# 1 Scattering of light at the plane interface between two media (lecture 3)

Two kinds of features can be distinguished in the reflection and refraction of light at the plane interface between two media:
i) Kinematical properties:
a) the angle of reflection coincides with the angle of incidence
b) the angle of refraction relates to the angle of incidence and the refractive indices of the media via Snel's law
ii) Dynamical properties:
a) the intensitities of reflected and refracted radiation
b) phase shifts and polarization

The kinematical properties follow from the wave nature of the phenomena and the existence of the boundary conditions. The dynamical properties depend fully of the characteristics of the waves and their boundary conditions.
The coordinate systems and symbols are defined in Fig. 1. The original plane wave (wave vector $\mathbf{k}$, angular frequency $\omega$) is incident on the interface from the medium $\mu$, $\epsilon$ (refractive index $m = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$). The refracted plane wave propagates in the medium $\mu'$, $\epsilon'$ ($m' = \sqrt{\epsilon'\mu'/\epsilon_0\mu_0}$) with wave vector $\mathbf{k}_t$ and the reflected plane wave in the medium $\mu$, $\epsilon$ with wave vector $\mathbf{k}_r$.

The kinematics are described by the angles of incidence $\theta_i$, reflection $\theta_r$, and refraction $\theta_t$. Assume first that $\mu$, $\epsilon$, $\mu'$, $\epsilon'$ and therefor also $m$ and $m'$ are real-valued.

Based on what has already been described before, we can write the incident, reflected, and refracted fields as follows:

$$\begin{aligned} \mathbf{E}_i &= \mathbf{E}_{0i} e^{i\mathbf{k}_i \cdot \mathbf{x} - i\omega t} \\ \mathbf{B}_i &= \sqrt{\epsilon\mu} \frac{\mathbf{k}_i \times \mathbf{E}_i}{k_i} \end{aligned} \tag{1}$$

$$\begin{aligned} \mathbf{E}_r &= \mathbf{E}_{0r} e^{i\mathbf{k}_r \cdot \mathbf{x} - i\omega t} \\ \mathbf{B}_r &= \sqrt{\epsilon\mu} \frac{\mathbf{k}_r \times \mathbf{E}_r}{k_r} \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbf{E}_t &= \mathbf{E}_{0t} e^{i\mathbf{k}_t \cdot \mathbf{x} - i\omega t} \\ \mathbf{B}_t &= \sqrt{\epsilon'\mu'} \frac{\mathbf{k}_t \times \mathbf{E}_t}{k_t} \end{aligned} \tag{3}$$

The lengths of the wave vectors are

$$\begin{aligned} |\mathbf{k}_i| &= |\mathbf{k}_r| = k_i = k_r = \omega\sqrt{\epsilon\mu} \\ |\mathbf{k}_t| &= k_t = \omega\sqrt{\epsilon'\mu'} \end{aligned} \tag{4}$$

The boundary conditions are to be valid at the interface $z = 0$ at all times. Therefor, the spatial dependences of the fields need to coincide at the interface and, in particular, the arguments of the phase factors

$$(\mathbf{k}_i \cdot \mathbf{x})_{z=0} = (\mathbf{k}_r \cdot \mathbf{x})_{z=0} = (\mathbf{k}_t \cdot \mathbf{x})_{z=0} \tag{5}$$

independently of the detailed properties of the boundary conditions. It follows, first, that the wave vectors must be confined to a single plane. Second, it follows that $\theta_i = \theta_r$ and, third, we obtain Snel's law

$$\begin{aligned} k_i \sin\theta_i &= k_t \sin\theta_t \\ \Leftrightarrow \qquad m \sin\theta_i &= m' \sin\theta_t. \end{aligned} \tag{6}$$

