1 Lecture 13

2 Scattering at the short-wavelength limit. Scalar diffraction theory.

Traditionally, diffraction entails those deviations from geometric optics that derive from the finite wavelength of the waves. Thereby, diffraction is connected to objects (e.g., holes, obstacles) that are large compared to the wavelength. The possible geometries are described in the figure below (see Jackson). The sources of the radiation are located in region I and we want to derive the diffracted fields in the diffraction region II. The regions are bounded by the interfaces S_1 and S_2. Kirchhoff was the first one to treat this topic systematically.

For simplicity, we will first study scalar fields, whereafter we will extend the analysis to vector fields. Let $\psi(\mathbf{x}, t)$ be a scalar field, for which we assume a harmonic time dependence $e^{-i\omega t}$. In essence, ψ is one of the components of the $\mathbf{E}$ or $\mathbf{B}$ fields. We assume that ψ fulfils the scalar Helmholtz wave equation

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0$$

in the volume V bounded by S_1 and S_2. We introduce the Green's function $G(\mathbf{x}, \mathbf{x}')$,

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$$

and start from Green's theorem

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3\mathbf{x}' = \oint_S [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]dA'$$

$$\frac{\partial\psi}{\partial n} \equiv \mathbf{n}' \cdot \nabla\psi$$

where $\mathbf{n}'$ is the unit inward normal vector of S. Let us now set $\psi = G$ and $\phi = \psi$ so that, with the help of the wave equations for ψ and G,

$$\psi(\mathbf{x}) = \oint_S dA'[\psi(\mathbf{x}')\mathbf{n}' \cdot \nabla' G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\mathbf{n}' \cdot \nabla'\psi(\mathbf{x}')]$$

Kirchhoff's diffraction integral follows from this relation when G is chosen to be the free-space Green's function describing outgoing waves,

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{ikR}}{4\pi R}, \qquad \mathbf{R} = \mathbf{x} - \mathbf{x}', R = |\mathbf{R}|$$

Then

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S dA' \frac{e^{ikR}}{R}\mathbf{n}' \cdot [\nabla'\psi + ik(1 + \frac{i}{kR})\frac{\mathbf{R}}{R}\psi]$$

The surface S is composed of S_1 and S_2 and the integration can be divided into two parts. In the proximity of S_2, ψ is an outgoing wave and fulfils the so-called radiation condition

$$\psi \to f(\theta,\varphi)\frac{e^{ikr}}{r}, \qquad \frac{1}{\psi}\frac{\partial\psi}{\partial r} \to (ik - \frac{1}{r}).$$

By inserting these results into the integral above, it is possible to show that the integral over S_2 vanishes at least as the inverse of the radius of the sphere when the radius approaches infinity. There remains the integral over S_1, giving the final form of the Kirchhoff integral relation,

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} dA' \frac{e^{ikR}}{R}\mathbf{n}' \cdot [\nabla'\psi + ik(1 + \frac{i}{kR})\frac{\mathbf{R}}{R}\psi]$$

In applying the integral relation, it is necessary to know both ψ and $\partial\psi/\partial n$ on the surface S_1. In general, these are not known, at least not precisely. Kirchhoff's approach was based on the idea that ψ and $\partial\psi/\partial n$ are approximated on S_1 for the computation of the diffracted wave. This so-called Kirchhoff's approximation consists of the following assumptions:

1. ψ and $\partial\psi/\partial n$ vanish everywhere else but the holes of S_1
2. ψ and $\partial\psi/\partial n$ in the holes are equal to the original field values when there are no diffracting elements in space.

These assumptions contain a serious mathemtical inconsistency: if ψ and $\partial\psi/\partial n$ are zero on a finite surface, then $\psi = 0$ everywhere. In spite of the inconsistency, the Kirchhoff approximation works in an excellent way in practical problems and constitutes the basis of all diffraction calculus in classical optics.

