

# Electromagnetic scattering 1: Finite-element method (FEM)

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# Finite-element method (FEM)

## Strengths:

- Solid mathematical background
- Applicability
- Sparse matrix
- Simple implementation

## Weaknesses:

- Matrix conditioning  $\rightarrow$  preconditioning needed
- PML is needed for open region problems

# FDTD literature

## Introduction to FDTD, FEM, IEM

- Sheng Xin-Qing, Song Wei, Essentials of computational electromagnetics, IEEE,Wiley, 2012.

## Some FEM books

- Jin, J., The Finite Element Method in Electromagnetics, John Wiley & Sons, Inc., New York, 2002.
- Monk, P., Finite Element Methods for Maxwell's Equations, Oxford Science Publications, Clarendon Press, Oxford, 2003.

# Maxwell's equation

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)$$

Ampères law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \sigma \mathbf{E} + \mathbf{J} \quad (2)$$

Gauss's law for electric field

$$\nabla \cdot \mathbf{D} = \rho \quad (3)$$

Gauss's law for magnetic field

$$\nabla \cdot \mathbf{B} = 0 \quad (4)$$

Constitutive relations:

$$\mathbf{D} = \epsilon * \mathbf{E} \quad (5)$$

$$\mathbf{B} = \mu * \mathbf{H} \quad (6)$$

$\epsilon$  electric permittivity

$\mu$  magnetic permeability

# Vector wave equation

Time-harmonic case ( $\exp(-i\omega t)$ )

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \quad (7)$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} + \mathbf{J}^s \quad (8)$$

Radiation condition

$$\lim_{|\mathbf{r}| \rightarrow \infty} |\mathbf{r}| \left( \eta \mathbf{H}(\mathbf{r}) \times \frac{\mathbf{r}}{|\mathbf{r}|} - \mathbf{E}(\mathbf{r}) \right) = 0 \quad (9)$$

Vector wave equation

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \epsilon\omega^2 \mathbf{E} = i\omega \mathbf{J}^s \quad (10)$$

## Function spaces

Spaces for potentials:

$$H^1(\Omega) := \{p \in L^2(\Omega), \nabla p \in L^2(\Omega)^3\} \quad (11)$$

$$H_0^1(\Omega) := \{p \in L^2(\Omega), \nabla p \in L^2(\Omega)^3, p|_{\partial\Omega} = 0\} \quad (12)$$

Spaces for fields:

$$H_{curl}(\Omega) := \{\mathbf{f} \in L^2(\Omega)^3, \nabla \times \mathbf{f} \in L^2(\Omega)^3\} \quad (13)$$

$$H_{0,curl}(\Omega) := \{\mathbf{f} \in L^2(\Omega)^3, \nabla \times \mathbf{f} \in L^2(\Omega)^3, \mathbf{n} \times \mathbf{f}|_{\partial\Omega} = 0\} \quad (14)$$

Spaces for flux densities:

$$H_{div}(\Omega) := \{\mathbf{g} \in L^2(\Omega)^3, \nabla \cdot \mathbf{g} \in L^2(\Omega)\} \quad (15)$$

$$H_{0,div}(\Omega) := \{\mathbf{g} \in L^2(\Omega)^3, \nabla \cdot \mathbf{g} \in L^2(\Omega), \mathbf{n} \cdot \mathbf{g}|_{\partial\Omega} = 0\} \quad (16)$$

$L^2(\Omega)$  is a space of square integrable functions in  $\Omega$  (if  $\Omega$  is unbounded, square integrability is defined locally on each bounded subset of  $\Omega$ )

## Inner products

$L^2$ -inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2, \Omega} = \int_{\Omega} \mathbf{f} \cdot \mathbf{g} \, d\mathbf{r} \quad (17)$$

$H^1$ -inner product

$$\langle f, g \rangle_{H^1, \Omega} = \int_{\Omega} fg \, d\mathbf{r} + \int_{\Omega} \nabla f \cdot \nabla g \, d\mathbf{r} \quad (18)$$

