Inhomogeneous plane waves

$$\nabla\cdot\bar{D} = 0 \qquad \nabla\cdot\bar{H} = 0$$

$$\nabla\times\bar{E} = -\partial_t\bar{B}$$

$$\nabla\times\bar{H} = \partial_t\bar{D}$$

$$\begin{cases}\bar{D} = \varepsilon\bar{E}\\ \bar{B} = \mu\bar{H}\end{cases} \qquad \begin{cases}\varepsilon,\mu \text{ constants}\\ \text{harmonic time dependence } e^{-i\omega t}\end{cases}$$

$$\Rightarrow \begin{cases}\nabla\cdot\bar{E} = 0\\ \nabla\cdot\bar{H} = 0\\ \nabla\times\bar{E} = +i\omega\mu\bar{H}\\ \nabla\times\bar{H} = -i\omega\varepsilon\bar{E}\end{cases}$$

$$\begin{cases}\bar{E} = \bar{E}_0 e^{i\bar{k}\cdot\bar{r}}\\ \bar{H} = \bar{H}_0 e^{i\bar{k}\cdot\bar{r}}\end{cases}$$

$$\Rightarrow \begin{cases}\bar{k}\cdot\bar{E}_0 = 0\\ \bar{k}\cdot\bar{H}_0 = 0\\ \bar{k}\times\bar{E}_0 = \omega\mu\bar{H}_0\\ \bar{k}\times\bar{H}_0 = -\omega\varepsilon\bar{E}_0\end{cases}$$

$$\Rightarrow \quad \bar{k}\times(\bar{k}\times\bar{E}_0) = -\omega^2\varepsilon\mu\bar{E}_0$$

$$\Rightarrow \quad (\bar{k}\cdot\bar{E}_0)\bar{k} - (\bar{k}\cdot\bar{k})\bar{E}_0 = -(\bar{k}\cdot\bar{k})\bar{E}_0 = -\omega^2\varepsilon\mu\bar{E}_0$$

$$\Rightarrow \quad \bar{k}\cdot\bar{k} = \omega^2\varepsilon\mu = k_0^2c^2\varepsilon\mu = k_0^2\frac{\varepsilon\mu}{\varepsilon_0\mu_0} \equiv k_0^2\tilde{m}^2$$

($k_0 = \frac{2\pi}{\lambda}$, λ is wavelength in free space)

$$\begin{cases} \tilde{m} = m + i\varkappa & \text{(refractive index)} \\ \bar{k} = k_0 (N\hat{e} + iK\hat{f}), & |\hat{e}| = |\hat{f}| = 1 \end{cases}$$

$\hat{e}, \hat{f}$ define planes of constant phase and amplitude

$$\Rightarrow \begin{cases} \bar{k}\cdot\bar{k} = k_0^2 (N^2 - K^2) + i 2 k_0^2 N K\, \hat{e}\cdot\hat{f} \\ \tilde{m}^2 = m^2 - \varkappa^2 + i 2 m \varkappa \end{cases}$$

$$\Rightarrow \begin{cases} N^2 - K^2 = m^2 - \varkappa^2 \\ NK\, \hat{e}\cdot\hat{f} = m\varkappa \end{cases}$$

(N, K "apparent refractive index")

i) $\cos\alpha = \hat{e}\cdot\hat{f} \neq 0$:

$$K = \frac{m\varkappa}{N\cos\alpha} \Rightarrow N^2 - \frac{m^2\varkappa^2}{N^2\cos^2\alpha} = m^2 - \varkappa^2$$

$$\Rightarrow N^4\cos^2\alpha - (m^2 - \varkappa^2) N^2 \cos^2\alpha - m^2\varkappa^2 = 0$$

$$\Rightarrow N^2 = \frac{(m^2 - \varkappa^2)\cos^2\alpha \,(\pm)\, \sqrt{(m^2-\varkappa^2)^2\cos^4\alpha + 4m^2\varkappa^2\cos^2\alpha}}{2\cos^2\alpha}$$

