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Computer exercises for Electromagnetic scattering I, autumn 2012.

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Snell's law and Fresnel coefficients

Study reflection and refraction (i.e. transmission) on plane-parallel interface. When incident (i), reflected (r), and refracted (t) waves are divided to components that are perpendicular (\perp) and parallel (\parallel) to the scattering plane (defined by the directions of incident and reflected waves), the waves are:

$$\mathbf{E}_i(\mathbf{x}, t) = (E_{\perp} \hat{\mathbf{e}}_{\perp} + E_{\parallel} \hat{\mathbf{e}}_{\parallel}) \exp(i\mathbf{k}_i \cdot \mathbf{x} - i\omega t), \quad (1)$$

and similarly for \mathbf{E}_r and \mathbf{E}_t .

Snell's law is:

$$m \sin \theta_i = m' \sin \theta_t \quad (2)$$

and Fresnel reflection (r) and transmission (t) coefficients are:

$$r_{\perp} = \frac{m \cos \theta_i - m' \cos \theta_t}{m \cos \theta_i + m' \cos \theta_t} \quad (3)$$

$$r_{\parallel} = \frac{m' \cos \theta_i - m \cos \theta_t}{m' \cos \theta_i + m \cos \theta_t} \quad (4)$$

$$t_{\perp} = \frac{2m \cos \theta_i}{m \cos \theta_i + m' \cos \theta_t} \quad (5)$$

$$t_{\parallel} = \frac{2m \cos \theta_i}{m' \cos \theta_i + m \cos \theta_t} \quad (6)$$

The actual measurable, reflected and transmitted irradiance coefficients R and T , can be computed from r and t (for both \perp and \parallel) by

$$R = |r|^2, \text{ and } T = \frac{m' \cos \theta_t}{m \cos \theta_i} t^2 \quad (7)$$

Ex. 1

Write a function that computes the angle of transmitted wave in the case of planar interface. Plot θ_t as a function of θ_i for two cases: when $m < m'$ and when $m > m'$ (m and m' real). Find out what means *total internal reflection* and where it is seen in the plots.

Ex. 2

Write a function or functions that compute the Fresnel amplitude coefficients $r_{\perp}, r_{\parallel}, t_{\perp}, t_{\parallel}$ and Fresnel irradiance coefficients $R_{\perp}, R_{\parallel}, T_{\perp}, T_{\parallel}$. Plot, e.g., r_{\perp}, r_{\parallel} and then R_{\perp}, R_{\parallel} as a function of θ_i for some (real) m and m' . Find out what means *Brewster angle* and where it is seen in the plots.

Ex. 3

Produce a tool for ray tracing computations by non-absorbing media with planar interfaces. This tool (i.e. function, library, ...) will compute the Stokes vectors of reflected \mathbf{I}_r and refracted \mathbf{I}_t waves and their directions $\mathbf{x}_r, \mathbf{x}_t$ when given the Stokes vector for incident wave $\mathbf{I}_i = (I_i, Q_i, U_i, V_i)$, the

3D unit vector \mathbf{x}_i of the direction of the incident wave, the 3D unit normal vector \mathbf{n} of the interface, and (real) refractive indices m_1, m_2 .

Hints: There are two separate problems: the directions $\mathbf{x}_r, \mathbf{x}_t$ from $\mathbf{x}_i, \mathbf{n}, m_1$ and m_2 , and the change in Stokes parameters at the interface.

Directions: See http://en.wikipedia.org/wiki/Snell%27s_law#Vector_form for Snell's law in vector form. The symbols used in Wikipedia are such that with our symbols $\mathbf{l} = \mathbf{x}_i, \theta_1 = \theta_i, \theta_2 = \theta_t, \mathbf{v}_{reflect} = \mathbf{x}_r$, and $\mathbf{v}_{refract} = \mathbf{x}_t$.

Stokes parameters: The 4×4 scattering matrices $\mathbf{S}_r, \mathbf{S}_t$ that map the incident Stokes parameters to scattered ones: $\mathbf{I}_r = \mathbf{S}_r \mathbf{I}_i$, are

$$\mathbf{S}_r = \frac{1}{2} \begin{pmatrix} r_{\parallel} r_{\parallel}^* + r_{\perp} r_{\perp}^* & r_{\parallel} r_{\parallel}^* - r_{\perp} r_{\perp}^* & 0 & 0 \\ r_{\parallel} r_{\parallel}^* - r_{\perp} r_{\perp}^* & r_{\parallel} r_{\parallel}^* + r_{\perp} r_{\perp}^* & 0 & 0 \\ 0 & 0 & 2 \operatorname{re}(r_{\parallel} r_{\perp}^*) & 2 \operatorname{im}(r_{\parallel} r_{\perp}^*) \\ 0 & 0 & -2 \operatorname{im}(r_{\parallel} r_{\perp}^*) & 2 \operatorname{re}(r_{\parallel} r_{\perp}^*) \end{pmatrix}, \quad (8)$$