$H_{curl}$ -inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{H_{curl}, \Omega} = \int_{\Omega} \mathbf{f} \cdot \mathbf{g} \, d\mathbf{r} + \int_{\Omega} \nabla \times \mathbf{f} \cdot \nabla \times \mathbf{g} \, d\mathbf{r} \quad (19)$$

$H_{div}$ -inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{H_{div}, \Omega} = \int_{\Omega} \mathbf{f} \cdot \mathbf{g} \, d\mathbf{r} + \int_{\Omega} \nabla \cdot \mathbf{f} \cdot \nabla \cdot \mathbf{g} \, d\mathbf{r} \quad (20)$$

## Norms

 $L^2$ -norm

$$\|\mathbf{v}\|_0 = \left( \int_{\Omega} |\mathbf{v}|^2 d\Omega \right)^{1/2} \quad (21)$$

 $H^1$ -norm

$$\|\rho\|_{1,\Omega} = \left( \int_{\Omega} |\nabla \rho|^2 d\Omega + \int_{\Omega} |\rho|^2 d\Omega \right)^{1/2} \quad (22)$$

 $H_{curl}$ -norm

$$\|\mathbf{f}\|_{curl,\Omega} = \left( \int_{\Omega} |\nabla \times \mathbf{f}|^2 d\Omega + \int_{\Omega} |\mathbf{f}|^2 d\Omega \right)^{1/2} \quad (23)$$

 $H_{div}$ -norm

$$\|\mathbf{g}\|_{div,\Omega} = \left( \int_{\Omega} |\nabla \cdot \mathbf{g}|^2 d\Omega + \int_{\Omega} |\mathbf{g}|^2 d\Omega \right)^{1/2} \quad (24)$$



# Trace operators

Tangential trace operator:

$$\gamma_t \mathbf{F} = -\mathbf{n} \times \mathbf{n} \times \mathbf{F}|_{\partial\Omega} \quad (25)$$

Defines mapping:  $H_{curl}(\Omega) \rightarrow H_{Curl}^{-1/2}(\partial\Omega)$

Rotated tangential trace operator:

$$\gamma_r \mathbf{F} = \mathbf{n} \times \mathbf{F}|_{\partial\Omega} \quad (26)$$

Defines mapping:  $H_{curl}(\Omega) \rightarrow H_{Div}^{-1/2}(\partial\Omega)$

Normal trace operator:

$$\gamma_n \mathbf{F} = \mathbf{n} \cdot \mathbf{F}|_{\partial\Omega} \quad (27)$$

Defines mapping:  $H_{div}(\Omega) \rightarrow H^{-1/2}(\partial\Omega)$

## Some useful vector identities

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \quad (28)$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \quad (29)$$

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F} \quad (30)$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \quad (31)$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \quad (32)$$

$$\nabla \times (\nabla f) = 0 \quad (33)$$

$$\int_V \nabla \cdot \mathbf{F} dV = \int_{\partial V} \mathbf{n} \cdot \mathbf{F} dS \quad (34)$$

$$\int_V \nabla \times \mathbf{F} dV = \int_{\partial V} \mathbf{n} \times \mathbf{F} dS \quad (35)$$

## Weak problem

Let  $\Omega$  be simply connected and bounded domain with the PEC boundary condition  $\mathbf{n} \times \mathbf{E} = 0$  on  $\partial\Omega$ . Find  $\mathbf{E} \in H_{0,\text{curl}}(\Omega)$  such that

$$\langle \mathbf{w}, \nabla \times \mu^{-1} \nabla \times \mathbf{E} - \epsilon\omega^2 \mathbf{E} \rangle = i\omega \langle \mathbf{w}, \mathbf{J}^s \rangle, \quad (36)$$

$\forall \mathbf{w} \in H_{0,\text{curl}}(\Omega)$ . Here  $\langle \cdot, \cdot \rangle$  denotes the  $L^2(\Omega)$ -inner product.