$$= \frac{1}{2}\left[(m^2-\varkappa^2) + \sqrt{(m^2-\varkappa^2)^2 + \frac{4m^2\varkappa^2}{\cos^2\alpha}}\right]$$

propagation

$$\Rightarrow N = \sqrt{\frac{1}{2}\left[m^2 - \varkappa^2 + \sqrt{(m^2-\varkappa^2)^2 + \frac{4m^2\varkappa^2}{\cos^2\alpha}}\right]}$$

$$K^2 = N^2 - m^2 + \varkappa^2 = \frac{1}{2}\left[-(m^2-\varkappa^2) + \sqrt{(m^2-\varkappa^2)^2 + \frac{4m^2\varkappa^2}{\cos^2\alpha}}\right]$$

absorption

$$\Rightarrow K = \sqrt{\frac{1}{2}\left[-m^2 + \varkappa^2 + \sqrt{(m^2-\varkappa^2)^2 + \frac{4m^2\varkappa^2}{\cos^2\alpha}}\right]}$$

ii) $\cos\alpha = \hat{e}\cdot\hat{f} \to 0$:

$$\hat{e}\cdot\hat{f} = \frac{n\varkappa}{NK} = \frac{n\varkappa}{N\sqrt{N^2-n^2+\varkappa^2}}$$

$\hat{e}\cdot\hat{f} \to 0 \iff NK \to \infty$ with the condition $N^2 - K^2 = n^2 - \varkappa^2$

Planar interface: $\begin{cases} \text{normal vector } \hat{n} = \hat{e}_z \\ \text{plane } z=0 \end{cases}$

$$\bar{k}_i\cdot\bar{r}\Big|_{z=0} = \bar{k}_r\cdot\bar{r}\Big|_{z=0} = \bar{k}_t\cdot\bar{r}\Big|_{z=0} \qquad \bar{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\iff x(\bar{k}_i\cdot\hat{e}_x) + y(\bar{k}_i\cdot\hat{e}_y) = x(\bar{k}_r\cdot\hat{e}_x) + y(\bar{k}_r\cdot\hat{e}_y) = x(\bar{k}_t\cdot\hat{e}_x) + y(\bar{k}_t\cdot\hat{e}_y)$$

$$\iff \begin{cases} \bar{k}_i\cdot\hat{e}_x = \bar{k}_r\cdot\hat{e}_x = \bar{k}_t\cdot\hat{e}_x \\ \bar{k}_i\cdot\hat{e}_y = \bar{k}_r\cdot\hat{e}_y = \bar{k}_t\cdot\hat{e}_y \end{cases} \iff \bar{k}_i\times\hat{n} = \bar{k}_r\times\hat{n} = \bar{k}_t\times\hat{n}$$

$$\begin{cases} \bar{k}_i = k_0(N_i\hat{e}_i + iK_i\hat{f}_i) \\ \bar{k}_r = k_0(N_r\hat{e}_r + iK_r\hat{f}_r) \\ \bar{k}_t = k_0(N_t\hat{e}_t + iK_t\hat{f}_t) \end{cases}$$

