1 Introduction to scattering theory (lecture 2)

1.1 Electromagnetic formulation of the problem

1.2 Amplitude scattering matrix

1.3 Stokes parameters and scattering matrix

1.4 Extinction, scattering and absorption

Let us assume that medium surrounding the scattering particle is non-absorbing. The total or extinction cross section is then the sum of the absorption and scattering cross sections:

$$\sigma_e = \sigma_s + \sigma_a, \tag{1}$$

where

$$\begin{aligned} \sigma_e &= -\frac{1}{I_i}\int_A dA\boldsymbol{S}_e \cdot \boldsymbol{e}_r, \\ \sigma_s &= \frac{1}{I_i}\int_A dA\boldsymbol{S}_s \cdot \boldsymbol{e}_r, \end{aligned} \tag{2}$$

when A is a spherical envelope of radius r containing the scattering particle.

Let the original field be of $\boldsymbol{e}_x$-polarized form $\boldsymbol{E}_0 = E\boldsymbol{e}_x$. In the radiation zone,

$$\begin{aligned} \boldsymbol{E}_s &\propto \frac{\exp[\mathrm{i}k(r-z)]}{-\mathrm{i}kr}\boldsymbol{X}E, \boldsymbol{e}_r \cdot \boldsymbol{X} = 0, \\ \boldsymbol{H}_s &\propto \frac{k}{\omega\mu}\boldsymbol{e}_r \times \boldsymbol{E}_s, \end{aligned} \tag{3}$$

where the vector scattering amplitude $\boldsymbol{X}$ is related to the amplitude scattering matrix as follows:

$$\boldsymbol{X} = (S_4\cos\phi + S_1\sin\phi)\boldsymbol{e}_{s\perp} + (S_2\cos\phi + S_3\sin\phi)\boldsymbol{e}_{s\parallel}. \tag{4}$$

By making use of the asymptotic forms of the scattered field shown above and $\boldsymbol{e}_x$-polarized original field, the so-called optical theorem can be derived: extinction depends only on scattering in the exact forward direction,

$$\sigma_e = \frac{4\pi}{k^2}\mathrm{Re}[(\boldsymbol{X}\cdot\boldsymbol{e}_x)_{\theta=0}]. \tag{5}$$

In addition,

$$\sigma_s = \int_{4\pi} d\Omega \frac{d\sigma_s}{d\Omega}, \tag{6}$$

where the differential scattering cross section is

$$\frac{d\sigma_s}{d\Omega} = \frac{|\boldsymbol{X}|^2}{k^2}. \tag{7}$$

The extinction, scattering, and absorption efficiencies are defined as the ratios of the corresponding cross sections to the geometric cross section of the particle $A_\perp$ as projected in the propagation direction of the original field:

$$\begin{aligned} q_e &= \frac{\sigma_e}{A_\perp}, \\ q_s &= \frac{\sigma_s}{A_\perp}, \\ q_a &= \frac{\sigma_a}{A_\perp}. \end{aligned} \tag{8}$$

For an unpolarized original field, the cross sections are

$$\begin{aligned} \sigma_e &= \frac{1}{2}(\sigma_e^{(1)} + \sigma_e^{(2)}), \\ \sigma_s &= \frac{1}{2}(\sigma_s^{(1)} + \sigma_s^{(2)}), \end{aligned} \tag{9}$$

where the indices 1 and 2 refer to two polarization states of the original field perpendicular to one another.