According to the boundary conditions of electromagnetic fields, the normal components of **D** and **B** and the tangential components of **E** and **H** must be continuous across the boundary. Then, at the interface $z = 0$, we have

$$\begin{aligned} \hat{\mathbf{n}} &\cdot [\epsilon(\mathbf{E}_{0i} + \mathbf{E}_{0r}) - \epsilon'\mathbf{E}_{0t}] = 0 \\ \hat{\mathbf{n}} &\cdot [\mathbf{k}_i \times \mathbf{E}_{0i} + \mathbf{k}_r \times \mathbf{E}_{0r} - \mathbf{k}_t \times \mathbf{E}_{0t}] = 0 \\ \hat{\mathbf{n}} &\times [\mathbf{E}_{0i} + \mathbf{E}_{0r} - \mathbf{E}_{0t}] = 0 \\ \hat{\mathbf{n}} &\times [\frac{1}{\mu}(\mathbf{k}_i \times \mathbf{E}_{0i} + \mathbf{k}_r \times \mathbf{E}_{0r}) - \frac{1}{\mu'}(\mathbf{k}_t \times \mathbf{E}_{0t})] = 0 \end{aligned} \tag{7}$$

Let us divide the scattering problem into two cases: first, the incident field is linearly poalrized so that the electric field is perpendicular to the plane defined by $\mathbf{k}_i$ and $\hat{\mathbf{n}}$; second, the electric field is within that plane. An arbitrary elliptic polarization can be treated as a linear sum of the results following for the two cases defined above.
First, let the elctric field be perpendicular to the plane of incidence (see Fig. 2). The choice of **B**-vectors guarantees a positive flow of energy in the direction of the wave vectors. With the help of the third and fourth boundary conditions above, we obtain

$$\begin{aligned} E_{0i} + E_{0r} - E_{0t} = 0 \\ \sqrt{\frac{\epsilon}{\mu}}(E_{0i} - E_{0r})\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}E_{0t}\cos\theta_t = 0 \end{aligned} \tag{8}$$

Denote the Fresnel coefficients by

$$r_\perp = \frac{E_{0r}}{E_{0i}}, \qquad t_\perp = \frac{E_{0t}}{E_{0i}}.$$

Then,

$$\begin{aligned} 1 + r_\perp - t_\perp = 0 \\ \sqrt{\frac{\epsilon}{\mu}}(1 - r_\perp)\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}t_\perp\cos\theta_t = 0 \end{aligned} \tag{9}$$

and it follows that

$$\begin{aligned} t_\perp &= 1 + r_\perp \\ \sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t &= (\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t)r_\perp \end{aligned} \tag{10}$$

and, furthermore, we obtain, for the Fresnel coefficients,

$$\begin{aligned} r_\perp &= \frac{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t}{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t} \\ t_\perp &= \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i}{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t} \end{aligned} \tag{11}$$

Second, let the electric field be within the plane of incidence (see Fig. 3). Again, based on the third and fourth boundary conditions above, we have

$$\begin{aligned} (E_{0i} - E_{0r})\cos\theta_i - E_{0t}\cos\theta_t &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(E_{0i} + E_{0r}) - \sqrt{\frac{\epsilon'}{\mu'}}E_{0t} &= 0 \end{aligned} \tag{12}$$

Denote the Fresnel coefficients by

$$r_{\parallel} = \frac{E_{0r}}{E_{0i}}, \qquad t_{\parallel} = \frac{E_{0t}}{E_{0i}}.$$

Then,

$$\begin{aligned} (1 - r_{\parallel}) \cos\theta_i - t_{\parallel} \cos\theta_t &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(1 + r_{\parallel}) - \sqrt{\frac{\epsilon'}{\mu'}} t_{\parallel} &= 0 \end{aligned} \tag{13}$$

and we obtain the following pair of equations,

$$\begin{aligned} t_{\parallel} &= \frac{\cos\theta_i}{\cos\theta_t}(1 - r_{\parallel}) \\ \sqrt{\frac{\epsilon}{\mu}} - \sqrt{\frac{\epsilon'}{\mu'}}\frac{\cos\theta_i}{\cos\theta_t} &= -\Big(\sqrt{\frac{\epsilon}{\mu}} + \sqrt{\frac{\epsilon'}{\mu'}}\frac{\cos\theta_i}{\cos\theta_t}\Big) r_{\parallel} \end{aligned} \tag{14}$$

allowing for the Fresnel coefficients to be explicitly solved for:

$$\begin{aligned} r_{\parallel} &= \frac{\sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_i - \sqrt{\frac{\epsilon}{\mu}} \cos\theta_t}{\sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_i + \sqrt{\frac{\epsilon}{\mu}} \cos\theta_t} \\ t_{\parallel} &= \frac{2\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i}{\sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_i + \sqrt{\frac{\epsilon}{\mu}} \cos\theta_t} \end{aligned} \tag{15}$$

