickness (for absorption) through which the beam has traveled and $I_\eta(0)$ is the intensity entering the medium at $s = 0$. Note that the (linear) absorption coefficient is the inverse of the mean free path for a photon until it undergoes absorption. One may also define an *absorptivity* for the participating medium (for a given path within the medium) as

$$\alpha_\eta \equiv \frac{I_\eta(0) - I_\eta(s)}{I_\eta(0)} = 1 - e^{-\tau_\eta}. \tag{8.8}$$

Scattering

Attenuation by scattering, or "out-scattering" (away from the direction under consideration), is very similar to absorption, i.e., a part of the incoming intensity is removed from the direction of propagation, $\hat{s}$. The only difference between the two phenomena is that absorbed energy is converted into internal energy, while scattered energy is simply redirected and appears as augmentation along another direction (discussed in the next section), also known as "in-scattering." Thus, we may write

$$(dI_\eta)_{\text{sca}} = -\sigma_{s\eta} I_\eta \, ds, \tag{8.9}$$

where the proportionality constant $\sigma_{s\eta}$ is the *(linear) scattering coefficient* for scattering from the pencil of rays under consideration into all other directions. Again, scattering coefficients based on density or pressure may be defined. It is also possible to define an optical thickness for scattering, where the scattering coefficient is the inverse of the mean free path for scattering.

Total Attenuation

The total attenuation of the intensity in a pencil of rays by both absorption and scattering is known as *extinction*. Thus, an *extinction coefficient* is defined[3] as

$$\beta_\eta = \kappa_\eta + \sigma_{s\eta}. \tag{8.10}$$

The optical distance based on extinction is defined as

$$\tau_\eta = \int_0^s \beta_\eta\, ds. \tag{8.11}$$

As for absorption and scattering, the extinction coefficient is sometimes based on density or pressure.

8.4 AUGMENTATION BY EMISSION AND SCATTERING

A light beam traveling through a participating medium in the direction of $\hat{s}$ loses energy by absorption and by scattering away from the direction of travel. But at the same time it also gains energy by emission, as well as by scattering from other directions into the direction of travel $\hat{s}$.

Emission

The rate of emission from a volume element will be proportional to the magnitude of the volume. Therefore, the emitted intensity (which is the rate of emitted energy per unit area) along any path again must be proportional to the length of the path, and it must be proportional to the local energy content in the medium. Since, at thermodynamic equilibrium, the intensity everywhere must be equal to the blackbody intensity, it will be shown in Chapter 9, equation (9.15), that

$$(dI_\eta)_{em} = \kappa_\eta I_{b\eta}\, ds, \tag{8.12}$$

that is, the proportionality constant for emission is the same as for absorption. Similar to absorptivity, one may also define an *emissivity of an isothermal medium* as the amount of energy emitted over a certain path s that escapes into a given direction (without having been absorbed between point of emission and point of exit), as compared to the maximum possible. Combining equations (8.4) and (8.12) gives the complete equation of transfer for an absorbing-emitting (but not scattering) medium as

$$\frac{dI_\eta}{ds} = \kappa_\eta (I_{b\eta} - I_\eta), \tag{8.13}$$

[3]Care must be taken to distinguish the dimensional extinction coefficient β_η from the absorptive index, i.e., the imaginary part of the index of refraction complex k (sometimes referred to in the literature as the "extinction coefficient").

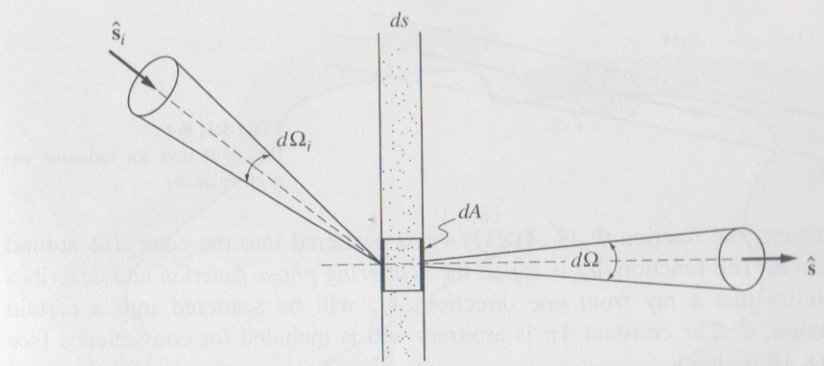

FIGURE 8-3
Redirection of radiative intensity by scattering.