The mathematical inconsistencies can be removed by a proper choice of the Green's function. In the setup of the figure below (see Jackson), (both P and P' are located several wavelengths away from the hole) we obtain

$$\psi(P) = \frac{k}{2\pi i}\int_{S_1} dA' \frac{e^{ikr}}{r}\frac{e^{ikr'}}{r'}\mathcal{O}(\theta,\theta')$$

$$\mathcal{O}(\theta,\theta') = \begin{cases} \cos\theta, & ; \\ \cos\theta', & ; \\ \frac{1}{2}(\cos\theta + \cos\theta'), & \text{(Kirchhoffin approksimaatio)}. \end{cases}$$

The obliquity factor $\mathcal{O}(\theta,\theta')$ assumes less significance than the phase factors, which partly explains the success of the Kirchhoff approximation.

3 Vector Kirchhoff integral relation

The scalar Kirchhoff integral relation is an exact relation between the scalar fields on the surface and at infinity. In a corresponding way, the vector Kirchhoff integral relation is an exact relation between the $\mathbf{E}, \mathbf{B}$ fields on the surface S and the diffracted or scattered fields at infinity. Such a relation is interesting in itself and it is a correct guess that the relation carries practical significance, too.

In what follows, we derive the vector relation for the electric field $\mathbf{E}$, starting from the generalization of Green's theorem already appearing in the scalar case for all components of the $\mathbf{E}$-field,

$$\mathbf{E}(\mathbf{x}) = \oint_S dA'[\mathbf{E}(\mathbf{n}' \cdot \nabla' G) - G(\mathbf{n}' \cdot \nabla')\mathbf{E}],$$

when $\mathbf{x} \in V$ and V is the volume bounded by S. Again, $\mathbf{n}'$ is the unit normal vector pointing into the volume V. Since G is singular at $\mathbf{x}' = \mathbf{x}$ and we make use of vector calculus valid for smooth functions, we assume that S is composed of the outer surface S' and an infinitesimally small inner surface S'' so that the point $\mathbf{x}' = \mathbf{x}$ is left out from volume V (but the point is inside S''). In such a case, the left-hand side of the previous equation disappears, but the integration over S'' on the right-hand side returns $-\mathbf{E}(\mathbf{x})$ when the radius of S'' goes to zero.

The vector relation can now be written in the form

$$0 = \oint_S dA'[2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) - \mathbf{n}' \cdot \nabla'(G\mathbf{E})]$$

and, with the help of the divergence theorem ja divergenssiteoreeman

$$\int_V dV' \nabla \cdot A = \oint_S dA' \mathbf{A} \cdot \mathbf{n},$$

the latter term can be transformed to a volume integral

$$0 = \oint_S dA' 2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) + \int_V dV' \nabla'^2 (G\mathbf{E})$$

Now

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

$$\int_V dV \nabla\phi = \oint_S dA \mathbf{n}\phi, \qquad (\mathbf{n} \text{ ulkonormaali})$$

$$\int_V dV \nabla \times \mathbf{A} = \oint_S dA (\mathbf{n} \times \mathbf{A})$$

and the volume integral can be returned back to a surface integral

$$0 = \oint_S dA'[2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) - \mathbf{n}'(\nabla' \cdot (G\mathbf{E})) + \mathbf{n}' \times (\nabla' \times (G\mathbf{E}))]$$

When the ∇-operations are carried out for $G\mathbf{E}$ and use is made of Maxwell's equations $\nabla' \cdot \mathbf{E} = 0, \nabla' \times \mathbf{E} = i\omega\mathbf{B}$, one obtains

$$0 = \oint_S dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]$$

and, furthermore,

$$\mathbf{E}(\mathbf{x}) = \oint_S dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]$$

where the volume bounded by S now again includes the point $\mathbf{x}$.

As in the case of the scalar relation, we can now derive the vector Kirchhoff integral relation

$$\mathbf{E}(\mathbf{x}) = \oint_{S_1} dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G],$$

where the integration extends over S_1 only.