The above equation can be written as

$$\langle \nabla \times \mathbf{w}, \mu^{-1} \nabla \times \mathbf{E} \rangle - \epsilon\omega^2 \langle \mathbf{w}, \mathbf{E} \rangle = i\omega \langle \mathbf{w}, \mathbf{J}^s \rangle. \quad (37)$$

It is clear that any solution for the wave-equation satisfies (37) but does a solution of (37) satisfy the wave-equation?

# Finite-element solution

- In the finite-element method, a weak problem is solved in a finite-dimensional space  $\mathcal{B}_h$
- The finite-element space is constructed by dividing the domain into smaller elements e.g. tetrahedral elements  $T_h$ .
- Basis functions (associated with elements  $T_h$ ) should span the finite-dimensional subspace “finite-element space”  $\mathcal{B}_h \subset H_{0,\text{curl}}(\Omega)$
- Testing functions should span the finite-dimensional subspace  $\mathcal{W}_h \subset H_{0,\text{curl}}(\Omega)$

## Projection method

Solve linear problem of the form

$$Lu = f, \quad L : U \rightarrow F, f \in F$$

$U$  and  $F$  are some Hilbert spaces, and the unknown  $u \in U$

Expand the unknown with a set of basis functions  $b_n$  span  $B_N \subset U$

$$u \approx \tilde{u}_N = \sum_{n=1}^N c_n b_n$$

Require the residual

$$R_N = L\tilde{u}_N - f,$$

to be orthogonal to the space  $T_M \subset F$  spanned by testing functions  $t_m$

$$\langle t_m, R_N \rangle_F = 0, \forall m = 1, 2, \dots, M$$

## Finite-element solution

Expand the unknown electric field as a linear combination of known basis functions  $\mathbf{b}_n$  as

$$\mathbf{E} \approx \sum_{n=1}^N c_n \mathbf{b}_n, \quad (38)$$

where  $c_n$  are unknown coefficients.

Taking the inner product with testing functions  $\mathbf{t}_m$ , gives rise to a matrix equation

$$A_{mn} c_n = f_m \quad (39)$$

where

$$A_{mn} = \langle \nabla \times \mathbf{t}_m, \mu^{-1} \nabla \times \mathbf{b}_n \rangle - \epsilon \omega^2 \langle \mathbf{t}_m, \mathbf{b}_n \rangle, \quad (40)$$

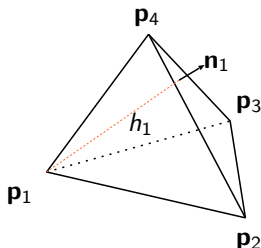
and the force vector read as

$$f_m = i\omega \langle \mathbf{t}_m, \mathbf{J}^s \rangle. \quad (41)$$

The unknown coefficients can be solve by inverting the matrix

# Shape functions

Consider tetrahedral element  $T_k$



Linear shape functions  $N_{p_i}(x, y, z)$ :

- $N_{p_i}(x, y, z) = 1$  at node  $\mathbf{p}_i$
- $N_{p_i}(x, y, z) = 0$  at other nodes
- linear inside the tetrahedron
- $\nabla N_{p_i} = -\frac{\mathbf{n}_i}{h_i}$

Any linear function inside the tetrahedron can be expressed as

$$f(x, y, z) = \sum_{i=1}^4 a_i N_{p_i}, \quad (42)$$

where  $a_i$  are some coefficients.

## Basis functions $H^1$

Basis functions should have the same differentiability and continuity properties as the original unknown functions e.g.  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$ ,  $\mathbf{B}$  etc.

