1 Plane waves

The electromagnetic plane wave

$$
\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t} \\
\boldsymbol{H} &= \boldsymbol{H}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}
\end{aligned} \tag{1}
$$

can, under certain conditions, fulfil Maxwell's equations. The physical fields correspond to the real parts of the complex-valued fields. The vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ above are constant vectors and can be complex-valued. Similarly, the wave vector $\boldsymbol{k}$ can be complex-valued:

$$
\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}'', \qquad \boldsymbol{k}', \boldsymbol{k}'' \in \mathbb{R}^n \tag{2}
$$

Inserting (2) into equation (1), we obtain

$$
\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t} \\
\boldsymbol{H} &= \boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t}
\end{aligned} \tag{3}
$$

In Eq. (3), $\boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ and $\boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ are amplitudes and $\boldsymbol{k}'\cdot\boldsymbol{x} - \omega t = \phi$ is the phase of the wave.

An equation of the form $\boldsymbol{k}\cdot\boldsymbol{x}$ =constant defines, in the case of a real-valued vector $\boldsymbol{k}$, a planar surface, whose normal is just the vector $\boldsymbol{k}$. Thus, $\boldsymbol{k}'$ is perpendicular to the planes of constant phase and $\boldsymbol{k}''$ is perpendicular to the planes of constant amplitude. If $\boldsymbol{k}' \parallel \boldsymbol{k}''$, the planes coincide and the wave is *homogeneous*. If $\boldsymbol{k}' \nparallel \boldsymbol{k}''$, the wave is *inhomogeneous*. A plane wave propagating in vacuum is homogeneous.

In the case of plane waves, Maxwell's equaitons can be written as

$$
\begin{aligned}
\boldsymbol{k}\cdot\boldsymbol{E}_0 &= 0 \\
\boldsymbol{k}\cdot\boldsymbol{H}_0 &= 0 \\
\boldsymbol{k}\times\boldsymbol{E}_0 &= \omega\mu\boldsymbol{H}_0 \\
\boldsymbol{k}\times\boldsymbol{H}_0 &= -\omega\epsilon\boldsymbol{E}_0
\end{aligned} \tag{4}
$$

The two upmost equations are conditions for the transverse nature of the waves: $\boldsymbol{k}$ is perpendicular to both $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. The two lowermost equations

show that $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are perpendicular to each other. Since $\boldsymbol{k}$, $\boldsymbol{E}_0$, and $\boldsymbol{H}_0$ are complex-valued, the geometric interpretaion is not simple unless the waves are homogeneous.
It follows from Maxwell's equations (4) that, on one hand,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = \omega\mu\boldsymbol{k} \times \boldsymbol{H}_0 = -\omega^2\epsilon\mu\boldsymbol{E}_0 \tag{5}$$

and, on the other hand,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}_0) - \boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{k}) = -\boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{k}), \tag{6}$$

so that

$$\boldsymbol{k} \cdot \boldsymbol{k} = \omega^2\epsilon\mu. \tag{7}$$

Plane waves solutions are in agreement with Maxwell's equations if

$$\boldsymbol{k} \cdot \boldsymbol{E}_0 = \boldsymbol{k} \cdot \boldsymbol{H}_0 = \boldsymbol{E}_0 \cdot \boldsymbol{H}_0 = 0 \tag{8}$$

and if

$$k'^2 - k''^2 + 2i\boldsymbol{k}' \cdot \boldsymbol{k}'' = \omega^2\epsilon\mu. \tag{9}$$

Note that ϵ and μ are properties of the medium, whereas $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are properties of the wave. Thus, ϵ and μ do not unambiguously determine the details of wave propagation.
In the case of a homogeneous plane wave ($\boldsymbol{k}' \| \boldsymbol{k}''$),

$$\boldsymbol{k} = (k' + ik'')\hat{\mathbf{e}}, \tag{10}$$

where k' and k'' are non-negative and $\hat{\mathbf{e}}$ is an arbitrary real-valued unit vector.

