Computational light scattering, fall 2020 (PAP315, 5 cr)
Exercise 4 (Introduction to multiple scattering)

1. (12p) Consider a plane-parallel medium of normal optical thickness τ_* composed of scatterers with a single-scattering albedo $\tilde{\omega}$ and a single-scattering phase function P. Show that, in the first-order multiple-scattering approximation, the diffuse upward (I_1^+) and downward intensities (I_1^-) are ($\tau \in [0, \tau_*]$)

$$\begin{aligned} I_1^+(\tau;\mu,\phi) &= \frac{\tilde{\omega}}{4\pi}\pi F_0 P(\mu,\phi;-\mu_0,\phi_0)\frac{\mu_0}{\mu+\mu_0}\left\{\exp\left[-\frac{\tau}{\mu_0}\right]-\exp\left[-\frac{\tau_*-\tau}{\mu}-\frac{\tau_*}{\mu_0}\right]\right\}, \\ I_1^-(\tau;-\mu,\phi) &= \frac{\tilde{\omega}}{4\pi}\pi F_0 P(-\mu,\phi;-\mu_0,\phi_0)\cdot\begin{cases} \frac{\tau}{\mu_0}\exp\left[-\frac{\tau}{\mu_0}\right], & \mu=\mu_0, \\ \frac{\mu_0}{\mu-\mu_0}\left\{\exp\left[-\frac{\tau}{\mu}\right]-\exp\left[-\frac{\tau}{\mu_0}\right]\right\}, & \mu\neq\mu_0, \end{cases} \end{aligned}$$

where $\mu = \cos\theta$ and $\mu_0 = \cos\theta_0$ and where (θ, ϕ) and (θ_0, ϕ_0) describe the directions of diffuse reflection and incidence, respectively. Here πF_0 is the incident flux density. Discuss the limiting case of a semi-infinite plane-parallel medium with $\tau_* \to \infty$.

2. (9 points) next page

3. (12 points) next next page

(9p) The generalized two-stream approximation can be expressed by

$$\frac{dF^{\uparrow}(\tau)}{d\tau} = \gamma_1 F^{\uparrow}(\tau) - \gamma_2 F^{\downarrow}(\tau) - \gamma_3 \tilde{\omega} F_{\odot} e^{-\tau/\mu_o}$$
$$\frac{dF^{\downarrow}(\tau)}{d\tau} = \gamma_2 F^{\uparrow}(\tau) - \gamma_1 F^{\downarrow}(\tau) + (1-\gamma_3) \tilde{\omega} F_{\odot} e^{-\tau/\mu_o}$$

Starting from the following RT equation,

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\tilde{\omega}}{2} \int_{-1}^{1} I(\tau,\mu^{'}) P(\mu,\mu^{'}) d\mu' - \frac{\tilde{\omega}}{4\pi} F_{\odot} P(\mu,-\mu_o) e^{-\tau/\mu_o} \tag{5}$$

Represent the specific intensity $I(\tau,\mu)$ and phase function $P(\mu,\mu^{'})$ by Legendre polynomials $P_l(\mu)$ up to degree N,

$$I(\tau,\mu) = \sum_{l=0}^{N} I_l(\tau) P_l(\mu).$$

$$P(\mu,\mu^{'}) = \sum_{l=0}^{N} P_l(\mu) P_l(\mu^{'}).$$

Show that when $N = 1$, equation (5) is equivalent to the generalized two-stream approximation and give out explicit expressions for the coefficients γ_i.

Hint: Use the orthogonal and recurrence properties of Legendre polynomials.

(12p) A plane wave is scattered by two small interacting spherical particles of radius $a \ll \lambda$, where λ is the wavelength of the incident plane wave (size parameter $x = 2\pi a/\lambda$). One particle is set to the origin and the location of the other particle is denoted by a vector $\mathbf{d}$. In the dipole approximation, the internal fields of the particles $\mathbf{E}_1$ and $\mathbf{E}_2$ are related through

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_{i1} + \beta \bar{\mathbf{T}}(u, v) \cdot \mathbf{E}_2, \\ \mathbf{E}_2 &= \mathbf{E}_{i2} + \beta \bar{\mathbf{T}}(u, v) \cdot \mathbf{E}_1, \end{aligned} \tag{6}$$

where $\mathbf{E}_{i1}$ and $\mathbf{E}_{i2}$ are the incident fields at the locations of the particles, and the polarizability (m is the refractive index)

$$\beta = x^3 \frac{m^2 - 1}{m^2 + 2}. \tag{7}$$

The transformation $\bar{\mathbf{T}}$ denotes the interaction between the particles:

$$\begin{aligned} \bar{\mathbf{T}}(u, v) &= u\bar{\mathbf{I}} + v\mathbf{dd}/d^2, \\ u &= e^{\mathrm{i}\rho}(\rho^2 + \mathrm{i}\rho - 1)/\rho^3, \\ v &= e^{\mathrm{i}\rho}(-\rho^2 - \mathrm{i}3\rho + 3)/\rho^3, \ \rho = kd. \end{aligned} \tag{8}$$

Solve the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$. (See Muinonen 1990, PhD thesis.)

Note the following rules for an operator $\mathbf{ab}$:
$(\mathbf{ab}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$
$\mathbf{c} \cdot (\mathbf{ab}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$
The unit operator $\bar{\mathbf{I}}$ has no effect on the operator $\mathbf{ab}$ or the vector $\mathbf{c}$.