2 Plane waves

The electromagnetic plane wave

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \end{aligned} \tag{10}$$

can, under certain conditions, fulfil Maxwell's equations. The physical fields correspond to the real parts of the complex-valued fields. The vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ above are constant vectors and can be complex-valued. Similarly, the wave vector $\boldsymbol{k}$ can be complex-valued:

$$\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}'', \qquad \boldsymbol{k}', \boldsymbol{k}'' \in \mathbb{R}^n \tag{11}$$

Inserting (11) into equation (10), we obtain

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{-\boldsymbol{k}'' \cdot \boldsymbol{x}} e^{i\boldsymbol{k}' \cdot \boldsymbol{x} - i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{-\boldsymbol{k}'' \cdot \boldsymbol{x}} e^{i\boldsymbol{k}' \cdot \boldsymbol{x} - i\omega t} \end{aligned} \tag{12}$$

In Eq. (12), $\boldsymbol{E}_0 e^{-\boldsymbol{k}'' \cdot \boldsymbol{x}}$ and $\boldsymbol{H}_0 e^{-\boldsymbol{k}'' \cdot \boldsymbol{x}}$ are amplitudes and $\boldsymbol{k}' \cdot \boldsymbol{x} - \omega t = \phi$ is the phase of the wave.

An equation of the form $\boldsymbol{k} \cdot \boldsymbol{x}$ =constant defines, in the case of a real-valued vector $\boldsymbol{k}$, a planar surface, whose normal is just the vector $\boldsymbol{k}$. Thus, $\boldsymbol{k}'$ is perpendicular to the planes of constant phase and $\boldsymbol{k}''$ is perpendicular to the planes of constant amplitude. If $\boldsymbol{k}' \parallel \boldsymbol{k}''$, the planes coincide and the wave is *homogeneous*. If $\boldsymbol{k}' \nparallel \boldsymbol{k}''$, the wave is *inhomogeneous*. A plane wave propagating in vacuum is homogeneous.

In the case of plane waves, Maxwell's equaitons can be written as

$$\begin{aligned} \boldsymbol{k} \cdot \boldsymbol{E}_0 &= 0 \\ \boldsymbol{k} \cdot \boldsymbol{H}_0 &= 0 \\ \boldsymbol{k} \times \boldsymbol{E}_0 &= \omega\mu \boldsymbol{H}_0 \\ \boldsymbol{k} \times \boldsymbol{H}_0 &= -\omega\epsilon \boldsymbol{E}_0 \end{aligned} \tag{13}$$

The two upmost equations are conditions for the transverse nature of the waves: $\boldsymbol{k}$ is perpendicular to both $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. The two lowermost equations show that $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are perpendicular to each other. Since $\boldsymbol{k}$, $\boldsymbol{E}_0$, and $\boldsymbol{H}_0$ are complex-valued, the geometric interpretaion is not simple unless the waves are homogeneous.

It follows from Maxwell's equations (13) that, on one hand,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = \omega\mu \boldsymbol{k} \times \boldsymbol{H}_0 = -\omega^2 \epsilon\mu \boldsymbol{E}_0 \tag{14}$$

and, on the other hand,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}_0) - \boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{k}) = -\boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{k}), \tag{15}$$

so that

$$\boldsymbol{k} \cdot \boldsymbol{k} = \omega^2 \epsilon\mu. \tag{16}$$

Plane waves solutions are in agreement with Maxwell's equations if

$$\boldsymbol{k} \cdot \boldsymbol{E}_0 = \boldsymbol{k} \cdot \boldsymbol{H}_0 = \boldsymbol{E}_0 \cdot \boldsymbol{H}_0 = 0 \tag{17}$$

and if

$$k'^2 - k''^2 + 2i\boldsymbol{k}' \cdot \boldsymbol{k}'' = \omega^2 \epsilon\mu. \tag{18}$$

Note that ϵ and μ are properties of the medium, whereas $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are properties of the wave. Thus, ϵ and μ do not unambiguously determine the details of wave propagation.
In the case of a homogeneous plane wave ($\boldsymbol{k}' \| \boldsymbol{k}''$),

$$\boldsymbol{k} = (k' + ik'')\hat{\mathbf{e}}, \tag{19}$$

where k' and k'' are non-negative and $\hat{\mathbf{e}}$ is an arbitrary real-valued unit vector.

