1 Electromagnetic field by a localized source

Consider the electromagnetic fields caused by time-dependent charge and current densities localized in a constrained region of space. Here we will mainly study the fields by an electric dipole. Later, the analysis is extended to the full multipole expansion.
Assume harmonic time dependence $e^{-i\omega t}$—arbitrary time dependences can be dealt with using Fourier analysis of their components. The charge density ρ and current density $\mathbf{j}$ are

$$\begin{aligned}\rho(\mathbf{x},t) &= \rho(\mathbf{x})e^{-i\omega t}\\ \mathbf{j}(\mathbf{x},t) &= \mathbf{j}(\mathbf{x})e^{-i\omega t}\end{aligned}$$

and the physical quantities correspond to the real parts of the complex quantities. The electromagnetic potentials and fields are also time-harmonic and the sources are assumed to be located in an otherwise empty space.

Let us start from the vector potential $\mathbf{A}$ in Lorentz gauge,

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}' \int dt' \frac{\mathbf{j}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta(t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c} - t) \tag{1}$$

and, by writing $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \qquad k = \frac{\omega}{c} \tag{2}$$

The magnetic field is, according to definitions, $\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}$ and, outside the source region, the electric field equals $\mathbf{E} = \frac{i\zeta_0}{k}\nabla \times \mathbf{H}$, where $\zeta_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.

When the current density $\mathbf{j}(\mathbf{x}')$ is given, the electromagnetic field can be calculated from the integral above, at least in principle. Let us study the case where the source region (size d) is much smaller than the wavelength: $d \ll \lambda = 2\pi c/\omega$. We can distinguish three regimes of interest:

(i) Near zone (static regime): $d \ll r \ll \lambda$
(ii) Intermediate zone (induction regime): $d \ll r \sim \lambda$
(iii) Far zone (radiation regime): $d \ll \lambda \ll r$

In the near zone (i) $kr \ll 1$ and the exponential part of the integrand for the vector potential can be set to unity, and the inverse distance can be presented using series of spherical harmonics Y_{lm}:

$$\lim_{kr\to 0} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \sum_{l,m} \frac{4\pi}{2l+1} \frac{\mathrm{Y}_{lm}(\theta,\varphi)}{r^{l+1}} \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}')(r')^l \mathrm{Y}^*_{lm}(\theta',\varphi') \tag{3}$$

We can see that the near fields vary harmonically in time but are static in their character: no wave solution follows for the spatial dependence. Above, we have made use of the relation

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \mathrm{Y}^*_{lm}(\theta',\varphi') \mathrm{Y}_{lm}(\theta,\varphi) \tag{4}$$

In the far zone (iii), $kr \gg 1$ and the exponential part of the vector potential varies strongly and dictates the character of the vector potential. We can approximate

$$|\mathbf{x} - \mathbf{x}'| \approx r - \hat{\mathbf{n}} \cdot \mathbf{x}', \qquad \hat{\mathbf{n}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{r} \tag{5}$$

When the leading term is desired in kr, the inverse distance can be replaced by r. The vector potential is of the form

$$\lim_{kr \to \infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}') e^{-ik\hat{\mathbf{n}} \cdot \mathbf{x}'} \tag{6}$$

Therefore, the vector potential behaves like an outgoing spherical wave (e^{ikr}/r) with angular dependence. It can be shown that the electromagnetic field is also of the form of a spherical wave and thus is a radiation field. (Note that this part of the analysis is valid for localized source regions of arbitrary size.)

Now that $kd \ll 1$ the integral can further be developed into series:

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(\hat{\mathbf{n}}\cdot\mathbf{x}')^n \tag{7}$$

where the magnitude for the nth term is $(1/n!)\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(k\hat{\mathbf{n}}\cdot\mathbf{x}')^n$ and thus becomes rapidly smaller with increasing n. In this case, the main contribution to radiation comes from the first non-vanishing term in the sum.