where the first term of the right-hand side is augmentation due to emission and the second term is attenuation due to absorption. The solution to the equation of transfer for an isothermal gas layer of thickness s is

$$I_\eta(s) = I_\eta(0)e^{-\tau_\eta} + I_{b\eta}\left(1 - e^{-\tau_\eta}\right), \tag{8.14}$$

where the optical distance has been defined in equation (8.7). If only emission is considered, $I_\eta(0) = 0$, and the emissivity is defined as

$$\epsilon_\eta = I_\eta(s)/I_{b\eta} = 1 - e^{-\tau_\eta}, \tag{8.15}$$

which, as is the case with surface radiation, is identical to the expression for absorptivity.

Scattering

Augmentation due to scattering, or "in-scattering," has contributions from all directions and, therefore, must be calculated by integration over all solid angles. Consider the radiative heat flux impinging on a volume element $dV = dA\, ds$, from an infinitesimal pencil of rays in the direction $\hat{s}_i$ as depicted in Fig. 8-3. Recalling the definition for radiative intensity as energy flux per unit area normal to the rays, per unit solid angle, and per unit wavenumber interval, one may calculate the total spectral radiative heat flux impinging on dA from within the solid angle $d\Omega_i$ as

$$I_\eta(\hat{s}_i)(dA\,\hat{s}_i \cdot \hat{s})\, d\Omega_i\, d\eta.$$

This flux travels through dV for a distance $ds/\hat{s}_i \cdot \hat{s}$. Therefore, the total amount of energy scattered away from $\hat{s}_i$ is, according to equation (8.9),

$$\sigma_{s\eta}\left[I_\eta(\hat{s}_i)(dA\,\hat{s}_i \cdot \hat{s})\, d\Omega_i\, d\eta\right]\left(\frac{ds}{\hat{s}_i \cdot \hat{s}}\right) = \sigma_{s\eta} I_\eta(\hat{s}_i)\, dA\, d\Omega_i\, d\eta\, ds. \tag{8.16}$$

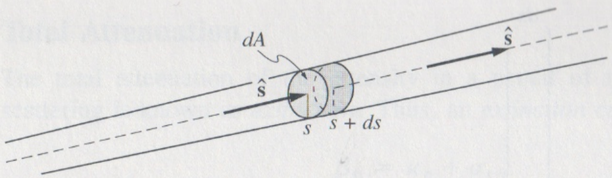

FIGURE 8-4
Pencil of rays for radiative energy balance.

Of this amount, the fraction $\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega/4\pi$ is scattered into the cone $d\Omega$ around the direction $\hat{\mathbf{s}}$. The function Φ_η is called the *scattering phase function* and describes the probability that a ray from one direction, $\hat{\mathbf{s}}_i$, will be scattered into a certain other direction, $\hat{\mathbf{s}}$. The constant 4π is arbitrary and is included for convenience [see equation (8.19) below].

The amount of energy flux from the cone $d\Omega_i$ scattered into the cone $d\Omega$ is then

$$\sigma_{s\eta} I_\eta(\hat{\mathbf{s}}_i)\, dA\, d\Omega_i\, d\eta\, ds \frac{\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})}{4\pi}\, d\Omega. \tag{8.17}$$

We can now calculate the energy flux scattered into the direction $\hat{\mathbf{s}}$ from *all* incoming directions $\hat{\mathbf{s}}_i$ by integrating:

$$(dI_\eta)_{\text{sca}}(\hat{\mathbf{s}})\, dA\, d\Omega\, d\eta = \int_{4\pi} \sigma_{s\eta} I_\eta(\hat{\mathbf{s}}_i)\, dA\, d\Omega_i\, d\eta\, ds\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}}) \frac{d\Omega}{4\pi},$$

or

$$(dI_\eta)_{\text{sca}}(\hat{\mathbf{s}}) = ds \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i. \tag{8.18}$$

Returning to equation (8.17), we find that the amount of energy flux scattered from $d\Omega_i$ into all directions is

$$\sigma_{s\eta} I_\eta(\hat{\mathbf{s}}_i)\, dA\, d\Omega_i\, d\eta\, ds \frac{1}{4\pi} \int_{4\pi} \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega,$$

which must be equal to the amount in equation (8.16). We conclude that

$$\frac{1}{4\pi} \int_{4\pi} \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega \equiv 1. \tag{8.19}$$

Therefore, if Φ_η = const, i.e., if equal amounts of energy are scattered into all directions (called *isotropic scattering*), then $\Phi_\eta \equiv 1$. This is the reason for the inclusion of the factor 4π.