Finally, we derive a relation between the scattering amplitude and the near fields. For the fields in the vector Kirchhoff integral relation, we choose the scattered fields $\mathbf{E}_s, \mathbf{B}_s$, that is, the total fields $\mathbf{E}, \mathbf{B}$ minus the original fields $\mathbf{E}_i, \mathbf{B}_i$. If the observation point is far away from the scatterer, both the Green's function and the scattered electric field can be given in their asymptotic forms

$$G(\mathbf{x}, \mathbf{x}') \Longrightarrow \frac{1}{4\pi}\frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

$$\mathbf{E}_s(\mathbf{x}) \to \frac{e^{ikr}}{r}\mathbf{F}(\mathbf{k}, \mathbf{k}_0)$$

where $\mathbf{k}$ is a wave vector pointing in the direction of the observer, $\mathbf{k}_0$ is the wave vector of the original field, and $\mathbf{F}(\mathbf{k}, \mathbf{k}_0)$ is the vector scattering amplitude. In this limit, $\nabla' G = -i\mathbf{k}G$ and we obtain an integral relation for the scattering amplitude,

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{i}{4\pi}\oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}[\omega(\mathbf{n}' \cdot \mathbf{B}_s) + \mathbf{k} \times (\mathbf{n}' \times \mathbf{E}_s) - \mathbf{k}(\mathbf{n}' \cdot \mathbf{E}_s)]$$

The relation depends explicitly on the direction of $\mathbf{k}$ and the dependence on $\mathbf{k}_0$ is implicit in $\mathbf{E}_s$ and $\mathbf{B}_s$. Since $\mathbf{k} \cdot \mathbf{F} = 0$, we can reduce the relation to

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{1}{4\pi i}\mathbf{k} \times \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}[\frac{c\mathbf{k} \times (\mathbf{n}' \times \mathbf{B}_s)}{k} - \mathbf{n}' \times \mathbf{E}_s]$$

Alternatively, one may want the scattering amplitude in direction $\mathbf{k}$ for a specific polarization state ϵ^*,

$$\epsilon^* \cdot \mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{i}{4\pi}\oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}[\omega\epsilon^* \cdot (\mathbf{n}' \times \mathbf{B}_s) + \epsilon^* \cdot (\mathbf{k} \times (\mathbf{n}' \times \mathbf{E}_s))]$$

These integral relations are useful in scattering problems entailing short wavelengths and in the derivation of the optical theorem.

4 Diffraction by a circular aperture

Diffraction is divided into Fraunhofer and Fresnel diffraction depending on the geometry under consideration. There are three length scales involved: the size of the diffracting system d, the distance from the system to the observation point r and the wavelength λ. The diffraction pattern is generated when $r >> d$. In tht case, the slowly changing parts of the vector integral relation can be kept constant. Particular attention needs to be paid to the phase factor e^{ikR}. When $r >> d$, we obtain

$$kR = kr - k\mathbf{n}\cdot\mathbf{x}' + \frac{k}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2] + \dots, \qquad \mathbf{n} = \frac{\mathbf{x}}{r}, \tag{1}$$

where $\mathbf{n}$ is a unit vector pointing in the direction of the observer. The magnitudes of the terms in the expansion are $kr, kd, (kd)^2/kr$. In Fraunhofer diffraction, the terms from the third one (inclusive) onwards are negligible. When the third term becomes significant (e.g., large diffracting systems), we enter the domain of Fresnel diffraction. Far enough from any diffracting system, we end up in the domain of Fraunhofer diffraction.

If the observation point is far away from the diffracting system, Kirchhoff's scalar integral relation assumes the form

$$\Psi(\mathbf{x}) = -\frac{e^{ikr}}{4\pi r}\int_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}\Big[\mathbf{n}\cdot\nabla'\Psi(\mathbf{x}') + i\mathbf{k}\cdot\mathbf{n}\Psi(\mathbf{x}'), \tag{2}$$

where $\mathbf{n}$ now is the unit normal vectoron, $\mathbf{x}'$ denotes the position of the element dA', and $r = |\mathbf{x}|, \mathbf{k} = k(\mathbf{x}/r)$. The so-called Smythe-Kirchhoff integral relation is an improved version of the pure Kirchhoff relation and, in the present limit, takes the form

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}}{2\pi r}\mathbf{k}\times\int_{S_1} dA'\mathbf{n}\times\mathbf{E}(\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}'} \tag{3}$$

Let us study next what the different diffraction formulae give for a circular hole (radius a) in an infinitesimally thin perfectly conducting slab.