In the intermediate zone (ii), all powers of kr need to be accounted for, and no simple limits can be taken. The vector potential is then written with the help of the expansion for the exact Green's function in the form

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik\sum_{l,m} h_l^{(1)}(kr)\mathrm{Y}_{lm}(\theta,\varphi)\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')j_l(kr')\mathrm{Y}^*_{lm}(\theta',\varphi') \tag{8}$$

where we have made use of the expansion

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} \mathrm{Y}_{lm}^*(\theta',\varphi')\mathrm{Y}_{lm}(\theta,\varphi) \tag{9}$$

where $r_< = \min(r, r')$, $r_> = \max(r, r')$, and j_l and $h_l^{(1)}$ are the spherical Bessel and Hankel functions.

Again when $kd \ll 1$, the j_l-functions can be approximated and the result is of the same form as the near zone result, when the following replacement is carried out:

$$\frac{1}{r^{l+1}} \to \frac{e^{ikr}}{r^{l+1}}[1 + a_1(ikr) + a_2(ikr)^2 + \ldots + a_l(ikr)^l] \tag{10}$$

The coefficients a_i derive from the explicit expansions of the Hankel functions. This end result allows us to see the transition from the near-zone $kr \ll 1$ static field to the far-zone $kr \gg 1$ radiation field.

2 Electromagnetic field of an electric dipole

If only the first term in kd is kept in the expansion of the vector potential, one obtains

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}') \tag{11}$$

which holds everywhere outside the source region (this follows from the intermediate-zone results above). With the help of partial integration,

$$\int d^3\mathbf{x}'\mathbf{j} = -\int d^3\mathbf{x}'\mathbf{x}'(\nabla\cdot\mathbf{j}) = -i\omega\int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}') \tag{12}$$

where the substitution term disappears (the source region is constrained) and, according to the continuity equation, $i\omega\rho(\mathbf{x}') = \nabla\cdot\mathbf{j}(\mathbf{x}')$. The vector potential is thus

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}, \tag{13}$$

where $\mathbf{p}$ is the electric dipole moment $\mathbf{p} = \int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}')$.

The electromagnetic fields are

$$
\begin{aligned}
\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r}(1 - \frac{1}{ikr}) \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\Big(k^2(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}\frac{e^{ikr}}{r} + (3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p})(\frac{1}{r^2} - \frac{ik}{r})\frac{e^{ikr}}{r}\Big)
\end{aligned}
$$

We note that the magnetic field is always transverse but that the electric field has both longitudinal and transverse components.

In the far zone,

$$
\begin{aligned}
\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r} \\
\mathbf{E} &= \zeta_0 \mathbf{H} \times \hat{\mathbf{n}}
\end{aligned}
$$

which shows the typical form of a spherical wave.

In the near zone,

$$
\begin{aligned}
\mathbf{H} &= \frac{i\omega}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{1}{r^2} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}(3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p})\frac{1}{r^3}
\end{aligned}
$$

The electric field is, except for the harmonic time dependence, that of a static electric dipole. The field $\zeta_0\mathbf{H}$ is smaller, by a factor of kr, than the field $\mathbf{E}$ so, in the near zone, the field is electric in its nature. In the static limit $k \to 0$, the magnetic field disappears and the near zone extends to infinity.

The power radiated by the vibrating dipole moment $\mathbf{p}$ as per solid angle is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2}Re(r^2\hat{\mathbf{n}}\cdot\mathbf{E}\times\mathbf{H}^*)\\ &= \frac{c^2\zeta_0}{32\pi^2}k^4|(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}|^2,\end{aligned}$$

where $\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}$ gives the polarization state. If all components of $\mathbf{p}$ are in the same phase,

$$\frac{dP}{d\Omega} = \frac{c^2\zeta_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta \tag{14}$$

which is the typical radiation pattern of an electric dipole (θ is here measured from the direction of $\mathbf{p}$). Independently of the phases of the components for $\mathbf{p}$:n, the total radiated power is

$$P = \frac{c^2\zeta_0 k^4}{12\pi}|\mathbf{p}|^2 \tag{15}$$

3 Scattering by small spherical particles in the electric dipole approximation

Light scattering by particles clearly smaller than the wavelength can be studied in the approximation, where the incident field induces an electric dipole moment to the particle. The dipole fluctuates in a certain phase with the incident field and thus scatters radiation in directions differing from the propagation direction of the incident field. In this case, the dipole moments can be computed using electrostatic methods.