8.5 THE EQUATION OF TRANSFER

We can now make an energy balance on the radiative energy traveling in the direction of $\hat{\mathbf{s}}$ within a small pencil of rays as shown in Fig. 8-4. The change in intensity is found by summing the contributions from emission, absorption, scattering away from the direction $\hat{\mathbf{s}}$, and scattering into the direction of $\hat{\mathbf{s}}$, from equations (8.4), (8.9) (8.12), and (8.18) as

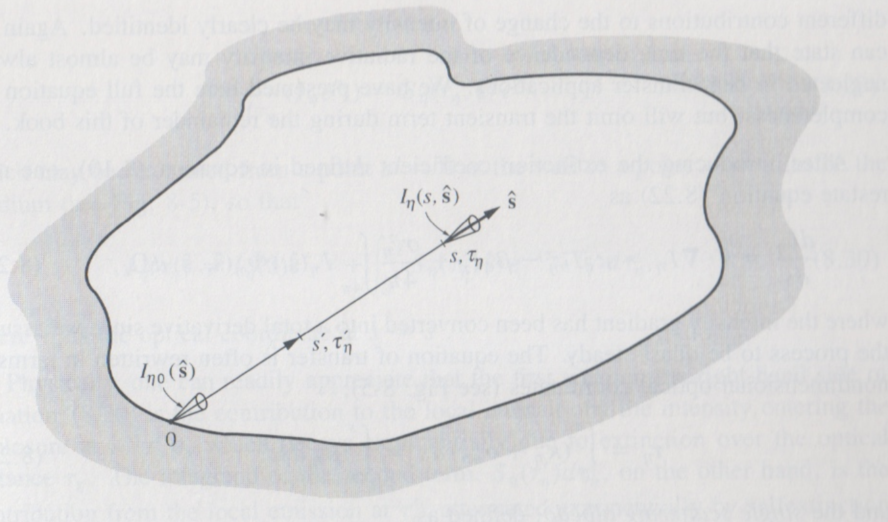

FIGURE 8-5
Enclosure for derivation of equation of transfer.

$$\begin{aligned} I_\eta(s+ds, \hat{\mathbf{s}}, t+dt) - I_\eta(s, \hat{\mathbf{s}}, t) &= \kappa_\eta I_{b\eta}(s, t)\, ds - \kappa_\eta I_\eta(s, \hat{\mathbf{s}}, t)\, ds \\ &- \sigma_{s\eta} I_\eta(s, \hat{\mathbf{s}}, t)\, ds + \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i\, ds. \end{aligned} \tag{8.20}$$

The outgoing intensity may be developed into a truncated Taylor series, or

$$I_\eta(s+ds, \hat{\mathbf{s}}, t+dt) = I_\eta(s, \hat{\mathbf{s}}, t) + dt \frac{\partial I_\eta}{\partial t} + ds \frac{\partial I_\eta}{\partial s}, \tag{8.21}$$

so that equation (8.20) may be simplified to

$$\frac{1}{c}\frac{\partial I_\eta}{\partial t} + \frac{\partial I_\eta}{\partial s} = \kappa_\eta I_{b\eta} - \kappa_\eta I_\eta - \sigma_{s\eta} I_\eta + \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i, \tag{8.22}$$

where $c = ds/dt$ is the speed with which the radiation intensity propagates. In this equation all quantities may vary with location in space, time and wavenumber, while the intensity and the phase function also depend on direction $\hat{\mathbf{s}}$ (and $\hat{\mathbf{s}}_i$). Only the directional dependence, and only whenever necessary, has been explicitly indicated in this and the following equations, to simplify notation. Equation (8.22) is valid anywhere inside an arbitrary enclosure. Its solution requires knowledge of the intensity for each direction at some location s, usually the intensity entering the medium through or from the enclosure boundary into the direction of $\hat{\mathbf{s}}$, as indicated in Fig. 8-5. We have not yet brought equation (8.22) into its most compact form so that the four different contributions to the change of intensity may be clearly identified. Again we can state that the time dependence of the radiative intensity may be almost always neglected in heat transfer applications. We have presented here the full equation for completeness, but will omit the transient term during the remainder of this book.