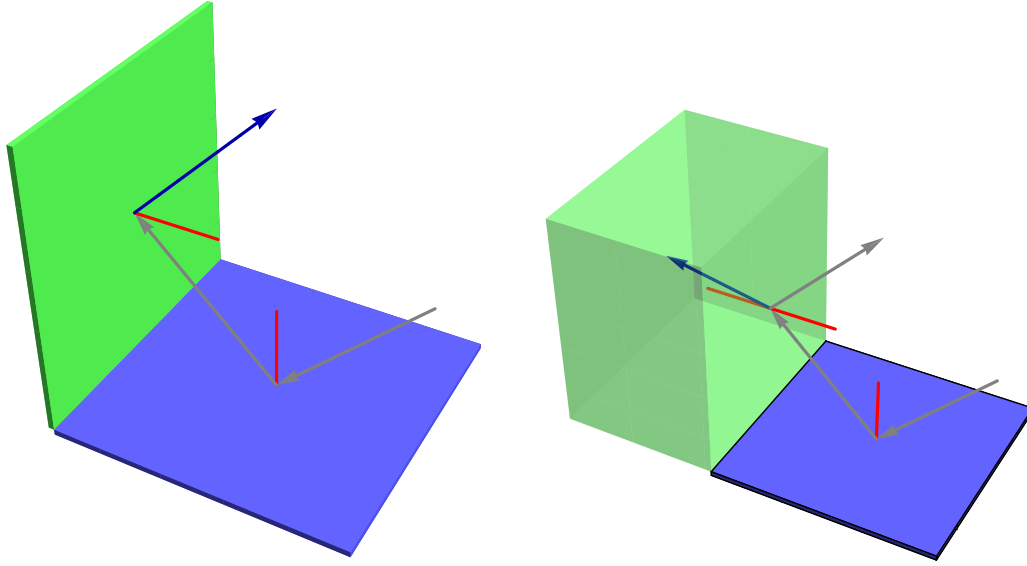
and similarly for \mathbf{S}_t , except that all Fresnel reflectance coefficients r are replaced with transmission coefficients t . Operation r^* is complex conjugate, which does nothing for real values. So, for reals $r r^* = r^2$.

The function could look something like this (in pseudocode):

```
function ScatteringAtInterface(xi, n, Ii, m1, m2){
  compute xt using Snell and xi,n,m1,m2
  compute xr using Snell and xi,n,m1,m2
  compute Fresnel r's and t's
  form matrix Sr from r's
  form matrix St from t's
  compute Ir = Sr . Ii
  compute It = St . Ii
  return xr, Ir, xt, It
}
```

Tasks: Use setup that is described in the figures below. We have vacuum ($m_0 = 1$), first media ($m_1 = 1.5$, blue) and second media ($m_2 = 1.3$, green). Incident light comes along the x -axis and has angle θ_1 with the first normal, $\mathbf{n}_1 = (0, 0, 1)$. Reflected wave will always hit the second interface with $\mathbf{n}_2 = (1, 0, 0)$. Plot the following quantities of the wave reflected from the second interface (left figure, blue arrow), or transmitted into the second media (right figure, blue arrow) as the angle of incidence θ_1 goes from 10° to 80° (note: $\mathbf{x}_i = (-\sin \theta_i, 0, -\cos \theta_i)$):

- i) Intensities ($I_{s\parallel}, I_{s\perp}$) of reflected wave when incident light is linearly polarized, either parallel $\mathbf{I}_{i\parallel} = (1, 1, 0, 0)$ or perpendicular $\mathbf{I}_{i\perp} = (1, -1, 0, 0)$ to the first interface.
- ii) Degree of linear polarization with reflected wave, i.e. $100\% (I_{\parallel} - I_{\perp}) / (I_{\parallel} + I_{\perp})$
- iii) Intensities ($I_{t\parallel}, I_{t\perp}$) of transmitted wave when incident light is linearly polarized, either parallel $\mathbf{I}_{i\parallel}$ or perpendicular $\mathbf{I}_{i\perp}$.
- iv) Degree of linear polarization with transmitted wave, i.e. $100\% (I_{\parallel} - I_{\perp}) / (I_{\parallel} + I_{\perp})$
- v) Angle between $-\mathbf{n}_2$ and transmitted wave.