Let a first consider a scalar space  $H^1(\Omega)$ , i.e., the space of square integrable functions whose gradients are also square integrable. These functions are **continuous!**

Functions in  $H^1(\Omega)$  can be approximated in tetrahedral mesh as a combinations of shape functions

$$\phi \approx \sum_{n=1}^N c_n N_n \quad (43)$$

where  $N$  is the number of nodes. (Degrees of freedom = number of nodes)

This is suitable space for e.g. Poisson equation (electrostatic)

$$\nabla^2 \phi = \rho \quad (44)$$



## Basis functions $H_{curl}$

$H_{curl}$  is a suitable space for fields.

The lowest order curl-conforming basis function associated into the edge  $e_{ij}$  (between nodes  $ij$ ) and can be expressed as

$$\mathbf{w}_{e_{ij}} = N_i \nabla N_j - N_j \nabla N_i \quad (45)$$

This function has a **continuous tangential component**, and its curl,

$$\nabla \times \mathbf{w}_{e_{ij}} = 2(\nabla N_i \times \nabla N_j), \quad (46)$$

is piecewise constant and square integrable.

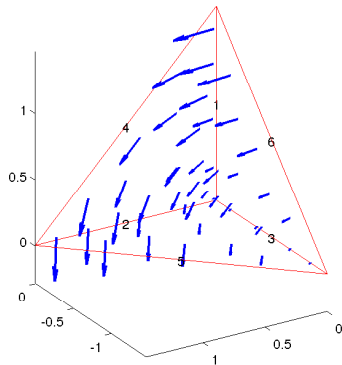
Therefore, the electric or magnetic field can be expanded as

$$\mathbf{E} \approx \sum_{e=1}^E c_e \mathbf{w}_e, \quad (47)$$

where  $E$  is the number of edges

$H_{curl}$ -function

Edge-element: continuous tangential component



## Basis functions $H_{div}$

$H_{div}$  is a suitable space for flux densities or currents.

The lowest order div-conforming basis function associated into the face  $f_{ijk}$  (between nodes  $ijk$ ) and can be expressed as

$$\mathbf{v}_{f_{ijk}} = N_i(\nabla N_j \times \nabla N_k) + N_j(\nabla N_k \times \nabla N_i) + N_k(\nabla N_i \times \nabla N_j) \quad (48)$$

This function has a **continuous normal component**, and its div,

$$\nabla \cdot \mathbf{v}_{f_{ijk}} = 1/V, \quad (49)$$

where  $V$  is the volume of tetrahedron.

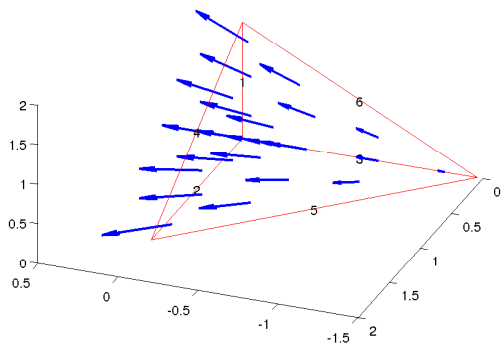
Therefore, the electric or magnetic flux can be expanded as

$$\mathbf{D} \approx \sum_{f=1}^F c_f \mathbf{v}_f, \quad (50)$$

where  $F$  is the number of faces

$H_{div}$ -function

Face-element: continuous normal component



Basis functions  $L^2$ 

Scalar  $L^2$ -functions for the charge density  $\rho$

Scalar case:

$$\rho = \sum_{t=1}^T c_t \frac{1}{\sqrt{V}}, \quad (51)$$

where  $T$  is the number of tetrahedra

Vector case (equivalent volumetric currents):