Assume that a monochromatic plane wave is incident on a small scatterer located in free space. Let the propagation direction and polarization vector of the incident field be $\hat{\mathbf{n}}_0$ and $\hat{\epsilon}_0$:

$$\begin{aligned}\mathbf{E}_i &= \hat{\epsilon}_0 E_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}} \\ \mathbf{H}_i &= \hat{\mathbf{n}}_0 \times \mathbf{E}_i/\zeta_0\end{aligned}$$

where $k = \omega/c$ and the time dependence has been assumed harmonic ($e^{-i\omega t}$). These fields induce a dipole momentn $\mathbf{p}$ in the small particle and the particle radiates energy in (almost) all directions. In the far zone, the scattered fields are of the form

$$\begin{aligned}\mathbf{E}_s &= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}((\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}) \\ \mathbf{H}_s &= \hat{\mathbf{n}}\times\mathbf{E}_s/\zeta_0\end{aligned}$$

where $\hat{\mathbf{n}}$ is the dirction of the observer and r the distance from the scatterer.

The power scattered in direction $\hat{\mathbf{n}}$ with polarization $\hat{\epsilon}$ per unit solid angle divided by the incident flux density is the so-called differential cross section

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \hat{\epsilon}, \hat{\mathbf{n}}_0, \hat{\epsilon}_0) = \frac{r^2 \frac{1}{2\zeta_0}|\hat{\epsilon}^* \cdot \mathbf{E}_s|^2}{\frac{1}{2\zeta_0}|\hat{\epsilon}_0^* \cdot \mathbf{E}_i|^2} \tag{16}$$

where the complex conjugation of the polarization vectors is important for proper treatment of circular polarization. Furthermore,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \hat{\epsilon}, \hat{\mathbf{n}}_0, \hat{\epsilon}_0) = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\hat{\epsilon}^* \cdot \mathbf{p}|^2, \tag{17}$$

where the $\hat{\mathbf{n}}_0, \hat{\epsilon}_0$ -dependence is implicit in $\mathbf{p}$. We can see that the differential and total cross sections of the dipole scatterer are both proportional to k^4 and λ^{-4} (Rayleigh's law).
Assume that the scatterer is a small sphere (radius a) with the relative permittivity $\epsilon_r = \epsilon/\epsilon_0$. According to electrostatics, the dipole moment of the sphere is

$$\mathbf{p} = 4\pi\epsilon_0\Big(\frac{\epsilon_r - 1}{\epsilon_r + 2}\Big)a^3\mathbf{E}_i \tag{18}$$

so that

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \tag{19}$$

The polarization dependence is purely that of electric dipole scattering. The scattered radiation is polarized in the plane defined by the dipole moment $\hat{\epsilon}_0$ and the vector $\hat{\mathbf{n}}$.

For unpolarized incident radiation, the differential cross sections in different polarization states of the scattered field are

$$\begin{aligned}\frac{d\sigma_{\parallel}}{d\Omega} &= \frac{k^4a^6}{2}|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \cos^2\theta \\ \frac{d\sigma_{\perp}}{d\Omega} &= \frac{k^4a^6}{2}|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2\end{aligned}$$

where θ is now the scattering angle.