$$\bar{k}_i\cdot\bar{k}_i = \bar{k}_r\cdot\bar{k}_r = k_0^2\tilde{n}_1^2$$

$\bar{k}_i\times\hat{n} = \bar{k}_r\times\hat{n}$

$$\Rightarrow \bar{k}_i\cdot\hat{e}_z = -\bar{k}_r\cdot\hat{e}_z$$

$$\Rightarrow \bar{k}_i\cdot\hat{n} = -\bar{k}_r\cdot\hat{n}$$

$\Rightarrow$

$$\begin{cases} N_i\, \hat{e}_i \cdot \hat{e}_x = N_r\, \hat{e}_r \cdot \hat{e}_x = N_t\, \hat{e}_t \cdot \hat{e}_x \\ K_i\, \hat{f}_i \cdot \hat{e}_x = K_r\, \hat{f}_r \cdot \hat{e}_x = K_t\, \hat{f}_t \cdot \hat{e}_x \\ N_i\, \hat{e}_i \cdot \hat{e}_y = N_r\, \hat{e}_r \cdot \hat{e}_y = N_t\, \hat{e}_t \cdot \hat{e}_y \\ K_i\, \hat{f}_i \cdot \hat{e}_y = K_r\, \hat{f}_r \cdot \hat{e}_y = K_t\, \hat{f}_t \cdot \hat{e}_y \\ \\ N_i\, \hat{e}_i \cdot \hat{e}_z = -N_r\, \hat{e}_r \cdot \hat{e}_z \\ K_i\, \hat{f}_i \cdot \hat{e}_z = -K_r\, \hat{f}_r \cdot \hat{e}_z \end{cases}$$

Assume, first, that $\hat{e}_i \cdot \hat{e}_y = 0$:

$$N_r\, \hat{e}_r \cdot \hat{e}_y = N_i\, \hat{e}_i \cdot \hat{e}_y = 0 \overset{N_r \neq 0}{\Longrightarrow} \hat{e}_r \cdot \hat{e}_y = 0$$

$$\left.\begin{aligned} N_i\, \hat{e}_i \cdot \hat{e}_x &= N_r\, \hat{e}_r \cdot \hat{e}_x \\ N_i\, \hat{e}_i \cdot \hat{e}_z &= -N_r\, \hat{e}_r \cdot \hat{e}_z \end{aligned}\right\} \Rightarrow \begin{cases} N_i = N_r \\ \hat{e}_i \cdot \hat{e}_x = \hat{e}_r \cdot \hat{e}_x \\ \hat{e}_i \cdot \hat{e}_z = -\hat{e}_r \cdot \hat{e}_z \end{cases}$$

Assume, second, that $\hat{f}_i \cdot \hat{e}_y = 0$: as above,

$$\begin{cases} K_i = K_r \\ \hat{f}_i \cdot \hat{e}_x = \hat{f}_r \cdot \hat{e}_x \\ \hat{f}_i \cdot \hat{e}_z = -\hat{f}_r \cdot \hat{e}_z \end{cases}$$

summary so far:
$$\begin{cases} N_r = N_i \;,\; K_r = K_i \\ \hat{e}_r \cdot \hat{f}_r = \hat{e}_i \cdot \hat{f}_i \\ \hat{e}_r = \hat{e}_i - 2(\hat{e}_i \cdot \hat{m})\, \hat{m} \\ \hat{f}_r = \hat{f}_i - 2(\hat{f}_i \cdot \hat{m})\, \hat{m} \end{cases}$$

Snell: assume $\hat{e}_i \cdot \hat{e}_y = 0$:

$$N_i \, \hat{e}_i \cdot \hat{e}_x = N_t \, \hat{e}_t \cdot \hat{e}_x$$

$$\Leftrightarrow \quad N_i \sin\theta_i = N_t \sin\theta_t$$

Assume $\hat{f}_i \cdot \hat{e}_y = 0$:

$$K_i \, \hat{f}_i \cdot \hat{e}_x = K_t \, \hat{f}_t \cdot \hat{e}_x$$

$$\Leftrightarrow \quad K_i \sin\psi_i = K_t \sin\psi_t$$

Return to the geometry shown in Fig. 1.

$$\begin{cases} \hat{e}_t = \sin\theta_t \, \hat{e}_x - \cos\theta_t \, \hat{e}_z \\ \hat{f}_t = \sin\psi_t \left(\cos\varphi_i \, \hat{e}_x + \sin\varphi_i \, \hat{e}_y \right) \\ \qquad - \cos\psi_t \, \hat{e}_z \end{cases}$$