According to Eq. (7),

$$(k' + ik'')^2 = \omega^2\epsilon\mu = \frac{\omega^2 m^2}{c^2}, \tag{11}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum and m is the complex-valued refractive index

$$m = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = m_r + im_i, \qquad m_r, m_i \geq 0. \tag{12}$$

In vacuum, the wave number is $\omega/c = 2\pi/\lambda$, where λ is the wavelength. The general homogeneous plane wave takes the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{2\pi m_i s}{\lambda}} e^{i\frac{2\pi m_r s}{\lambda} - i\omega t} \tag{13}$$

where $s = \boldsymbol{e} \cdot \boldsymbol{x}$. The imaginary and real parts of the refractive index determine the attenuation and phase velocity $v = c/m_r$ of the wave, respectively.

2 Poynting vector

Let us study the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ that is time harmonic. For the physical fields (the real parts of the complex-valued fields), the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{14}$$

describes the direction and amount of energy transfer everywhere in the space.

Let $\boldsymbol{n}$ be the unit normal vector of the planar surface element A. Electromagnetic energy is transferred through the planar surface with power $\boldsymbol{S} \cdot \boldsymbol{n}\, A$, where $\boldsymbol{S}$ is assumed constant on the surface. For an arbitrary surface and $\boldsymbol{S}$ depending on location, the power is

$$W = -\int_A \boldsymbol{S} \cdot \boldsymbol{n} dA, \tag{15}$$

where $\boldsymbol{n}$ is the outward unit normal vector and the sign has been chosen so that positive W corresponds to absorption in the case of a closed surface.
The time-averaged Poynting vector

$$\langle \boldsymbol{S} \rangle = \frac{1}{\tau} \int_t^{t+\tau} \boldsymbol{S}(t') dt' \qquad \tau >> 1/\omega \tag{16}$$

is more important than the momentary Poynting vector (cf. measurements).

The time-averaged Poynting vector for time-harmonic fields is

$$\langle \boldsymbol{S} \rangle = \frac{1}{2} \mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} \tag{17}$$

and, in what follows, this is the Poynting vector meant even though the averaging is not always shown explicitly.
For a plane wave field, the Poynting vector is

$$\boldsymbol{S} = \frac{1}{2}\mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} = \mathbf{Re}\left\{\frac{\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*)}{2\omega\mu^*}\right\}, \tag{18}$$

where

$$\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*) = \boldsymbol{k}^*(\boldsymbol{E} \cdot \boldsymbol{E}^*) - \boldsymbol{E}^*(\boldsymbol{k}^* \cdot \boldsymbol{E}). \tag{19}$$

For a homogeneous plane wave,

$$\boldsymbol{k} \cdot \boldsymbol{E} = \boldsymbol{k}^* \cdot \boldsymbol{E} = 0 \tag{20}$$

and

$$\boldsymbol{S} = \frac{1}{2}\mathbf{Re}\left\{\frac{\sqrt{\epsilon\mu}}{\mu^*}\right\}|\boldsymbol{E}_0|^2 e^{-\frac{4\pi \mathbf{Im}(m) z}{\lambda}}\hat{\mathbf{e}}_z. \tag{21}$$

3 Stokes parameters

Consider the following experiment for an arbitrary monochromatic light source (see Bohren & Huffman p. 46). In the experiment, we make use of a measuring apparatus and polarizers with ideal performance: the measuring apparatus detects energy flux density independently of the state of polarization and the polarizers do not change the amplitude of the transmitted wave.