After introducing the extinction coefficient defined in equation (8.10), one may restate equation (8.22) as

$$\frac{dI_\eta}{ds} = \hat{\mathbf{s}} \cdot \nabla I_\eta = \kappa_\eta I_{b\eta} - \beta_\eta I_\eta + \frac{\sigma_{s\eta}}{4\pi}\int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i, \tag{8.23}$$

where the intensity gradient has been converted into a total derivative since we assume the process to be quasi-steady. The equation of transfer is often rewritten in terms of nondimensional optical coordinates (see Fig. 8-5),

$$\tau_\eta = \int_0^s (\kappa_\eta + \sigma_{s\eta})\, ds = \int_0^s \beta_\eta\, ds, \tag{8.24}$$

and the *single scattering albedo*, defined as

$$\omega_\eta \equiv \frac{\sigma_{s\eta}}{\kappa_\eta + \sigma_{s\eta}} = \frac{\sigma_{s\eta}}{\beta_\eta}, \tag{8.25}$$

leading to

$$\frac{dI_\eta}{d\tau_\eta} = -I_\eta + (1 - \omega_\eta)I_{b\eta} + \frac{\omega_\eta}{4\pi}\int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i. \tag{8.26}$$

The last two terms in equation (8.26) are often combined and are then known as the *source function* for radiative intensity,

$$S_\eta(\tau_\eta, \hat{\mathbf{s}}) = (1 - \omega_\eta)I_{b\eta} + \frac{\omega_\eta}{4\pi}\int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i. \tag{8.27}$$

Equation (8.26) then assumes the deceptively simple form of

$$\frac{dI_\eta}{d\tau_\eta} + I_\eta = S_\eta(\tau_\eta, \hat{\mathbf{s}}), \tag{8.28}$$

which is, of course, an integro-differential equation (in space, and in two directional coordinates with local origin). Furthermore, the Planck function $I_{b\eta}$ is generally not known and must be found by considering the overall energy equation (adding derivatives in the three space coordinates and integrations over two more directional coordinates and the wavenumber spectrum).

8.6 FORMAL SOLUTION TO THE EQUATION OF TRANSFER

If the source function is known (or assumed known), equation (8.28) can be formally integrated by the use of an integrating factor. Thus, multiplying through by e^{τ_η} results in

$$\frac{d}{d\tau_\eta}(I_\eta e^{\tau_\eta}) = S_\eta(\tau_\eta, \hat{\mathbf{s}})\, e^{\tau_\eta}, \tag{8.29}$$

which may be integrated from a point $s' = 0$ at the wall to a point $s' = s$ inside the medium (see Fig. 8-5), so that

$$I_\eta(\tau_\eta) = I_\eta(0)e^{-\tau_\eta} + \int_0^{\tau_\eta} S_\eta(\tau'_\eta, \hat{\mathbf{s}})\, e^{-(\tau_\eta - \tau'_\eta)}\, d\tau'_\eta, \tag{8.30}$$

where τ'_η is the optical coordinate at $s = s'$.

Physically, one can readily appreciate that the first term on the right-hand side of equation (8.30) is the contribution to the local intensity by the intensity entering the enclosure at $s = 0$, which decays exponentially due to extinction over the optical distance τ_η. The integrand of the second term, $S_\eta(\tau'_\eta)d\tau'_\eta$, on the other hand, is the contribution from the local emission at τ'_η, attenuated exponentially by selfextinction over the optical distance between the emission point and the point under consideration, $\tau_\eta - \tau'_\eta$. The integral, finally, sums all the contributions over the entire emission path.

Equation (8.30) is a third-order integral equation in intensity I_η. The integral over the source function must be carried out over the optical coordinate (for all directions), while the source function itself is also an integral over a set of direction coordinates (with varying local origin) containing the unknown intensity. Furthermore, usually the temperature and, therefore, the blackbody intensity are not known and must be found in conjunction with overall conservation of energy. There are, however, a few cases for which the equation of transfer becomes considerably simplified.

Nonscattering Medium

If the medium only absorbs and emits, the source function reduces to the local blackbody intensity, and

$$I_\eta(\tau_\eta) = I_\eta(0)e^{-\tau_\eta} + \int_0^{\tau_\eta} I_{b\eta}(\tau'_\eta)\, e^{-(\tau_\eta - \tau'_\eta)} d\tau'_\eta. \tag{8.31}$$

This equation is an explicit expression for the radiation intensity if the temperature field is known. However, generally the temperature is not known and must be found in conjunction with overall conservation of energy.