Snell and Fresnel in absorbing media

You have, hopefully, gone through somehow the case of absorbing media and its effect to the Snell law and Fresnel coefficients. With notations where all (possibly) complex values are marked with \sim , vectors with bold font and unit vectors with $\hat{\cdot}$, the plane wave is of the form

$$\tilde{\mathbf{E}}(\mathbf{x}, t) = \tilde{\mathbf{E}}_0 \exp(i\tilde{\mathbf{k}} \cdot \mathbf{x} - i\omega t), \quad (9)$$

where the wave vector $\tilde{\mathbf{k}}$ can have real and complex parts, $\tilde{\mathbf{k}} = k_0(N\hat{\mathbf{e}} + iK\hat{\mathbf{f}})$. For homogeneous waves, i.e. waves traveling in non-absorbing media, the $\hat{\mathbf{e}}$ and $\hat{\mathbf{f}}$ are parallel, with a special case where $K\hat{\mathbf{f}} = 0$ and $\tilde{\mathbf{k}}$ is real \mathbf{k} .

For inhomogeneous waves we can write

$$\tilde{\mathbf{E}}(\mathbf{x}, t) = \tilde{\mathbf{E}}_0 \exp(-k_0K\hat{\mathbf{f}} \cdot \mathbf{x}) \exp(ik_0N\hat{\mathbf{e}} \cdot \mathbf{x} - i\omega t), \quad (10)$$

where $\tilde{\mathbf{E}}_0 \exp(-k_0K\hat{\mathbf{f}} \cdot \mathbf{x})$ is the amplitude of the wave, and $k_0N\hat{\mathbf{e}} \cdot \mathbf{x} - \omega t$ is the phase.

Now, following Chang et al. (2005) and the note "Inhomogeneous plane waves" by Muinonen & Lindqvist, I will list the equations needed for generalized Snell's law and Fresnel coefficients in the case of absorbing media.

Incident wave travels in medium 1 with refractive index $\tilde{n}_1 = n_1 + i\kappa_1$. The wave vector of the incident wave is

$$\tilde{\mathbf{k}} = k_0(N_1\hat{\mathbf{e}}_i + iK_1\hat{\mathbf{f}}_i), \quad (11)$$

where N_1 and K_1 are so-called apparent refractive index components and $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{f}}_i$ are the directions of the incident wave. The angles of incidences with the interface normal $\hat{\mathbf{n}}$ are:

$$\theta_i = \arccos(-\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}) \quad (12)$$

$$\psi_i = \arccos(-\hat{\mathbf{f}} \cdot \hat{\mathbf{n}}) \quad (13)$$

and the angle between \hat{e} and \hat{f} projected into plane with normal \hat{n} is

$$\varphi_i = \arccos \left(\frac{\mathbf{e}_n \cdot \mathbf{f}_n}{\|\mathbf{e}_n\| \|\mathbf{f}_n\|} \right), \text{ where} \quad (14)$$

$$\mathbf{e}_n = (-\hat{e} \cdot \hat{n})\hat{n} + \hat{e} \quad (15)$$

$$\mathbf{f}_n = (-\hat{f} \cdot \hat{n})\hat{n} + \hat{f} \quad (16)$$

With definition $\cos \alpha = \hat{e} \cdot \hat{f}$, we can solve apparent refractive index:

$$N_1 = \sqrt{\frac{1}{2} \left(n_1^2 - \kappa_1^2 + \sqrt{(n_1^2 - \kappa_1^2)^2 + \frac{4n_1^2 \kappa_1^2}{\cos^2 \alpha}} \right)} \quad (17)$$

$$K_1 = \sqrt{N_1^2 - n_1^2 + \kappa_1^2} \quad (18)$$

To solve apparent refractive index of the medium 2 we need to solve the quadratic equation:

$$-N_2^2(-K_2^2 + K_s^2 + N_2^2 + N_s^2) + N_s^2(N_2^2 - K_2^2) - (K_2 N_2 - K_s N_s \cos(\varphi_i))^2 + K_s^2 N_s^2 + N_2^4 = 0, \quad (19)$$

where $N_s = N_1 \sin \theta_i$ and $K_s = K_1 \sin \psi_i$. It seems to me that the physically realistic solution is

$$N_2 = \frac{1}{\sqrt{2}} \text{sqrt} \left\{ -\kappa_2^2 + \text{sqrt} \left[\kappa_2^4 + 2\kappa_2^2(-K_s^2 + n_2^2 + N_s^2) + 2K_s N_s (K_s N_s \cos(2\varphi_i) - 4\kappa_2 n_2 \cos(\varphi_i)) + (K_s^2 + n_2^2)^2 - 2n_2^2 N_s^2 + N_s^4 \right] + K_s^2 + n_2^2 + N_s^2 \right\} \quad (20)$$

and then $K_2 = \sqrt{N_2^2 - n_2^2 + \kappa_2^2}$.

