1 Rayleigh-Gansin or Born approximation (lecture 6)

The integral equation derived above allows for a solution via perturbation series, where the internal field of the scatterer is first approximated by the incident field. What follows is the so-called Rayleigh-Gans approximation or the first Born approximation based on the corresponding integral equation in quantum mechanics.

Consider purely spatial fluctuations from an otherwise uniform medium and assume, in addition, that the fluctuations are linear, $\mathbf{D}(\mathbf{x}) = [\epsilon_m + \delta\epsilon(\mathbf{x})]\mathbf{E}(\mathbf{x})$, where $\delta\epsilon(\mathbf{x})$ is small compared to ϵ_m. The difference $\mathbf{D} - \epsilon_m \mathbf{E}$ showing up in the integral equation is proportional to $\delta\epsilon(\mathbf{x})$. In the lowest order,

$$\mathbf{D} - \epsilon_m \mathbf{E} \approx \frac{\delta\epsilon(\mathbf{x})}{\epsilon_m}\mathbf{D}^{(0)}. \qquad (1)$$

Let the incident field be a plane wave so that $\mathbf{D}^{(0)}(\mathbf{x}) = \hat{\epsilon}_0 D_0 e^{ik\hat{n}_0\cdot\mathbf{x}}$. Then

$$\begin{aligned} \frac{\hat{\epsilon}^* \cdot \mathbf{A}_s^{(0)}}{D_0} &= \frac{k^2}{4\pi}\int d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}}\hat{\epsilon}^* \cdot \hat{\epsilon}_0 \frac{\delta\epsilon(\mathbf{x})}{\epsilon_m} \\ \mathbf{q} &= k(\hat{n}_0 - \hat{n}), \end{aligned} \qquad (2)$$

the square of which, in absolute terms, gives the differential cross section. If the wavelength is much larger than the size of the region where $\delta\epsilon \neq 0$, the exponent in the integral can be set to unity. This results in the dipole approximation that was treated before for a small spherical particle.

Let us study the situation where the particle continues to be spherical and is located in free space. Thus, $\delta\epsilon \neq 0$ inside a sphere of radius a. We obtain

$$\begin{aligned} \frac{\hat{\epsilon}^* \cdot \mathbf{A}_s^{(1)}}{D_0} &= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}'} \\ &= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \int_0^a dr' r'^2 e^{iqr'\cos\theta'} \\ &= \frac{k^2}{2}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int_0^a dr' r'^2 /_{-1}^{1} \frac{1}{iqr'} e^{iqr'\mu'}, \qquad \mu' = \cos\theta' \\ &= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\frac{1}{iq}\Big\{ /_0^a r' \frac{1}{iq}(e^{iqr'} + e^{-iqr'}) - \int_0^a dr' \frac{1}{iq}(e^{iqr'} + e^{-iqr'})\Big\} \\ &= k^2 \frac{\delta\epsilon}{\epsilon_0}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\Big(\frac{\sin qa - qa\cos qa}{q^3}\Big), \qquad q = |\mathbf{q}| = \sqrt{2}k\sqrt{1 - \hat{n}\cdot\hat{n}_0}. \end{aligned}$$

In the limit $a \to 0$, the term inside the parentheses approaches $a^3/3$ so that, for scatterers much smaller than the wavelength or for q approaching zero,

$$\lim_{q\to 0}\Big(\frac{d\sigma}{d\Omega}\Big)_{R-G} = k^4 a^6 \Big|\frac{\delta\epsilon}{3\epsilon_0}\Big|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2. \qquad (3)$$

This is in agreement with the long-wavelength limit studied earlier. The integral $\int_S d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}'}$ is commonly called the form factor.

2 Why is the sky blue?

In the present context, we can consider the blueness of the sky and redness of the sunrises and sunsets. Assume that the atmosphere is composed of individual molecules with locations $\mathbf{x}_j$ and that have the dipole moment $\mathbf{p}_j = \hat{\epsilon}_0 \gamma_{\text{mol}} \mathbf{E}(\mathbf{x}_j)$, where γ_{mol} is the molecular polarizability. Then, the fluctuations of the electric permittivity can be described with the sum

$$\delta\epsilon(\mathbf{x}) = \epsilon_0 \sum_j \gamma_{mol} \delta(\mathbf{x} - \mathbf{x}_j) \tag{4}$$

The differential scattering cross section is of the form

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} |\gamma_{mol}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 F(\mathbf{q}), \tag{5}$$

where F is the structure factor treated before. For randomly distributed scatterers, $F(\mathbf{q})$ is directly the number of the molecules. For low-density gas, the relative permittivity is $\epsilon_r = \epsilon/\epsilon_0 = 1 + N\gamma_{\text{mol}}$, where N is now the number of molecules in unit volume. The total scattering cross section as per molecule is

$$\sigma_s \approx \frac{k^4}{6\pi N^2} |\epsilon_r - 1|^2 \cong \frac{2k^4}{3\pi N^2} |m - 1|^2, \tag{6}$$

where m is the refractive index and $|m - 1| \ll 1$.

When the radiation propagates a distance dx in the atmoshpere, the relative change in its intensity is $N\sigma dx$ and $I(x) = I_0 e^{-k_e x}$, where k_e is the so-called extinction coefficient:

$$k_e = N\sigma_s \cong \frac{2k^4}{3\pi N} |m - 1|^2 \tag{7}$$

This is called Rayleigh scattering that is incoherent scattering by gas molecules and other dipole scatterers, where each scatterer scatters radiation based on Rayleigh's $1/\lambda^4$ -law.

The $1/\lambda^4$ -law means that blue light is scattered much more efficiently than red light. In practice, this shows up so that blue color predominates when looking in directions other than the light source whereas, in the direction of the light source, red color predominates.

For visible light, $\lambda = 0.41 - 0.65\mu$m and, under normal conditions, $m - 1 \approx 2.78 \cdot 10^{-4}$. When $N = 2.69 \cdot 10^{19}$ molecules/cm^3, we obtain for the mean free path $1/k_e$ =30, 77, and 188 km at wavelengths 0.41 μm (violet), 0.52 μm (green), and 0.65 μm (red), respectively.

Polarization reaches its maximum of 75 % at the wavelength of 0.55 μm. The deviation from 100 % derives from multiple scattering (6 %), the anisotropy of the molecules (6 %), reflection from the surface (5 %, in particular, for green light in the case of vegetation), and aerosols (8 %).