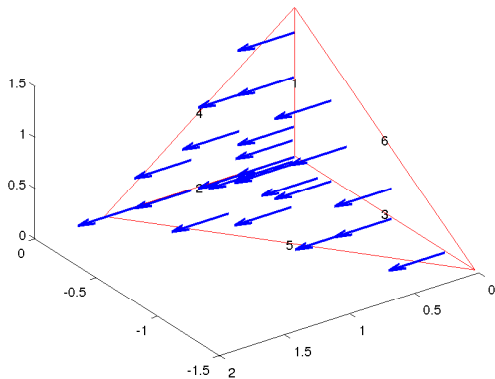
$$\mathbf{p} = \sum_{t=1}^T c_t^x \frac{1}{\sqrt{V_t}} \hat{\mathbf{e}}_x + c_t^y \frac{1}{\sqrt{V_t}} \hat{\mathbf{e}}_y + c_t^z \frac{1}{\sqrt{V_t}} \hat{\mathbf{e}}_z, \quad (52)$$

where  $T$  is the number of tetrahedra

These functions are **discontinuous!**

$H_{div}$ -function

L2-element: discontinuous



## Reference element

In practise, all integrals are evaluate in a reference element.

$$N_1 = 1 - \xi - \eta - \zeta$$

$$N_2 = \xi$$

$$N_3 = \eta$$

$$N_4 = \zeta$$

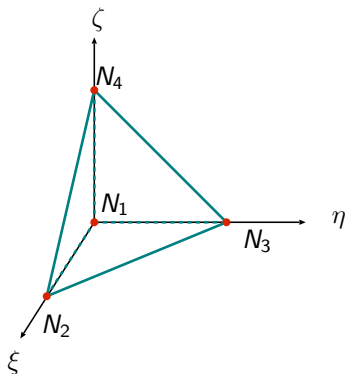
(53)

$$\nabla N_1 = [-1, -1, -1]$$

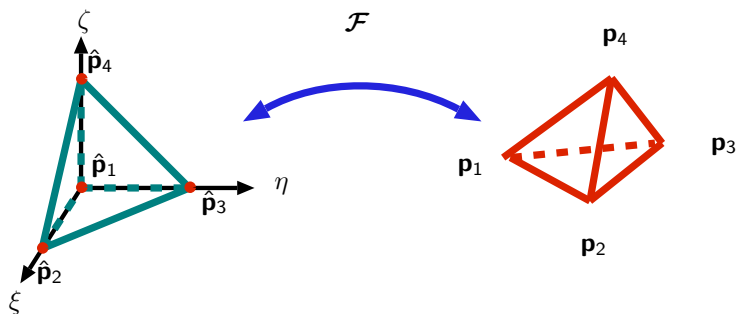
$$\nabla N_2 = [1, 0, 0]$$

$$\nabla N_3 = [0, 1, 0]$$

$$\nabla N_4 = [0, 0, 1]$$



## Linear mapping



$$\mathcal{F}(\xi, \eta, \zeta) =$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (\mathbf{p}_2 - \mathbf{p}_1)_x & (\mathbf{p}_3 - \mathbf{p}_1)_x & (\mathbf{p}_4 - \mathbf{p}_1)_x \\ (\mathbf{p}_2 - \mathbf{p}_1)_y & (\mathbf{p}_3 - \mathbf{p}_1)_y & (\mathbf{p}_4 - \mathbf{p}_1)_y \\ (\mathbf{p}_2 - \mathbf{p}_1)_z & (\mathbf{p}_3 - \mathbf{p}_1)_z & (\mathbf{p}_4 - \mathbf{p}_1)_z \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} + \begin{pmatrix} (\mathbf{p}_1)_x \\ (\mathbf{p}_1)_y \\ (\mathbf{p}_1)_z \end{pmatrix} \quad (54)$$