According to Eq. (16),

$$(k' + ik'')^2 = \omega^2 \epsilon\mu = \frac{\omega^2 m^2}{c^2}, \tag{20}$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum and m is the complex-valued refractive index

$$m = \sqrt{\frac{\epsilon\mu}{\epsilon_0 \mu_0}} = m_r + im_i, \qquad m_r, m_i \geq 0. \tag{21}$$

In vacuum, the wave number is $\omega/c = 2\pi/\lambda$, where λ is the wavelength. The general homogeneous plane wave takes the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{2\pi m_i s}{\lambda}} e^{i\frac{2\pi m_r s}{\lambda} - i\omega t} \tag{22}$$

where $s = \boldsymbol{e} \cdot \boldsymbol{x}$. The imaginary and real parts of the refractive index determine the attenuation and phase velocity $v = c/m_r$ of the wave, respectively.

3 Poynting vector

Let us study the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ that is time harmonic. For the physical fields (the real parts of the complex-valued fields), the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{23}$$

describes the direction and amount of energy transfer everywhere in the space.

Let $\boldsymbol{n}$ be the unit normal vector of the planar surface element A. Electromagnetic energy is transferred through the planar surface with power $\boldsymbol{S} \cdot \boldsymbol{n}\ A$, where $\boldsymbol{S}$ is assumed constant on the surface. For an arbitrary surface and $\boldsymbol{S}$ depending on location, the power is

$$W = -\int_A \boldsymbol{S} \cdot \boldsymbol{n} dA, \tag{24}$$

where $\boldsymbol{n}$ is the outward unit normal vector and the sign has been chosen so that positive W corresponds to absorption in the case of a closed surface.
The time-averaged Poynting vector

$$\langle \boldsymbol{S} \rangle = \frac{1}{\tau} \int_t^{t+\tau} \boldsymbol{S}(t')dt' \qquad \tau >> 1/\omega \tag{25}$$

is more important than the momentary Poynting vector (cf. measurements).

The time-averaged Poynting vector for time-harmonic fields is

$$\langle \boldsymbol{S} \rangle = \frac{1}{2} \mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} \tag{26}$$

and, in what follows, this is the Poynting vector meant even though the averaging is not always shown explicitly.
For a plane wave field, the Poynting vector is

$$\boldsymbol{S} = \frac{1}{2} \mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} = \mathbf{Re}\Big\{ \frac{\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*)}{2\omega\mu^*} \Big\}, \tag{27}$$

where

$$\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*) = \boldsymbol{k}^*(\boldsymbol{E} \cdot \boldsymbol{E}^*) - \boldsymbol{E}^*(\boldsymbol{k}^* \cdot \boldsymbol{E}). \tag{28}$$

For a homogeneous plane wave,

$$\boldsymbol{k} \cdot \boldsymbol{E} = \boldsymbol{k}^* \cdot \boldsymbol{E} = 0 \tag{29}$$

and

$$\boldsymbol{S} = \frac{1}{2} \mathbf{Re}\Big\{ \frac{\sqrt{\epsilon\mu}}{\mu^*} \Big\} |\boldsymbol{E}_0|^2 e^{-\frac{4\pi \mathbf{Im}(m) z}{\lambda}} \hat{\mathbf{e}}_z. \tag{30}$$

4 Stokes parameters

Consider the following experiment for an arbitrary monochromatic light source (see Bohren & Huffman p. 46). In the experiment, we make use of a measuring apparatus and polarizers with ideal performance: the measuring apparatus detects energy flux density independently of the state of polarization and the polarizers do not change the amplitude of the transmitted wave.

Denote

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{ikz-i\omega t}, \qquad \boldsymbol{E}_0 = E_\perp \hat{\mathbf{e}}_\perp + E_\parallel \hat{\mathbf{e}}_\parallel \\ E_\perp &= a_\perp e^{-i\delta_\perp} \\ E_\parallel &= a_\parallel e^{-i\delta_\parallel} \qquad a_\perp, a_\parallel \geq 0, \delta_\perp, \delta_\parallel \in \mathbb{R} \end{aligned} \tag{31}$$