Example 8.1. What is the spectral intensity emanating from an isothermal sphere bounded by vacuum or a cold black wall?

Solution

Because of the symmetry in this problem, the intensity emanating from the sphere surface is only a function of the exit angle. Examining Fig. 8-6, we see that equation (8.31)

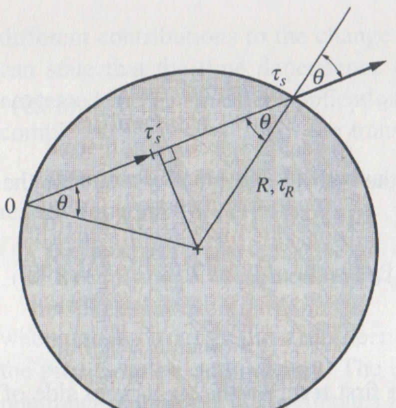

FIGURE 8-6
Isothermal sphere for Example 8.1.

reduces to

$$I_\eta(\tau_R, \theta) = \int_0^{\tau_s} I_{b\eta}(\tau_s')\, e^{-(\tau_s - \tau_s')}\, d\tau_s'.$$

But for a sphere

$$\tau_s = 2\tau_R \cos\theta,$$

regardless of the azimuthal angle. Therefore, with $I_{b\eta}(\tau_s') = I_{b\eta}$ = const, the desired intensity turns out to be

$$I_\eta(\tau_R, \theta) = I_{b\eta}\, e^{-(2\tau_R \cos\theta - \tau_s')}\Big|_0^{2\tau_R \cos\theta} = I_{b\eta}\left(1 - e^{-2\tau_R \cos\theta}\right).$$

Thus, for $\tau_R \gg 1$ the isothermal sphere emits equally into all directions, like a black surface at the same temperature.

The Cold Medium

If the temperature of the medium is so low that the blackbody intensity at that temperature is small as compared with incident intensity, then the equation of transfer is decoupled from other modes of heat transfer. However, the governing equation remains a third-order integral equation, namely,

$$I_\eta(\tau_\eta, \hat{s}) = I_\eta(0) e^{-\tau_\eta} + \int_0^{\tau_\eta} \frac{\omega_\eta}{4\pi} \int_{4\pi} I_\eta(\tau_\eta', \hat{s}_i)\, \Phi_\eta(\hat{s}_i, \hat{s})\, d\Omega_i\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{8.32}$$

If the scattering is isotropic, or $\Phi \equiv 1$, the directional integration in equation (8.32) may be carried out, so that

$$I_\eta(\tau_\eta, \hat{s}) = I_\eta(0) e^{-\tau_\eta} + \frac{1}{4\pi} \int_0^{\tau_\eta} \omega_\eta\, G_\eta(\tau_\eta')\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta', \tag{8.33}$$

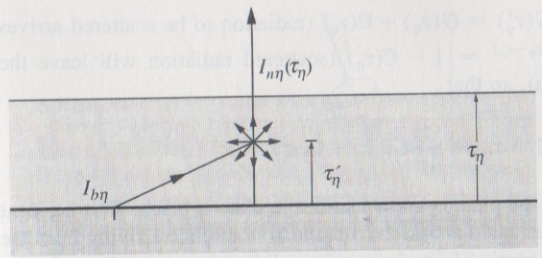

FIGURE 8-7
Geometry for Example 8.2.

where

$$G_\eta(\tau) \equiv \int_{4\pi} I_\eta(\tau_\eta', \hat{s}_i)\, d\Omega_i \tag{8.34}$$

is known as the *incident radiation function* (since it is the total intensity impinging on a point from all sides). The problem is then much simplified since it is only necessary to find a solution for G [by direction-integrating equation (8.33)] rather than determining the direction-dependent intensity.