## Integration on the reference element

Shape function on element  $T_k$ :  $N_{n_i^k}(x, y, z) = \hat{N}_i(\mathcal{F}_k^{-1}(x, y, z))$

$$\int_{T_k} N_{n_i^k}(x, y, z) N_{n_j^k}(x, y, z) dx dy dz = \int_{\hat{T}} \hat{N}_i(\xi, \eta, \zeta) \hat{N}_j(\xi, \eta, \zeta) |\det(J_{\mathcal{F}_k})| d\xi d\eta d\zeta \quad (55)$$

where  $J_{\mathcal{F}_k}$  is the Jacobian of the mapping  $\mathcal{F}_k$

$$J_{\mathcal{F}_k} = \left[ \frac{\partial \mathcal{F}_k}{\partial \xi}, \frac{\partial \mathcal{F}_k}{\partial \eta}, \frac{\partial \mathcal{F}_k}{\partial \zeta} \right] = [\mathbf{p}_2^k - \mathbf{p}_1^k, \mathbf{p}_3^k - \mathbf{p}_1^k, \mathbf{p}_4^k - \mathbf{p}_1^k] \quad (56)$$

The above integral can be numerically evaluated as

$$I_{ij} = |\det(J_{\mathcal{F}_k})| \sum_{m=1}^M \hat{N}_i(\xi_m, \eta_m, \zeta_m) \hat{N}_j(\xi_m, \eta_m, \zeta_m) w_m \quad (57)$$

where  $(\xi_m, \eta_m, \zeta_m)$  are the Gaussian quadrature points for the reference tetrahedron and  $w_m$  are the corresponding weights.

## Mapping of derivatives

Functions and their derivatives transforms differently!

$$\hat{N}_i(\xi, \eta, \zeta) = N_{n_i^k}(\mathcal{F}_k(\xi, \eta, \zeta)) \quad (58)$$

Chain rule:

$$\begin{aligned} \frac{\partial \hat{N}_i}{\partial \xi} &= \frac{\partial N_{n_i^k}}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial z} \frac{\partial z}{\partial \xi} = \\ & \frac{\partial N_{n_i^k}}{\partial x} \frac{\partial \mathcal{F}_k^x}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial y} \frac{\partial \mathcal{F}_k^y}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial z} \frac{\partial \mathcal{F}_k^z}{\partial \xi} \end{aligned} \quad (59)$$

$$\begin{aligned} \frac{\partial \hat{N}_i}{\partial \eta} &= \frac{\partial N_{n_i^k}}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial z} \frac{\partial z}{\partial \eta} = \\ & \frac{\partial N_{n_i^k}}{\partial x} \frac{\partial \mathcal{F}_k^x}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial y} \frac{\partial \mathcal{F}_k^y}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial z} \frac{\partial \mathcal{F}_k^z}{\partial \eta} \end{aligned} \quad (60)$$

## Mapping of derivatives

$$\frac{\partial \hat{N}_i}{\partial \zeta} = \frac{\partial N_{n_i^k}}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial z} \frac{\partial z}{\partial \zeta} =$$

$$\frac{\partial N_{n_i^k}}{\partial x} \frac{\partial \mathcal{F}_k^x}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial y} \frac{\partial \mathcal{F}_k^y}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial z} \frac{\partial \mathcal{F}_k^z}{\partial \zeta} \quad (61)$$

In other words

$$\hat{\nabla} \hat{N}_i = J_{\mathcal{F}_k}^T \nabla N_{n_i^k} \quad (62)$$

So we can write an expression for gradients in a general tetrahedron as

$$\nabla N_{n_i^k} = (J_{\mathcal{F}_k}^T)^{-1} \hat{\nabla} \hat{N}_i \quad (63)$$

## Data structures

3D tetrahedral mesh can be stored as follows:

Coordinates of nodes ( $3 \times N_n$ ):

$$\mathit{mesh.coord} = \begin{bmatrix} p_x^1 & p_x^2 & p_x^3 & \dots & p_x^{N_n} \\ p_y^1 & p_y^2 & p_y^3 & \dots & p_y^{N_n} \\ p_z^1 & p_z^2 & p_z^3 & \dots & p_z^{N_n} \end{bmatrix}$$