Figure (see Jackson)

In the vector relation,

$$(\mathbf{n}\times\mathbf{E}_i)_{z=0} = E_0\epsilon_2\cos\alpha e^{ik\sin\alpha x'} \tag{4}$$

and, in polar coordinates,

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}E_0\cos\alpha}{2\pi r}(\mathbf{k}\times\epsilon_2)\int_0^a d\zeta\zeta\int_0^{2\pi} d\beta e^{ik\zeta[\sin\alpha\cos\beta - \sin\theta\cos(\varphi-\beta)]} \tag{5}$$

Define

$$\xi \equiv \frac{1}{k}|\mathbf{k}_\perp - \mathbf{k}_{0,\perp}| = \sqrt{\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\varphi}, \tag{6}$$

in which case the integral takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-ik\zeta\xi\cos\beta'} = J_0(k\zeta\xi) \tag{7}$$

that is, the result is the Bessel function J_0. Hereafter, the integration over the radial part can be calculated analytically, and

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}}{r}a^2 E_0 \cos\alpha(\mathbf{k}\times\epsilon_2)\frac{J_1(ka\xi)}{ka\xi} \tag{8}$$

The time-averaged power as per unit solid angle is then

$$\frac{dP}{d\Omega} = P_i \cos\alpha\frac{(ka)^2}{4\pi}(\cos^2\theta + \cos^2\varphi\sin^2\theta)|\frac{2J_1(ka\xi)}{ka\xi}|^2 \tag{9}$$

$$P_i = (E_0^2/2z_0)\pi a^2\cos\alpha, \tag{10}$$

where P_i is the total power normally incident on the hole. If $ka >> 1$, the function $[(2J_1(ka\xi)/ka\xi)^2]$ peaks sharply at 1 with the argument $\xi = 0$ and falls down to zero at $\Delta\xi \approx 1/ka$. The main part of the wave propagates according to geometric optics and only modest diffraction effects show up. If, however, $ka \approx 1$, the Bessel function varies slowly as a function of the angles and the transmitted wave bends into directions considerably deviting from the propagation direction of the incident field. In the extreme limit $ka << 1$, the angular dependence derives from the polarization factor $\mathbf{k}\times\epsilon_2$, but the analysis fails because the field in the hole can no longer be the original undisturbed field as assumed earlier.

let us study the scalar solution assuming that Ψ corresponds the magnitude of the $\mathbf{E}$ field,

$$\begin{aligned}\Psi(\mathbf{x}) &= -ik\frac{e^{ikr}}{r}a^2E_0\frac{1}{2}(\cos\alpha+\cos\theta)\frac{J_1(ka\xi)}{ka\xi} \\ \frac{dP}{d\Omega} &\cong P_i\frac{(ka)^2}{4\pi}\cos\alpha(\frac{\cos\alpha+\cos\theta}{2\cos\alpha})|\frac{2J_1(ka\xi)}{ka\xi}|^2\end{aligned} \tag{11}$$

Both the vector and scalar results include the Bessel part $[(2J_1(ka\xi)/ka\xi)^2]$ and the same wave number dependence. But whereas there is no azimuthal dependence in the scalar result, the vector result is significantly affected by the azimuthal dependence. The dependence derives from the polarization of the vector field. For an original field propagating in the direction of the normal vector, the polarization effects are not important, when additionally $ka >> 1$. Then, all the results reduce into the familiar expression

$$\frac{dP}{d\Omega} \cong P_i\frac{(ka)^2}{\pi}|\frac{J_i(ka\sin\theta)}{ka\sin\theta}|^2 \tag{12}$$

However, for oblique directions, there are large deviations and, for very small holes, the analysis fails completely.

5 Scattering in detail

Let us now consider a small particle that is much larger than the wavelength and study what kind of tools the vector Kirchhoff integral relation offers, if the fields close to the surface can be estimated somehow.

For example, the surface of the scatterer is divided into the illuminated and shadowed parts. The boundary between the two parts is sharp only in the limit of geometric optics and, in the transition zone, the breadth of the boundary is of the order of $(2/kR)^{1/3} \cdot R$, where R is a typical radius of curvature on the surface of the particle.

On the shadow side, the scattered field must be equal to the original field but opposite in sign, in which case the total field vanishes. On the illuminated side, the field depends in a detailed way on the properties of the scattering particle. If the curvature radii are large compared to the wavelength, we can make use of Fresnel's coefficients and geometric optics in general. The analysis can be generalized into the case of a transparent particle and the method is known as the physical-optics approximation (or Kirchhoff approximation).