Purely Scattering Medium

If the medium scatters radiation, but does not absorb or emit, then the radiative transfer is again decoupled from other heat transfer modes. In this case $\omega_\eta \equiv 1$, and the equation of transfer reduces to a form essentially identical to equation (8.32), i.e.,

$$I_\eta(\tau_\eta, \hat{s}) = I_\eta(0)\, e^{-\tau_\eta} + \frac{1}{4\pi} \int_0^{\tau_\eta} \int_{4\pi} I_\eta(\tau_\eta', \hat{s}_i)\, \Phi_\eta(\hat{s}_i, \hat{s})\, d\Omega_i\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{8.35}$$

Again, for isotropic scattering, this equation may be simplified by introducing the incident radiation, so that

$$I_\eta(\tau_\eta, \hat{s}) = I_\eta(0)\, e^{-\tau_\eta} + \frac{1}{4\pi} \int_0^{\tau_\eta} G_\eta(\tau_\eta', \hat{s})\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{8.36}$$

Example 8.2. A large isothermal black plate is covered with a thin layer of isotropically scattering, nonabsorbing (and, therefore, nonemitting) material with unity index of refraction. Assuming that the layer is so thin that any ray emitted from the plate is scattered at most once before leaving the scattering layer, estimate the radiative intensity above the layer in the direction normal to the plate.

Solution

The exiting intensity in the normal direction (see Fig. 8-7) may be calculated from equation (8.36) by retaining only terms of order τ_η or higher (since $\tau_\eta \ll 1$). This process leads to $e^{-\tau_\eta} = 1 - \tau_\eta + \mathcal{O}(\tau_\eta^2)$, $G(\tau'_\eta) = G(\tau_\eta) + \mathcal{O}(\tau_\eta)$ (radiation to be scattered arrives unattenuated at a point), and $e^{-(\tau_\eta - \tau'_\eta)} = 1 - \mathcal{O}(\tau_\eta)$ (scattered radiation will leave the medium without further attenuation), so that

$$I_{n\eta} = I_{b\eta}(1 - \tau_\eta) + \frac{1}{4\pi} G_\eta \tau_\eta + \mathcal{O}(\tau_\eta^2),$$

where the intensity emanating from the plate is known since the plate is black. The incident radiation at any point is due to unattenuated emission from the bottom plate arriving from the lower 2π solid angles, and nothing coming from the top 2π solid angles, i.e., $G_\eta \approx 2\pi I_{b\eta}$ and

$$I_{n\eta} = I_{b\eta}(1 - \tau_\eta) + \frac{1}{2} I_{b\eta} \tau_\eta + \mathcal{O}(\tau_\eta^2) = I_{b\eta}\left(1 - \frac{\tau_\eta}{2}\right) + \mathcal{O}(\tau_\eta^2).$$

Physically this result tells us that the emission into the normal direction is attenuated by the fraction τ_η (scattered away from normal direction), and augmented by the fraction $\tau_\eta/2$ (scattered into the normal direction): Since scattering is isotropic, exactly half of the attenuation is scattered upward and half downward; the latter is then absorbed by the emitting plate. Thus, the scattering layer acts as a heat shield for the hot plate.

8.7 BOUNDARY CONDITIONS FOR THE EQUATION OF TRANSFER

The equation of transfer in its quasi-steady form, equation (8.23), is a first-order differential equation in intensity (for a fixed direction $\hat{\mathbf{s}}$). As such, the equation requires knowledge of the radiative intensity at a single point in space, into the direction of $\hat{\mathbf{s}}$. Generally, the point where the intensity can be specified independently lies on the surface of an enclosure surrounding the participating medium, as indicated by the formal solution in equation (8.30). This intensity, leaving a wall into a specified direction, may be determined by the methods given in Chapter 5 (diffusely emitting and reflecting surfaces), Chapter 6 (diffusely emitting and specularly reflecting surfaces) and Chapter 7 (surfaces with arbitrary characteristics).

Diffusely Emitting and Reflecting Opaque Surfaces

For a surface that emits and reflects diffusely, the exiting intensity is independent of direction. Therefore, at a point $\mathbf{r}_w$ on the surface, from equations (5.18) and (5.19),

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = I(\mathbf{r}_w) = J(\mathbf{r}_w)/\pi = \epsilon(\mathbf{r}_w) I_b(\mathbf{r}_w) + \rho(\mathbf{r}_w) H(\mathbf{r}_w)/\pi, \tag{8.37}$$

where $H(\mathbf{r}_w)$ is the hemispherical irradiation (i.e., incoming radiative heat flux) defined by equation (3.39), leading to

