# Mie series

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# Mie series — build your own I

For implementing Mie series, you will need: spherical Bessel ($j$) and spherical Hankel type 1 ($h^{(1)}$) functions

$$j_n(x) = \sqrt{\frac{\pi/2}{x}}\, J_{n+1/2}(x) \quad h_n^{(1)}(x) = \sqrt{\frac{\pi/2}{x}}\, H_{n+1/2}^{(1)}(x), \tag{1}$$

where $J$ and $H^{(1)}$ are ordinary Bessel and Hankel type 1 -functions.

Ricatti-Bessel functions and their derivatives:

$$\psi_n(x) = x j_n(x) \quad \xi_n(x) = -x h_n^{(1)}(x) \tag{2}$$

and

$$\psi_n'(x) = (n+1) j_n(x) - x j_{n+1}(x) \quad \xi_n'(x) = -(n+1) h_n^{(1)}(x) + x h_{n+1}^{(1)}(x) \tag{3}$$

# Mie series — build your own II

Then, the scaled coefficients for the Mie series $a_n$ and $b_n$ are given by:

$$a_n(x,m) = \frac{m\psi_n(mx)\psi_n'(x) - \psi_n(x)\psi_n'(mx)}{m\psi_n(mx)\xi_n'(x) - \xi_n(x)\psi_n'(mx)} \tag{4}$$

$$b_n(x,m) = \frac{\psi_n(mx)\psi_n'(x) - m\psi_n(x)\psi_n'(mx)}{\psi_n(mx)\xi_n'(x) - m\xi_n(x)\psi_n'(mx)}, \tag{5}$$

which concludes the part that is not angle-dependent. The $x$ is the size parameter of the sphere, and $m$ its refractive index.

# Mie series — build your own III

For the angle-dependent part of Mie scattering, we need derivative of associated Legendre function $P_n^1$

$$P'^1_n(\theta) = \frac{dP_n^1(\cos(\theta))}{d\theta} = -\frac{(n+1)\cos(\theta)}{\sin(\theta)} P_n^1(\cos(\theta)) + \frac{n}{\sin(\theta)} P_{n+1}^1(\cos(\theta)) \tag{6}$$

then, we can write angle-dependent coefficients $\pi_n$ and $\tau_n$ as

$$\pi_n(\theta) = -\frac{1}{\sin(\theta)} P_n^1(\cos(\theta)) \quad \tau_n(\theta) = -P'^1_n(\theta) \tag{7}$$

# Mie series — build your own IV

Finally, when a sphere is excited by a plane wave, the elements $S_1$ and $S_2$ of the amplitude scattering matrix are:

$$S_1(\theta, x, m) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n(x,m)\pi_n(\theta) + b_n(x,m)\tau_n(\theta)\right) \quad (8)$$

$$S_2(\theta, x, m) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(b_n(x,m)\pi_n(\theta) + a_n(x,m)\tau_n(\theta)\right) \quad (9)$$

and the real-valued $4\times4$ Mueller matrix elements that are non-zero for a sphere are:

$$M_{11} = M_{22} = (|S_1|^2 + |S_2|^2)/2, \quad M_{12} = M_{21} = (|S_2|^2 - |S_1|^2)/2,$$
$$M_{33} = M_{44} = \mathrm{Re}(S_1 S_2^*), \quad M_{34} = M_{43} = \mathrm{Im}(S_1 S_2^*) \quad (10)$$

# Mie series — build your own V

The extinction and scattering cross-sections can be computed directly from the series coefficients $a_n$ and $b_n$ as

$$C_{ext} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) \mathrm{Re}(a_n + b_n) \tag{11}$$

$$C_{sca} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1)(|a_n|^2 + |b_n|^2) \tag{12}$$

# Mie series — build your own VI

Computational notes: The number of terms $n_{max}$ in the series is roughly of order $x$ (size parameter of the sphere). There are recurrence relations for the special functions that can be used for efficient computation, but one should try to avoid numerical round-off errors. The recurrence for spherical Bessel function $j_n$ is stable in downward recurrence scheme. The spherical Hankel type 1 function can be written as $h_n^{(1)}(x) = j_n(x) + iy_n(x)$, where $y_n$ is the spherical Bessel function of the second kind. Recurrence for $y_n$ is stable upwards.