<u>Experiment I</u>

No polarizer: the flux density is proportional to

$$|\boldsymbol{E}_0|^2 = E_\parallel E_\parallel^* + E_\perp E_\perp^* \tag{32}$$

<u>Experiment II</u>

Linear polarizers $\parallel$ and $\perp$:

1) $\parallel$: the amplitude of the transmitted wave is $E_\parallel$ and the flux density is $E_\parallel E_\parallel^*$

2) $\perp$: the amplitude of the transmitted wave is $E_\perp$ and the flux density is $E_\perp E_\perp^*$

The difference of the two measurements is $I_\parallel - I_\perp = E_\parallel E_\parallel^* - E_\perp E_\perp^*$.

<u>Experiment III</u>

Linear polarizers $+45°$ ja $-45°$: The new basis vectors are

$$\begin{cases} \hat{\mathbf{e}}_+ = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + \hat{\mathbf{e}}_\perp) \\ \hat{\mathbf{e}}_- = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - \hat{\mathbf{e}}_\perp) \end{cases}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_+ \hat{\mathbf{e}}_+ + E_- \hat{\mathbf{e}}_- \\ E_+ &= \frac{1}{\sqrt{2}}(E_\parallel + E_\perp) \\ E_- &= \frac{1}{\sqrt{2}}(E_\parallel - E_\perp). \end{aligned}$$

1) $+45°$: the amplitude of the transmitted wave is E_+ and the flux density is $E_+E_+^* = \frac{1}{2}(E_\parallel E_\parallel^* + E_\parallel E_\perp^* + E_\perp E_\parallel^* + E_\perp E_\perp^*)$
2) $-45°$: the amplitude of the transmitted wave is E_- and the flux density is $E_-E_-^* = \frac{1}{2}(E_\parallel E_\parallel^* - E_\parallel E_\perp^* - E_\perp E_\parallel^* + E_\perp E_\perp^*)$

The difference os the measurements is $I_+ - I_- = E_\parallel E_\perp^* + E_\perp E_\parallel^*$.

Experiment IV

Circular polarizers R and L:

$$\begin{aligned}
\hat{\mathbf{e}}_R &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_R^* = 1 \\
\hat{\mathbf{e}}_L &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_L \cdot \hat{\mathbf{e}}_L^* = 1 \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_L^* = 0
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{E}_0 &= E_R\hat{\mathbf{e}}_R + E_L\hat{\mathbf{e}}_L \\
E_R &= \frac{1}{\sqrt{2}}(E_\parallel - iE_\perp) \\
E_L &= \frac{1}{\sqrt{2}}(E_\parallel + iE_\perp).
\end{aligned}$$

1) R: the amplitude of the transmitted wave is E_R and the flux density is $E_RE_R^* = \frac{1}{2}(E_\parallel E_\parallel^* - iE_\parallel^* E_\perp + iE_\perp^* E_\parallel + E_\perp E_\perp^*)$
2) L: the amplitude of the transmitted wave is E_L and the flux density is $E_LE_L^* = \frac{1}{2}(E_\parallel E_\parallel^* + iE_\parallel^* E_\perp - iE_\perp^* E_\parallel + E_\perp E_\perp^*)$

The difference of the measurements is $I_R - I_L = i(E_\perp^* E_\parallel - E_\parallel^* E_\perp)$.

With the help of Experiments I-IV, we have determined the Stokes parameters I, Q, U, and V:

$$\begin{aligned}
I &= E_\parallel E_\parallel^* + E_\perp E_\perp^* = a_\parallel^2 + a_\perp^2 \\
Q &= E_\parallel E_\parallel^* - E_\perp E_\perp^* = a_\parallel^2 - a_\perp^2 \\
U &= E_\parallel E_\perp^* + E_\perp E_\parallel^* = 2a_\parallel a_\perp \cos\delta \\
V &= i(E_\parallel E_\perp^* - E_\perp E_\parallel^*) = 2a_\parallel a_\perp \sin\delta \qquad \delta = \delta_\parallel - \delta_\perp
\end{aligned} \tag{33}$$