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = \epsilon(\mathbf{r}_w) I_b(\mathbf{r}_w) + \frac{\rho(\mathbf{r}_w)}{\pi} \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}'<0} I(\mathbf{r}_w, \hat{\mathbf{s}}') |\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}'| \, d\Omega', \tag{8.38}$$

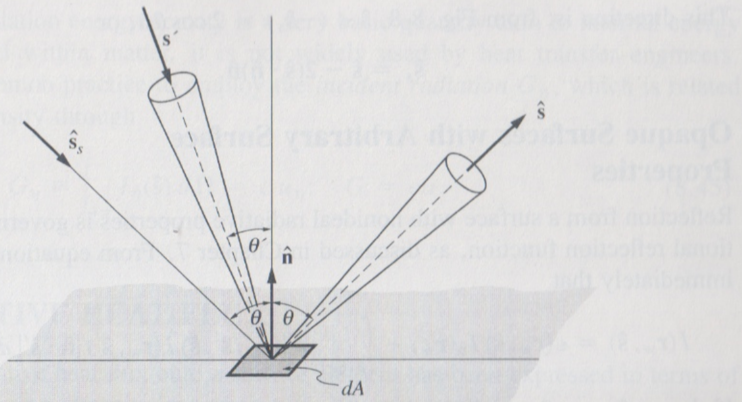

FIGURE 8-8
Radiative intensity reflected from a surface.

where $\hat{\mathbf{n}}$ is the local outward surface normal and $\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}' = \cos\theta'$ is the cosine of the angle between any incoming direction $\hat{\mathbf{s}}'$ and the surface normal, as indicated in Fig. 8-8. Therefore, the outgoing intensity is not generally known explicitly, but is related to the incoming intensity. An exception is the black surface, for which (with $\rho = 0$),

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = I_b(\mathbf{r}_w). \tag{8.39}$$

Diffusely Emitting, Specularly Reflecting, Opaque Surfaces

If the reflectivity of the surface has a specular as well as a diffuse component, i.e., the reflectivity obeys equation (6.1), then the outgoing intensity also consists of two components. One part of the outgoing intensity is due to diffuse emission as well as the diffuse fraction of reflected energy, as described by equation (8.38). In addition, the outgoing intensity has a specularly reflected component,[4] so that

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = \epsilon(\mathbf{r}_w) I_b(\mathbf{r}_w) + \frac{\rho^d(\mathbf{r}_w)}{\pi} \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}'<0} I(\mathbf{r}_w, \hat{\mathbf{s}}') |\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}'| \, d\Omega' + \rho^s(\mathbf{r}_w) I(\mathbf{r}_w, \hat{\mathbf{s}}_s), \tag{8.40}$$

where $\hat{\mathbf{s}}_s$ is the "specular direction," defined as the direction from which a light beam must hit the surface in order to travel into the direction of $\hat{\mathbf{s}}$ after a specular reflection.

[4] Note that the specularly reflected component cannot be "assigned" to the surface where it leaves in diffuse fashion, as was done for surface transport in Chapter 6. The reason is that the intensity changes while radiation travels from surface to surface within a participating medium.

This direction is, from Fig. 8-8, $\hat{\mathbf{s}} + (-\hat{\mathbf{s}}_s) = 2\cos\theta\hat{\mathbf{n}}$, or

$$\hat{\mathbf{s}}_s = \hat{\mathbf{s}} - 2(\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}. \tag{8.41}$$

Opaque Surfaces with Arbitrary Surface Properties

Reflection from a surface with nonideal radiative properties is governed by the bidirectional reflection function, as discussed in Chapter 7. From equation (7.10) it follows immediately that

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = \epsilon(\mathbf{r}_w, \hat{\mathbf{s}}) I_b(\mathbf{r}_w) + \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}<0} \rho''(\mathbf{r}_w, \hat{\mathbf{s}}', \hat{\mathbf{s}}) I(\mathbf{r}_w, \hat{\mathbf{s}}') |\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}'| \, d\Omega'. \tag{8.42}$$

If the surface reflects diffusely, $\rho'' = \rho^d/\pi$ and equation (8.42) reduces to equation (8.38). For specular reflection the development of equation (7.15) shows that it reduces to equation (8.40).