Denote

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{ikz - i\omega t}, \qquad && \boldsymbol{E}_0 = E_\perp \hat{\mathbf{e}}_\perp + E_\parallel \hat{\mathbf{e}}_\parallel \\ E_\perp &= a_\perp e^{-i\delta_\perp} && \\ E_\parallel &= a_\parallel e^{-i\delta_\parallel} && a_\perp, a_\parallel \geq 0, \delta_\perp, \delta_\parallel \in \mathbb{R} \end{aligned} \tag{22}$$

Experiment I

No polarizer: the flux density is proportional to

$$|\boldsymbol{E}_0|^2 = E_\parallel E_\parallel^* + E_\perp E_\perp^* \tag{23}$$

Experiment II

Linear polarizers $\|$ and $\perp$:
1) $\|$: the amplitude of the transmitted wave is $E_\|$ and the flux density is $E_\| E_\|^*$
2) $\perp$: the amplitude of the transmitted wave is $E_\perp$ and the flux density is $E_\perp E_\perp^*$

The difference of the two measurements is $I_\| - I_\perp = E_\| E_\|^* - E_\perp E_\perp^*$.

Experiment III
Linear polarizers $+45°$ ja $-45°$: The new basis vectors are

$$\begin{cases} \hat{\mathbf{e}}_+ = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\| + \hat{\mathbf{e}}_\perp) \\ \hat{\mathbf{e}}_- = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\| - \hat{\mathbf{e}}_\perp) \end{cases}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_+\hat{\mathbf{e}}_+ + E_-\hat{\mathbf{e}}_- \\ E_+ &= \frac{1}{\sqrt{2}}(E_\| + E_\perp) \\ E_- &= \frac{1}{\sqrt{2}}(E_\| - E_\perp). \end{aligned}$$

1) $+45°$: the amplitude of the transmitted wave is E_+ and the flux density is
$E_+E_+^* = \frac{1}{2}(E_\| E_\|^* + E_\| E_\perp^* + E_\perp E_\|^* + E_\perp E_\perp^*)$
2) $-45°$: the amplitude of the transmitted wave is E_- and the flux density is
$E_-E_-^* = \frac{1}{2}(E_\| E_\|^* - E_\| E_\perp^* - E_\perp E_\|^* + E_\perp E_\perp^*)$

The difference os the measurements is $I_+ - I_- = E_\| E_\perp^* + E_\perp E_\|^*$.

Experiment IV
Circular polarizers R and L:

$$\begin{aligned} \hat{\mathbf{e}}_R &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\| + i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_R^* = 1 \\ \hat{\mathbf{e}}_L &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\| - i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_L \cdot \hat{\mathbf{e}}_L^* = 1 \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_L^* = 0 \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_R \hat{\mathbf{e}}_R + E_L \hat{\mathbf{e}}_L \\ E_R &= \frac{1}{\sqrt{2}}(E_\parallel - iE_\perp) \\ E_L &= \frac{1}{\sqrt{2}}(E_\parallel + iE_\perp). \end{aligned}$$

1) R: the amplitude of the transmitted wave is E_R and the flux density is $E_R E_R^* = \frac{1}{2}(E_\parallel E_\parallel^* - iE_\parallel^* E_\perp + iE_\perp^* E_\parallel + E_\perp E_\perp^*)$
2) L: the amplitude of the transmitted wave is E_L and the flux density is $E_L E_L^* = \frac{1}{2}(E_\parallel E_\parallel^* + iE_\parallel^* E_\perp - iE_\perp^* E_\parallel + E_\perp E_\perp^*)$

The difference of the measurements is $I_R - I_L = i(E_\perp^* E_\parallel - E_\parallel^* E_\perp)$.

With the help of Experiments I-IV, we have determined the Stokes parameters I, Q, U, and V:

$$\begin{aligned} I &= E_\parallel E_\parallel^* + E_\perp E_\perp^* = a_\parallel^2 + a_\perp^2 \\ Q &= E_\parallel E_\parallel^* - E_\perp E_\perp^* = a_\parallel^2 - a_\perp^2 \\ U &= E_\parallel E_\perp^* + E_\perp E_\parallel^* = 2a_\parallel a_\perp \cos\delta \\ V &= i(E_\parallel E_\perp^* - E_\perp E_\parallel^*) = 2a_\parallel a_\perp \sin\delta \qquad \delta = \delta_\parallel - \delta_\perp \end{aligned} \tag{24}$$