In the case of a plane wave normally incident on the interface ($\theta_i = 0$), we obtain

$$\begin{aligned} r_{\parallel} &= -r_{\perp} = \frac{\sqrt{\frac{\epsilon'}{\mu'}} - \sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon'}{\mu'}} + \sqrt{\frac{\epsilon}{\mu}}} \rightarrow \frac{m' - m}{m' + m}, \mu = \mu' \\ t_{\parallel} &= t_{\perp} = \frac{2\sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon'}{\mu'}} + \sqrt{\frac{\epsilon}{\mu}}} \rightarrow \frac{2m}{m' + m}, \mu = \mu' \end{aligned} \tag{16}$$

The Fresnel coefficients derived above are also valid for complex-valued $\epsilon$, $\mu$, $\epsilon'$, and $\mu'$. Usually, for visible light, $\mu = \mu' = \mu_0$. The generalization of Snel's law for complex $m'$ is left for an exercise. In addition, the derivation of the $4 \times 4$ reflection and refraction matrices relating the Stokes parameters of incident, reflected, and refracted light is left for an exercise. In the case of incident electric field polarized in the plane of incidence, we can find the so-called Brewster angle, at which there is no reflected wave. Let $\mu = \mu'$. At the Brewster angle,

$$\begin{aligned}
m' \cos\theta_{iB} &= m\sqrt{1 - \frac{m^2}{m'^2}\sin^2\theta_{iB}} \\
(\frac{m'}{m})^2 \cos^2\theta_{iB} &= 1 - (\frac{m}{m'})^2 \sin^2\theta_{iB} \\
(\frac{m'}{m})^2 &= 1 + \tan^2\theta_{iB} - (\frac{m}{m'})^2 \tan^2\theta_{iB} \\
\tan^2\theta_{iB} &= \frac{(\frac{m'}{m})^2 - 1}{1 - (\frac{m}{m'})^2} = (\frac{m'}{m})^2
\end{aligned}$$

The physical solution is

$$\theta_{iB} = \arctan(\frac{m'}{m}) \tag{17}$$

As a rule for other angles of incidence, too, the reflected light tends to be polarized perpendicular to the plane of incidence.

Total internal reflection can occur when $m > m'$ (the incident wave is "internals"). If $m > m'$, $\theta_t > \theta_{i0}$ according to Snel's law and

$$\theta_{i0} = \arcsin\frac{m'}{m} \tag{18}$$

When the angle of incidence is $\theta_{i0}$, the refracted wave is propagating parallel to the interface and there is no energy flow across the interface. Thus, all the incident energy is reflected back. When $\theta_i > \theta_{i0}$, $\sin\theta_t > 1$ and $\theta_t$ must be a complex-valued angle that has a purely imaginary cosine,

$$\cos\theta_t = i\sqrt{(\frac{\sin\theta_i}{\sin\theta_{i0}})^2 - 1} \tag{19}$$

The refracted wave is of the form

$$
\begin{aligned}
e^{i\mathbf{k}_t\cdot\mathbf{x}} &= e^{ik_t(x\sin\theta_t - z\cos\theta_t)} \\
&= e^{-k_t\sqrt{(\frac{\sin\theta_i}{\sin\theta_{i0}})^2-1}|\mathbf{z}|}e^{ik_t(\frac{\sin\theta_i}{\sin\theta_{i0}})x}
\end{aligned}
\tag{20}
$$

and, thus, attenuates exponentially in the medium $m'$ and propagates only in the direction of the interface.