Let us write the scattering amplitude explicitly in two parts,

$$\epsilon^* \cdot \mathbf{F} = \epsilon^* \cdot \mathbf{F}_{sh} + \epsilon^* \cdot \mathbf{F}_{ill} \tag{13}$$

and assume that the incident fields is a plane wave

$$\begin{aligned} \mathbf{E}_i &= E_0 \epsilon_0 e^{i\mathbf{k}_0 \cdot \mathbf{x}} \\ \mathbf{B}_i &= \mathbf{k}_0 \times \mathbf{E}_i / kc \end{aligned}$$

The shadow scattering amplitude is then ($\mathbf{E}_s \approx -\mathbf{E}_i, \mathbf{B}_s \approx -\mathbf{B}_i$)

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{E_0}{4\pi i} \int_{sh} dA' \epsilon^* \cdot [\mathbf{n}' \times (\mathbf{k}_0 \times \epsilon_0) + \mathbf{k} \times (\mathbf{n}' \times \epsilon_0)] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'} \tag{14}$$

where the integration is over the shadowed region. The amplitude can be rearranged into the form

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{E_0}{4\pi i} \int_{sh} dA' \epsilon^* \cdot [(\mathbf{k} + \mathbf{k}_0) \times (\mathbf{n}' \times \epsilon_0) + (\mathbf{n}' \cdot \epsilon_0)\mathbf{k}_0] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'} \tag{15}$$

In the short-wavelength limit, $\mathbf{k}_0 \cdot \mathbf{x}'$ and $\mathbf{k} \cdot \mathbf{x}'$ vary across a large regime and the exponential factor fluctuates rapidly and eliminates the integral everywhere else but the forward-scattering direction $\mathbf{k} \approx \mathbf{k}_0$. In that direction ($\theta \lesssim 1/kR$), the second factor is negligible compared to the first one since $(\epsilon^* \cdot \mathbf{k}_0)/k$ is of the order of $\sin\theta << 1, (\epsilon^* \cdot \mathbf{k} = 0, \mathbf{k}_0 \approx \mathbf{k})$. Thus,

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{iE_0}{2\pi} (\epsilon^* \cdot \epsilon_0) \int_{sh} dA' (\mathbf{k}_0 \cdot \mathbf{n}') e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'} \tag{16}$$

In this approximation, the integral over the shadow side only depends on the projected area against the propagation direction of the original field. This can be seen from the fact that

$\mathbf{k}_0 \cdot \mathbf{n}' dA' = k dx' dy' = k d^2 \mathbf{x}'_\perp$ ja $(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}' = k(1-\cos\theta) z' - \mathbf{k}_\perp \cdot \mathbf{x}'_\perp \approx -\mathbf{k}_\perp \cdot \mathbf{x}'_\perp$. The final form of the shadow scattering amplitude is thus

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{ik}{2\pi} E_0 (\epsilon^* \cdot \epsilon_0) \int_{sh} d^2 \mathbf{x}'_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}'_\perp} \tag{17}$$

In this limit, all scatterers producing the same prohjected area will have the same shadow scattering amplitude. For example, in the case of a circular cylindrical slab (radius a)

$$\int_{sh} d^2 \mathbf{x}'_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}'_\perp} = 2\pi a^2 \frac{J_1(ka \sin\theta)}{ka \sin\theta}, \tag{18}$$

$$\epsilon^* \cdot \mathbf{F}_{sh} \cong ika^2 E_0 (\epsilon^* \cdot \epsilon_0) \frac{J_1(ka \sin\theta)}{ka \sin\theta}. \tag{19}$$

This explains nicely the forward diffraction pattern in scattering by small particles.