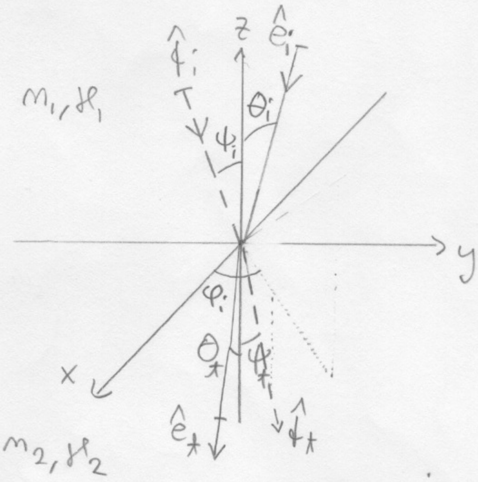

Fig. 1

$$N_t^2 - K_t^2 = n_2^2 - \varkappa_2^2$$

$$\Rightarrow K_t^2 = N_t^2 - n_2^2 + \varkappa_2^2$$

$$\Rightarrow \sin\psi_t = \frac{K_i \sin\psi_i}{\sqrt{N_t^2 - n_2^2 + \varkappa_2^2}} \equiv \frac{K_s}{\sqrt{N_t^2 - n_2^2 + \varkappa_2^2}}$$

$$\sin\theta_t = \frac{N_i \sin\theta_i}{N_t} \equiv \frac{N_s}{N_t}$$

$$\hat{e}_t \cdot \hat{f}_t = \sin\theta_t \sin\psi_t \cos\varphi_i + \cos\theta_t \cos\psi_t$$

$$= \frac{N_s K_s \cos\varphi_i}{N_t \sqrt{N_t^2 - n_2^2 + \varkappa_2^2}} + \sqrt{1 - \frac{N_s^2}{N_t^2}} \sqrt{1 - \frac{K_s^2}{N_t^2 - n_2^2 + \varkappa_2^2}}$$

$$N_t K_t \, \hat{e}_t \cdot \hat{f}_t = m_2 \varkappa_2$$

$$= N_t \sqrt{N_t^2 - m_2^2 + \varkappa_2^2} \cdot \left[\frac{N_s K_s \cos\varphi_i}{N_t \sqrt{N_t^2 - m_2^2 + \varkappa_2^2}} + \sqrt{1 - \frac{N_s^2}{N_t^2}} \sqrt{1 - \frac{K_s^2}{N_t^2 - m_2^2 + \varkappa_2^2}} \right]$$

$$= N_s K_s \cos\varphi_i + \sqrt{N_t^2 - N_s^2} \sqrt{N_t^2 - m_2^2 + \varkappa_2^2 - K_s^2}$$

$$\Rightarrow \; m_2^2 \varkappa_2^2 + N_s^2 K_s^2 \cos^2\varphi_i - 2 m_2 \varkappa_2 N_s K_s \cos\varphi_i$$

$$= (N_t^2 - N_s^2)(N_t^2 - m_2^2 + \varkappa_2^2 - K_s^2)$$

$$\Rightarrow \; N_t^4 - N_t^2 (m_2^2 - \varkappa_2^2 + K_s^2 + N_s^2) + N_s^2 (m_2^2 - \varkappa_2^2 + K_s^2)$$

$$- m_2^2 \varkappa_2^2 - N_s^2 K_s^2 \cos^2\varphi_i + 2 m_2 \varkappa_2 N_s K_s \cos\varphi_i = 0$$

$$\Rightarrow \; N_t^4 - N_t^2 (N_s^2 + K_s^2 + m_2^2 - \varkappa_2^2) + N_s^2 K_s^2 + (m_2^2 - \varkappa_2^2) N_s^2$$

$$- (m_2 \varkappa_2 - N_s K_s \cos\varphi_i)^2 = 0$$

This gives N_t, whereafter everything fully follows.