The directions of reflected wave are given by

$$\hat{e}_r = \hat{e}_i + 2 \cos(\theta_i)\hat{n}, \quad \hat{f}_r = \hat{f}_i + 2 \cos(\psi_i)\hat{n} \quad (21)$$

and, when angles of transmitted wave are first solved from Snell's law: $N_1 \sin \theta_i = N_2 \sin \theta_t$, $K_1 \sin \psi_i = K_2 \sin \psi_t$, and $\varphi_t = \varphi_i$, the directions of transmitted wave are given by

$$\hat{e}_t = \frac{N_1}{N_2} \hat{e}_i + \left(\frac{N_1}{N_2} \cos \theta_i - \cos \theta_t \right) \hat{n}, \quad \hat{f}_t = \frac{K_1}{K_2} \hat{f}_i + \left(\frac{K_1}{K_2} \cos \psi_i - \cos \psi_t \right) \hat{n} \quad (22)$$

Now that the directions $\hat{e}_r, \hat{f}_r, \hat{e}_t, \hat{f}_t$ and apparent index for transmitted wave N_2, K_2 are worked out, we can move to the Fresnel coefficients. Generalized coefficients are:

$$\tilde{r}_\perp = \frac{\tilde{k}_i - \tilde{k}_t}{\tilde{k}_i + \tilde{k}_t}, \quad \tilde{t}_\perp = \frac{2\tilde{k}_i}{\tilde{k}_i + \tilde{k}_t} \quad (23)$$

$$\tilde{r}_\parallel = \frac{\tilde{n}_2^2 \tilde{k}_i - \tilde{n}_1^2 \tilde{k}_t}{\tilde{n}_2^2 \tilde{k}_i + \tilde{n}_1^2 \tilde{k}_t}, \quad \tilde{t}_\parallel = \frac{2\tilde{n}_1 \tilde{n}_2 \tilde{k}_i}{\tilde{n}_2^2 \tilde{k}_i + \tilde{n}_1^2 \tilde{k}_t}, \quad (24)$$

where $\tilde{k}_i = (N_1 \cos \theta_i + iK_1 \cos \psi_i)$ and $\tilde{k}_t = (N_2 \cos \theta_t + iK_2 \cos \psi_t)$.

The wave is also attenuated while it travels inside absorbing media. During a path with length l (in size parameter units) along the direction of propagation \hat{e} , the wave is attenuated by a factor of

$$\exp(-Kl \cos \alpha), \quad (25)$$

and the influence in Stokes vector of the wave is that $\mathbf{I} = \exp(-Kl \cos \alpha)^2 \mathbf{I}_0$.

Ex. 4

Program these generalized Snell's law and Fresnel coefficients so that you can call something like `AbsSnell($\hat{e}_i, \hat{f}_i, \hat{n}, \tilde{n}_1, \tilde{n}_2$)` and get $\hat{e}_r, \hat{f}_r, \hat{e}_t, \hat{f}_t, N_2, K_2$ and then call something like `AbsFresnel($\hat{e}_i, \hat{f}_i, \hat{n}, \hat{e}_t, \hat{f}_t, \tilde{n}_1, \tilde{n}_2, \tilde{N}_2$)` and get $\tilde{r}_\perp, \tilde{r}_\parallel, \tilde{t}_\perp, \tilde{t}_\parallel$.

Mie scattering

See course web page at <http://wiki.helsinki.fi/display/53919/>, there are several possible Mie programs for you to use. Choose one suitable to your platform. Notice that some compute scattering amplitude matrix, some also scattering (mueller) matrix. Some give (extinction, scattering, absorption) cross-sections (C_{ext} etc.), some efficiencies (Q_{ext} etc.).

Ex. 5

Learn how to use the code and how it produces output. Then, compute Q_{ext} and scattering matrix \mathbf{S} for ice sphere with $r = 0.25 \mu\text{m}$ in vacuum. Use wavelength of $0.55 \mu\text{m}$, where the refractive index of ice is $n = 1.311 + i 2.289 \times 10^{-9}$.

- i) Plot intensity (S_{11}) and degree of linear polarization ($-100\% S_{21}/S_{11}$) as a function of phase or scattering angle. You might need logarithmic scale for intensity.
- ii) Try if you can create a polar plot of the intensity, again probably in log-scale.
- iii) Compute and plot Q_{ext} for a range of sizes (or size parameters X , say X from 0.5 to 20). Try to remember what is Q_{ext} and think what does this plot tell us.