The scattering amplitude due to the illuminated side of the scatterer cannot be calculated without defining the shape and optical properties of the particle. Let us assume in the following example that the illuminated region is perfectly conducting. Then, the tangential components of the fields $\mathbf{E}_s$ and $\mathbf{B}_s$ on S_1 are approximately opposite and similar to those of the original fields, respectively. The scattering amplitude due to the illuminated part is then

$$\epsilon^* \cdot \mathbf{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill} dA' \epsilon^* \cdot [-\mathbf{n}' \times (\mathbf{k}_0 \times \epsilon_0) + \mathbf{k} \times (\mathbf{n}' \times \epsilon_0)] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'}$$

When this is compared with the shadow amplitude, the only notable difference is the sign in the first term. This sign difference results in a completely different scattering amplitude that can also be written in the form muodossa

$$\epsilon^* \cdot \mathbf{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill} dA' \epsilon^* \cdot [(\mathbf{k} - \mathbf{k}_0) \times (\mathbf{n}' \times \epsilon_0) - (\mathbf{n}' \cdot \epsilon_0) \mathbf{k}_0] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'}$$

When again $kR >> 1$, the exponential factor fluctuates rapidly and one would expect a strong contribution in the forward direction; however, the first term goes to zero in the forward direction and no strong contribution can follow. The illuminated region contributes to scattering in the form of a reflected wave.

Assume next that the scattering particle is spherical (radius a). The predominating contribution to the scattering amplitude now derives from a region of integration where the phase of the exponential factor is stationary. If (θ, φ) are the coordinates of $\mathbf{k}$ and (α, β) those of $\mathbf{n}'$ (with respect to $\mathbf{k}_0$), the phase factor is

$$\phi(\alpha, \beta) = (\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}' = ka[(1-\cos\theta)\cos\alpha - \sin\theta \sin\alpha \cos(\beta - \varphi)]$$

The stationary point can be found at angles α_0, β_0, where $\alpha_0 = \pi/2 + \theta/2$ and $\beta_0 = \varphi$. These angles correspond exactly to the angles of reflection on the surface of the sphere as dictated by geometric optics. At that point, the vector $\mathbf{n}'$ points in the direction of $(\mathbf{k} - \mathbf{k}_0)$. In the proximity of angles $\alpha = \alpha_0$ and $\beta = \beta_0$

$$\phi(\alpha, \beta) = -2ka \sin\frac{\theta}{2}[1 - \frac{1}{2}(x^2 + \cos^2\frac{\theta}{2}y^2) + \ldots]$$

where $x = \alpha - \alpha_0$ and $y = \beta - \beta_0$. The integration can be carried out approximately:

$$\epsilon^* \cdot \mathbf{F}_{ill} \cong ka^2 E_0 \sin\theta e^{-2ika\sin\frac{\theta}{2}}(\epsilon^* \cdot \epsilon_r) \cdot \int dx e^{i[ka\sin\frac{\theta}{2}]x^2} \int dy e^{i[ka\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}]y^2}$$

$$\epsilon_r = -\epsilon_0 + 2(\mathbf{n}_r \cdot \epsilon_0)\mathbf{n}_r, \qquad \mathbf{n}_r = \frac{\mathbf{k} - \mathbf{k}_0}{|\mathbf{k} - \mathbf{k}_0|}$$

When $2ka\sin\frac{\theta}{2} >> 1$, the integrals can be calculated using the result $\int_{-\infty}^{\infty} dx e^{i\alpha x^2} = \sqrt{\pi i/\alpha}$,

$$\epsilon^* \cdot \mathbf{F}_{ill} \cong E_0 \frac{a}{2} e^{-2ika\sin\frac{\theta}{2}} \epsilon^* \cdot \epsilon_r$$

For large $2ka\sin\frac{\theta}{2}$, the intensity of the reflected part of the radiation is constant as a function of the angle, but the part has a rapidly varying phase. When $\theta \to 0$, the intensity vanishes as θ^2 (see the integral above).

Comparison of the amplitudes due to the shadowed and illuminated parts of the surface shows that, in the forward direction, the former amplitude predominates over the latter by a factor $ka >> 1$ whereas, at the scattering angles $2ka\sin\theta >> 1$, the ratio of the amplitudes is of the order of $1/(ka\sin^3\theta)^{1/2}$. The differential scattering cross section (summed over the polarization states of the original and scattered waves) is

$$\frac{d\sigma}{d\Omega} \cong \begin{cases} a^2(ka)^2|\frac{J_1(ka\sin\theta)}{ka\sin\theta}|^2, & \theta \lesssim \frac{10}{ka} \quad ; \\ \frac{a^2}{4}, \quad \theta >> \frac{1}{ka} & . \end{cases}$$

The total scattering cross section is twice the geometric cross section of the particle.