The degree of polarization is

$$P(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta} = -\frac{S_{21}(\theta)}{S_{11}(\theta)} \tag{20}$$

and the differential cross section summed over the polarization states of the scattered field is

$$\frac{d\sigma}{d\Omega} = k^4a^6|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2\frac{1}{2}(1+\cos^2\theta) \propto S_{11}(\theta) \tag{21}$$

where $S_{11}(\theta)$ and $S_{21}(\theta)$ are elements of the scattering matrix. The total scattering cross section is

$$\sigma = \int_{(4\pi)} \frac{d\sigma}{d\Omega}d\Omega = \frac{8\pi}{3}k^4a^6|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \tag{22}$$

The scattered radiation is 100% positively polarized at the scattering angle $\theta = 90°$. It was the polarization characteristics of the blue sky that got Rayleigh interested in scattering by small particles.

4 Scattering by an ensemble of small particles in the dipole approximation

Consider an ensemble of numerous small particles which have fixed locations in space and the scattering amplitudes of which can be expressed in the dipole approximation. Assume presently that the particles do not interact with each other. Since the induced dipole moments are proportional to the incident field, the moments will depend on the phase factor $e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}_j}$, where $\mathbf{x}_j$ is the location of the jth scatterer. When the observer is located far away from the scatterer, the exponential part of the Green's function results in an additional phase factor for the jth scatterer, $e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}_j}$. In the dipole approximation, the ensemble of particles scatters as follows:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\sum_j \hat{\epsilon}^* \cdot \mathbf{p}_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|, \qquad \mathbf{q} = k(\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}) \tag{23}$$

Except for the forward-scattering direction ($\mathbf{q} = 0$), scattering will depend sensitively on how the small particles are located in space.

Assume now that all the particles are identical so that $\mathbf{p} = \mathbf{p}_j$ for all j and

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\hat{\epsilon}^* \cdot \mathbf{p}|^2 F(\mathbf{q}), \tag{24}$$

where $F(\mathbf{q})$ is the so-called structure factor,

$$F(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2 = \sum_{j,j'} e^{i\mathbf{q}\cdot(\mathbf{x}_j - \mathbf{x}_{j'})} \tag{25}$$

If the small particles are located in random positions, the terms $j \neq j'$ will cause a negligible contribution to the sum. Only the terms $j = j'$ are significant and $F(\mathbf{q}) = N$, where N is the number of scatterers. In this case, the total scattering is the incoherent superposition of the individual contributions.

If the small particles are regularly located in space, the structure factor disappears almost everywhere except for the proximity of the forward-scattering direction. Therefor, large regular arrays of small particles do not scatter (for example, individual transparent crystals of rock salt and quartz).

Consider scatterers located in a regular cubic lattice. The structure factor can be calculated analytically, since

$$\begin{aligned}
\left|\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}\right|^2 &= \left|\sum_{j_1=0}^{N_1-1} e^{iq_1 j_1 a} \sum_{j_2=0}^{N_2-1} e^{iq_2 j_2 a} \sum_{j_3=0}^{N_3-1} e^{iq_3 j_3 a}\right|^2 \\
&= \left|(\frac{1-e^{iq_1 N_1 a}}{1-e^{iq_1 a}})(\frac{1-e^{iq_2 N_2 a}}{1-e^{iq_2 a}})(\frac{1-e^{iq_3 N_3 a}}{1-e^{iq_3 a}})\right|^2 \\
&= N^2[(\frac{\sin^2 \frac{1}{2}N_1 q_1 a}{N_1^2 \sin^2 \frac{1}{2}q_1 a})(\frac{\sin^2 \frac{1}{2}N_2 q_2 a}{N_2^2 \sin^2 \frac{1}{2}q_2 a})(\frac{\sin^2 \frac{1}{2}N_3 q_3 a}{N_3^2 \sin^2 \frac{1}{2}q_3 a})], \qquad (26)
\end{aligned}$$

where a is the lattice constant (distance between the lattice points) and where N_1, N_2, and N_3 are the numbers of lattice points in each direction hilapisteiden so that the total number of lattice points equals $N = N_1 N_2 N_3$ (this was utilized to obtain the final result above). The components of the vector $\mathbf{q}$ in each direction are q_1, q_2, and q_3.