Semitransparent Boundaries

If the boundary is a semitransparent wall, external radiation may penetrate into the enclosure and must be added to equations (8.38), (8.40), and (8.42) as $I_o(\mathbf{r}_w, \hat{\mathbf{s}})$. The emissivity ϵ in these boundary conditions is then an effective value for the internal emission from the entire semitransparent wall thickness. If the bounding surface is totally transparent (or simply an opening), then there is no emission from the boundary and $\epsilon = 0$. This type of boundary condition was discussed in some detail in Section 6.6.

8.8 RADIATION ENERGY DENSITY

A volume element inside an enclosure is irradiated from all directions and, at any instant in time t, contains a certain amount of radiative energy in the form of photons. Consider, for example, an element $dV = dA ds$ irradiated perpendicularly to dA with intensity $I_\eta(\hat{\mathbf{s}})$ as shown in Fig. 8-4. Therefore, per unit time radiative energy in the amount of $I_\eta(\hat{\mathbf{s}})\, d\Omega\, dA$ enters dV. From equation (8.1) we see that this energy remains inside dV for a duration of $dt = ds/c$, before exiting at the other side. Thus, due to irradiation from a single direction, the volume contains the amount of radiative energy $I_\eta(\hat{\mathbf{s}})\, d\Omega\, dA\, ds/c = I_\eta(\hat{\mathbf{s}})\, d\Omega\, dV/c$ at any instant in time. Adding the contributions from all possible directions, we find the total radiative energy stored within dV is $u_\eta\, dV$, where u_η is the *spectral radiation energy density*

$$u_\eta \equiv \frac{1}{c}\int_{4\pi} I_\eta(\hat{\mathbf{s}})\, d\Omega. \tag{8.43}$$

Integration over the spectrum gives the *total radiation energy density,*

$$u = \int_0^\infty u_\eta\, d\eta = \frac{1}{c}\int_{4\pi}\int_0^\infty I_\eta(\hat{\mathbf{s}})\, d\eta\, d\Omega = \frac{1}{c}\int_{4\pi} I(\hat{\mathbf{s}})\, d\Omega. \tag{8.44}$$

Although the radiation energy density is a very basic quantity akin to internal energy for energy stored within matter, it is not widely used by heat transfer engineers. Instead, it is common practice to employ the *incident radiation* G_η, which is related to the energy density through

$$G_\eta \equiv \int_{4\pi} I_\eta(\hat{\mathbf{s}})\, d\Omega = c\, u_\eta; \quad G = cu. \tag{8.45}$$

8.9 RADIATIVE HEAT FLUX

The spectral radiative heat flux onto a surface element has been expressed in terms of incident and outgoing intensity in equation (1.36) as

$$\mathbf{q}_\eta \cdot \hat{\mathbf{n}} = \int_{4\pi} I_\eta\, \hat{\mathbf{n}} \cdot \hat{\mathbf{s}}\, d\Omega. \tag{8.46}$$

This relationship also holds, of course, for a hypothetical (i.e., totally transmissive) surface element placed arbitrarily inside an enclosure. Removing the surface normal from equation (1.36), we obtain the definition for the *spectral, radiative heat flux vector* inside a participating medium. To obtain the *total radiative heat flux*, equation (8.46) needs to integrated over the spectrum, and

$$\mathbf{q} = \int_0^\infty \mathbf{q}_\eta\, d\eta = \int_0^\infty \int_{4\pi} I_\eta(\hat{\mathbf{s}})\, \hat{\mathbf{s}}\, d\Omega\, d\eta. \tag{8.47}$$

Depending on the coordinate system used, or the surface being described, the radiative heat flux vector may be separated into its coordinate components, for example q_x, q_y, and q_z (for a Cartesian coordinate system), or into components normal and tangential to a surface, and so on.

Example 8.3. Evaluate the total heat loss from an isothermal spherical medium bounded by vacuum, assuming that κ_η = const (i.e., does not vary with location, temperature or wavenumber).

Solution

Here we are dealing with a spherical coordinate system, and we are interested in the radial component of the radiative heat flux (the other two being equal to zero by symmetry). We saw in the last example that the intensity emanating from the sphere is

$$I_\eta(\tau_R, \theta) = I_{b\eta}\left(1 - e^{-2\tau_R\cos\theta}\right), \quad 0 \le \theta \le \frac{\pi}{2},$$

where θ is measured from the surface normal pointing away from the sphere (Fig. 8-6). Since the sphere is bounded by vacuum, there is no incoming radiation and

$$I_\eta(\tau_R, \theta) = 0, \quad \frac{\pi}{2} \le \theta \le \pi.$$