Nodes for each tetrahedron ( $4 \times N_t$ )

$$\mathit{mesh.etopol} = \begin{bmatrix} n_1^1 & n_1^2 & n_1^3 & \dots & n_1^{N_t} \\ n_2^1 & n_2^2 & n_2^3 & \dots & n_2^{N_t} \\ n_3^1 & n_3^2 & n_3^3 & \dots & n_3^{N_t} \\ n_4^1 & n_4^2 & n_4^3 & \dots & n_4^{N_t} \end{bmatrix}$$

$$\mathit{mesh.param} = [ \epsilon_r^1 \quad \epsilon_r^2 \quad \epsilon_r^3 \quad \dots \quad \epsilon_r^{N_t} ] \quad (64)$$

$N_t$  is the number of tetrahedra

$N_n$  is the number of nodes

## Data structures

Some additional data structures:

Nodes of edges + boundary ( $3 \times N_e$ ) (last line: 1 if boundary edge and 0 otherwise):

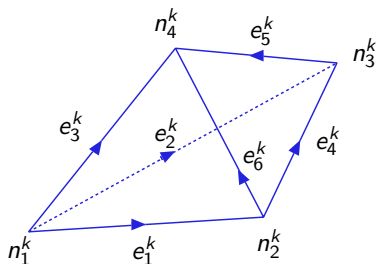
$$\mathit{mesh.edges} = \begin{bmatrix} n_1^1 & n_1^2 & n_1^3 & \dots & n_1^{N_e} \\ n_2^1 & n_2^2 & n_2^3 & \dots & n_2^{N_e} \\ 0 & 0 & 1 & \dots & 0 \end{bmatrix}$$

Edges for each tetrahedron ( $6 \times N_t$ )

$$\mathit{mesh.etopol2} = \begin{bmatrix} e_1^1 & e_1^2 & e_1^3 & \dots & e_1^{N_t} \\ e_2^1 & e_2^2 & e_2^3 & \dots & e_2^{N_t} \\ e_3^1 & e_3^2 & e_3^3 & \dots & e_3^{N_t} \\ e_4^1 & e_4^2 & e_4^3 & \dots & e_4^{N_t} \\ e_5^1 & e_5^2 & e_5^3 & \dots & e_5^{N_t} \\ e_6^1 & e_6^2 & e_6^3 & \dots & e_6^{N_t} \end{bmatrix}$$

$N_e$  is the number of edges

## Data structures

Local edges in tetrahedron  $k$ 

$e^k$	$n^k$	$n^k$
1	1	2
2	1	3
3	1	4
4	2	3
5	3	4
6	2	4

Edge orientation may be different in global and local notations  
 $\Rightarrow$  we need to calculate a correct sign for each basis function

e.g. Global edge 19

$node1 = mesh.edges(1, 19)$

$node2 = mesh.edges(2, 19)$

## Calculation of matrix elements

Recall:

$$A_{mn} = \langle \nabla \times \mathbf{t}_m, \mu^{-1} \nabla \times \mathbf{b}_n \rangle - \epsilon \omega^2 \langle \mathbf{t}_m, \mathbf{b}_n \rangle, \quad (65)$$

where  $\mathbf{t}_m$  and  $\mathbf{b}_n \in H_{0, \text{curl}}(\Omega)^3$

We use lowest order edge elements (edge  $m$  between nodes  $n_j$  and  $n_i$ ):

$$\mathbf{t}_m = \mathbf{b}_m = N_{n_i^k} \nabla N_{n_j^k} - N_{n_j^k} \nabla N_{n_i^k} \quad (66)$$

$$\nabla \times \mathbf{t}_m = \nabla \times \mathbf{b}_m = 2(\nabla N_{n_i^k} \times \nabla N_{n_j^k}) \quad (67)$$

Loop over elements, compute local inner products elementwise and add local contributions to the global matrix