We note that, at short wavelengths ($ka \geq \pi$), the structure factor has peaks when the Bragg condition is fulfilled: $q_i a = 0, 2\pi, 4\pi \ldots$, where $i = 1, 2, 3 \ldots$. This is typical in X-ray diffraction. At long wavelengths, only the peak $q_i a = 0$ is relevant, since $\max|q_i a| = 2ka << 1$. In this limit, the structure factor is a product of three $\sin^2 x_i / x_i^2$-type factors ($x_i = \frac{1}{2}N_i q_i a$), and scattering is confined to the region $q_i \leq 2\pi/N_i a$, corresponding to the angles λ/L, where L is the size of the lattice.

5 Volume integral equation for scattering

In a uniform medium, the electromagnetic wave propagates undisturbed and wiythout changing its direction of propagation. If there are fluctuations in the medium depending on space or time, the wave is scattered, and part of its energy is redirected. If the fluctuations in the medium are small, scattering is weak and one may utilize methods based on perturbation series.

Consider a uniform isotropic medium with electric permittivity ϵ_m and magnetic permeability equal to the permeability of vacuum, $\mu_m = \mu_0$. Fluctuations in the medium result in $\mathbf{D} \neq \epsilon_m \mathbf{E}$ in some constrained region. Let us start from Maxwell's equations in sourceless space:

$$\nabla \cdot \mathbf{B} = 0 \quad , \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{D} = 0 \quad , \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

Then

$$\nabla \times (\mathbf{D} - \mathbf{D} + \epsilon_m \mathbf{E}) = -\epsilon_m \frac{\partial \mathbf{B}}{\partial t}, \tag{27}$$

so that

$$\nabla \times (\nabla \times \mathbf{D}) = \nabla \times [\nabla \times (\mathbf{D} - \epsilon_m \mathbf{E})] - \epsilon_m \frac{\partial}{\partial t} \mu_0 \nabla \times \mathbf{H}. \tag{28}$$

Moreover, after further manipulation,

$$-\nabla^2\mathbf{D} = \nabla \times \nabla \times (\mathbf{D} - \epsilon_m \mathbf{E}) - \epsilon_m \mu_0 \frac{\partial^2}{\partial t^2}\mathbf{D}, \tag{29}$$

which can be written in the form

$$\nabla^2\mathbf{D} - \epsilon_m \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = -\nabla \times \nabla \times (\mathbf{D} - \epsilon_m \mathbf{E}), \tag{30}$$

that is the exact wave equation for the **D**-field derived without any approximations. Later, the right-hand side of the equation is treated as a small perturbation.

If the right-hand side of the equation were known, the solution of the wave equation could be written an a suitable integral of it. Although the right-hand side is usually unknown, the integral form is useful, since it allows the derivation of important approximations.

Assume again harmonic time dependence $e^{-i\omega t}$, in which case

$$\begin{aligned} (\nabla^2 + k^2)\mathbf{D} &= -\nabla \times \nabla \times (\mathbf{D} - \epsilon_m \mathbf{E}) \\ k^2 &= \mu_0 \epsilon_m \omega^2, \end{aligned} \tag{31}$$

where ϵ_m is the permittivity corresponding to the angular frequency ω. The solution of the undisturbed problem is obtained by setting the right-hand side equal to zero; denote this solution by $\mathbf{D}^{(0)}$. The formal complete solution is then, in an exact way,

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}^{(0)}(\mathbf{x}) + \frac{1}{4\pi} \int d^3\mathbf{x}' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \nabla' \times \nabla' \times (\mathbf{D}(\mathbf{x}') - \epsilon_m \mathbf{E}(\mathbf{x}')) \tag{32}$$