6 Optical theorem

The optical theorem is a fundamental relation that connects the exticntion cross section to the imaginary part of the forward-scattering amplitude. Consider a plane wave with a wave vector $\mathbf{k}_0$ and field components $\mathbf{E}_i, \mathbf{B}_i$. The plane wave is incident on a finite-sized scatterer inside the surface S_1. The scattered field $\mathbf{E}_s, \mathbf{B}_s$ propagates away from the scatterer and is observed in the far zone in the direction $\mathbf{k}$. The total field outside the surface S_1 is, by definition,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_i + \mathbf{E}_s \\ \mathbf{B} &= \mathbf{B}_i + \mathbf{B}_s. \end{aligned}$$

In the general case, the scatterer absorbs energy from the original field. The absorbed power can be calculated by integrating the inward-directed Poynting-vector component of the total field over the surface S_1:

$$P_{abs} = -\frac{1}{2\mu_0}\oint_{S_1} dA'\mathrm{Re}(\mathbf{E}\times\mathbf{B}^*)\cdot\mathbf{n}'$$

The scattered power is computed in the usual way from the asymptotic form of the Poynting vector for the scattered fields in the regime, where the fields are simple transverse spherical waves that attenuate as $1/r$. But since there are no sources between S_1 and infinity, the scattered power can as well be calculated as an integral of the outward-directed component of the Poynting vector for the scattered field over S_1:

$$P_{sca} = \frac{1}{2\mu_0}\oint_{S_1} dA'\mathrm{Re}(\mathbf{E}_s\times\mathbf{B}_s^*)\cdot\mathbf{n}'$$

The total power is the sum of the absorbed and scattered power so that, after rearranging,

$$P = P_{abs} + P_{sca} = -\frac{1}{2\mu_0}\oint_{S_1} dA'\mathrm{Re}(\mathbf{E}_s\times\mathbf{B}_i^* + \mathbf{E}_i^*\times\mathbf{B}_s)\cdot\mathbf{n}'$$

When the original field in written explicitly in the form

$$\begin{aligned}\mathbf{E}_i &= E_0\epsilon_0 e^{i\mathbf{k}_0\cdot\mathbf{x}}\\ c\mathbf{B}_i &= \frac{1}{k}\mathbf{k}_0\times\mathbf{E}_i\end{aligned}$$

the total power can be transformed to the form

$$P = \frac{1}{2\mu_0}ReE_0^*\oint_{S_1} dA' e^{-i\mathbf{k}_0\cdot\mathbf{x}}[\epsilon_0^*\cdot(\mathbf{n}'\times\mathbf{B}_s) + \epsilon_0^*\cdot\frac{\mathbf{k}_0\times(\mathbf{n}'\times\mathbf{E}_s)}{kc}]$$

By comparing this with the scattering amplitude $\mathbf{F}(\mathbf{k},\mathbf{k}_0)$ derived earlier, we can recognize that the total power is proportional to the value of $\mathbf{F}$ in the forward-scattering direction $\mathbf{k}=\mathbf{k}_0$ in the polarization state coinciding with that of the original field:

$$P = \frac{2\pi}{kZ_0}\mathrm{Im}[E_0^*\epsilon_0^*\cdot\mathbf{F}(\mathbf{k}=\mathbf{k}_0)],$$

which is the basic form of the optical theorem.

The total or extinction cross section σ_e is defined as the ratio of the total and original flux densities ($|E_0|^2/2Z_0$, power as per unit surface area).

In a corresponding way, one can define a normalized scattering amplitude $\mathbf{f}$ (against the original field value at origin)

$$f(\mathbf{k}=\mathbf{k}_0) = \frac{\mathbf{F}(\mathbf{k},\mathbf{k}_0)}{E_0}$$

The final form of the optical theorem is then

$$\sigma_e = \frac{4\pi}{k}\mathrm{Im}[\epsilon_0^*\cdot\mathbf{f}(\mathbf{k}=\mathbf{k}_0)].$$