## Matrix assembly

First term:

$$\begin{aligned}
 & \langle \nabla \times \mathbf{t}_m, \mu^{-1} \nabla \times \mathbf{b}_n \rangle_{T_k} \\
 &= \int_{T_k} 2(\nabla N_{n_i^k} \times \nabla N_{n_j^k}) \cdot \mu_k^{-1} 2(\nabla N_{n_p^k} \times \nabla N_{n_l^k}) dV \\
 &= 4(\nabla N_{n_i^k} \times \nabla N_{n_j^k}) \cdot \mu_k^{-1} (\nabla N_{n_p^k} \times \nabla N_{n_l^k}) V_k
 \end{aligned} \tag{68}$$

Second term:

$$\begin{aligned}
 \langle \mathbf{t}_m, \mathbf{b}_n \rangle_{T_k} &= \int_{T_k} (N_{n_i^k} \nabla N_{n_j^k} - N_{n_j^k} \nabla N_{n_i^k}) \cdot (N_{n_p^k} \nabla N_{n_l^k} - N_{n_l^k} \nabla N_{n_p^k}) dV \\
 &= \nabla N_{n_j^k} \cdot \nabla N_{n_l^k} \int_{T_k} N_{n_i^k} N_{n_p^k} dV \\
 &\quad - \nabla N_{n_j^k} \cdot \nabla N_{n_p^k} \int_{T_k} N_{n_i^k} N_{n_l^k} dV \\
 &\quad - \nabla N_{n_i^k} \cdot \nabla N_{n_l^k} \int_{T_k} N_{n_j^k} N_{n_p^k} dV \\
 &\quad + \nabla N_{n_i^k} \cdot \nabla N_{n_p^k} \int_{T_k} N_{n_j^k} N_{n_l^k} dV
 \end{aligned} \tag{69}$$



## Example: Computation of resonant frequencies

Consider the eigenvalue problem

$$\begin{aligned}\nabla \times \mu^{-1} \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} &= 0, \text{ in } \Omega \\ \mathbf{n} \times \mathbf{E} &= 0, \text{ on } \partial\Omega\end{aligned}\quad (70)$$

Corresponding weak problem: Find  $\mathbf{E} \in H_{0,\text{curl}}$  and  $\omega \in \mathbb{R} \setminus \{0\}$  such that

$$\langle \nabla \times \mathbf{t}, \mu^{-1} \nabla \times \mathbf{E} \rangle_{L^2(\Omega)} - \omega^2 \langle \mathbf{t}, \epsilon \mathbf{E} \rangle_{L^2(\Omega)} = 0 \quad (71)$$

for all  $\mathbf{t} \in H_{0,\text{curl}}$ .

Use curl conforming edge-elements as basis and testing functions (boundary edges removed), and solve the generalised eigenvalue problem

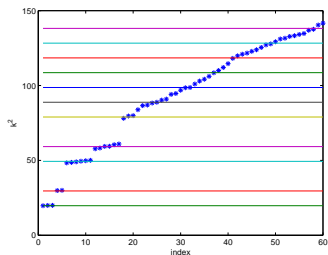
$$SE = \omega^2 ME \quad (72)$$

where  $S$  is the “stiffness” matrix arising from  $\langle \nabla \times \mathbf{t}, \mu^{-1} \nabla \times \mathbf{E} \rangle$ , and  $M$  is the “mass” matrix arising from  $\langle \mathbf{t}, \epsilon \mathbf{E} \rangle$ .

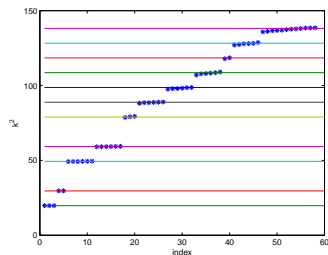
# Example: Computation of resonant frequencies

Analytical solution (solid lines)  $k^2 = \pi^2(l^2 + m^2 + n^2)$ ,  $l, m, n = 0, 1, \dots$

Numerical solution (stars) FEM with lowest order edge-elements



638 tetrahedra



2433 tetrahedra