In a scattering problem, the integral on the right-hand side is taken over a constrained region of space and $\mathbf{D}^{(0)}$ describes the incident field. Then, in the far zone,

$$\mathbf{D}(\mathbf{x}) \to \mathbf{D}^{(0)}(\mathbf{x}) + \frac{e^{ikr}}{r}\mathbf{A}_s, \tag{33}$$

where the scattering amplitude $\mathbf{A}_s$ is

$$\mathbf{A}_s = \frac{1}{4\pi}\int d^3\mathbf{x}' e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'}\nabla' \times \nabla' \times (\mathbf{D}(\mathbf{x}') - \epsilon_m \mathbf{E}(\mathbf{x}')). \tag{34}$$

After some partial integration and noticing that the substitution terms diappear, one obtains

$$\mathbf{A}_s = \frac{k^2}{4\pi}\int d^3\mathbf{x}' e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'}\{[\hat{\mathbf{n}} \times (\mathbf{D}(\mathbf{x}') - \epsilon_m \mathbf{E}(\mathbf{x}'))] \times \hat{\mathbf{n}}\}. \tag{35}$$

The vector characteristics of the integrand can be compared with the field scattered by an electric dipole: the contribution from the term $\mathbf{D} - \epsilon_m \mathbf{E}$ is precisely the field of the electric dipole so that the scattering amplitude is a vector sum from all induced electric dipole moments. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{|\hat{\epsilon}^* \cdot \mathbf{A}_s|}{|\mathbf{D}^{(0)}|^2}, \tag{36}$$

where $\hat{\epsilon}$ is the polarization vector of scattered radiation. In principle, we have solved the scattering problem for an arbitrary scatterer in an exact way. The caveat is that we do not know the field inside the scatterer.

6 Rayleigh-Gans or Born approximation

The integral equation derived above allows for a solution via perturbation series, where the internal field of the scatterer is first approximated by the incident field. What follows is the so-called Rayleigh-Gans approximation or the first Born approximation based on the corresponding integral equation in quantum mechanics.

Consider purely spatial fluctuations from an otherwise uniform medium and assume, in addition, that the fluctuations are linear, $\mathbf{D}(\mathbf{x}) = [\epsilon_m + \delta\epsilon(\mathbf{x})]\mathbf{E}(\mathbf{x})$, where $\delta\epsilon(\mathbf{x})$ is small compared to ϵ_m. The difference $\mathbf{D} - \epsilon_m\mathbf{E}$ showing up in the integral equation is proportional to $\delta\epsilon(\mathbf{x})$. In the lowest order,

$$\mathbf{D} - \epsilon_m\mathbf{E} \approx \frac{\delta\epsilon(\mathbf{x})}{\epsilon_m}\mathbf{D}^{(0)}. \tag{37}$$

Let the incident field be a plane wave so that $\mathbf{D}^{(0)}(\mathbf{x}) = \hat{\epsilon}_0 D_0 e^{ik\hat{n}_0\cdot\mathbf{x}}$. Then

$$\begin{aligned} \frac{\hat{\epsilon}^* \cdot \mathbf{A}_s^{(0)}}{D_0} &= \frac{k^2}{4\pi}\int d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}}\hat{\epsilon}^* \cdot \hat{\epsilon}_0 \frac{\delta\epsilon(\mathbf{x})}{\epsilon_m} \\ \mathbf{q} &= k(\hat{n}_0 - \hat{n}), \end{aligned} \tag{38}$$

the square of which, in absolute terms, gives the differential cross section. If the wavelength is much larger than the size of the region where $\delta\epsilon \neq 0$, the exponent in the integral can be set to unity. This results in the dipole approximation that was treated before for a small spherical particle.

Let us study the situation where the particle continues to be spherical and is located in free space. Thus, $\delta\epsilon \neq 0$ inside a sphere of radius a. We obtain

$$\begin{aligned}
\frac{\hat{\epsilon}^* \cdot \mathbf{A}_s^{(1)}}{D_0} &= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}'} \\
&= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \int_0^a dr' r'^2 e^{iqr'\cos\theta'} \\
&= \frac{k^2}{2}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int_0^a dr' r'^2 /_{-1}^{1} \frac{1}{iqr'} e^{iqr'\mu'}, \qquad \mu' = \cos\theta' \\
&= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\frac{1}{iq}\Big\{ /_0^a r' \frac{1}{iq}(e^{iqr'} + e^{-iqr'}) - \int_0^a dr' \frac{1}{iq}(e^{iqr'} + e^{-iqr'}) \Big\} \\
&= k^2 \frac{\delta\epsilon}{\epsilon_0}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\Big(\frac{\sin qa - qa\cos qa}{q^3}\Big), \qquad q = |\mathbf{q}| = \sqrt{2}k\sqrt{1 - \hat{n}\cdot\hat{n}_0}.
\end{aligned}$$

In the limit $a \to 0$, the term inside the parentheses approaches $a^3/3$ so that, for scatterers much smaller than the wavelength or for q approaching zero,

$$\lim_{q\to 0}(\frac{d\sigma}{d\Omega})_{R-G} = k^4 a^6 |\frac{\delta\epsilon}{3\epsilon_0}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2. \tag{39}$$

This is in agreement with the long-wavelength limit studied earlier. The integral $\int_S d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}'}$ is commonly called the form factor.

7 Why is the sky blue?

In the present context, we can consider the blueness of the sky and redness of the sunrises and sunsets. Assume that the atmosphere is composed of individual molecules with locations $\mathbf{x}_j$ and that have the dipole moment $\mathbf{p}_j = \hat{\epsilon}_0 \gamma_{\mathrm{mol}} \mathbf{E}(\mathbf{x}_j)$, where γ_{mol} is the molecular polarizability. Then, the fluctuations of the electric permittivity can be described with the sum

$$\delta\epsilon(\mathbf{x}) = \epsilon_0 \sum_j \gamma_{mol} \delta(\mathbf{x} - \mathbf{x}_j) \tag{40}$$

The differential scattering cross section is of the form

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} |\gamma_{mol}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 F(\mathbf{q}), \tag{41}$$

where F is the structure factor treated before. For randomly distributed scatterers, $F(\mathbf{q})$ is directly the number of the molecules. For low-density gas, the relative permittivity is $\epsilon_r = \epsilon/\epsilon_0 = 1 + N\gamma_{\mathrm{mol}}$, where N is now the number of molecules in unit volume. The total scattering cross section as per molecule is

$$\sigma_s \approx \frac{k^4}{6\pi N^2} |\epsilon_r - 1|^2 \cong \frac{2k^4}{3\pi N^2} |m - 1|^2, \tag{42}$$

where m is the refractive index and $|m - 1| \ll 1$.

When the radiation propagates a distance dx in the atmoshpere, the relative change in its intensity is $N\sigma dx$ and $I(x) = I_0 e^{-k_e x}$, where k_e is the so-called extinction coefficient:

$$k_e = N\sigma_s \cong \frac{2k^4}{3\pi N}|m-1|^2 \tag{43}$$

This is called Rayleigh scattering that is incoherent scattering by gas molecules and other dipole scatterers, where each scatterer scatters radiation based on Rayleigh's $1/\lambda^4$ -law.

The $1/\lambda^4$ -law means that blue light is scattered much more efficiently than red light. In practice, this shows up so that blue color predominates when looking in directions other than the light source whereas, in the direction of the light source, red color predominates.

For visible light, $\lambda = 0.41 - 0.65\mu$m and, under normal conditions, $m - 1 \approx 2.78 \cdot 10^{-4}$. When $N = 2.69 \cdot 10^{19}$ molecules/cm^3, we obtain for the mean free path $1/k_e$ =30, 77, and 188 km at wavelengths 0.41 μm (violet), 0.52 μm (green), and 0.65 μm (red), respectively.

Polarization reaches its maximum of 75 % at the wavelength of 0.55 μm. The deviation from 100 % derives from multiple scattering (6 %), the anisotropy of the molecules (6 %), reflection from the surface (5 %, in particular, for green light in the case of vegetation), and aerosols (8 %).