1 Vector spherical harmonics expansion for a plane wave

In the scattering and absorption problem for localized objects, we need the vector spherical-harmonics expansion of the electromagnetic plane wave.

Let us first derive the spherical-harmonics expansion of the scalar plane wave using the Green's function $e^{ikR}/4\pi R$:

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

In the limit $|\mathbf{x}'| \to \infty$ pätee $|\mathbf{x}-\mathbf{x}'| \approx r' - \frac{\mathbf{x}'}{r'}\cdot\mathbf{x}$ ja $r_> = r'$, $r_< = r$ and $h_l^{(1)}(kr_>) \approx (-i)^{l+1}\frac{e^{ikr_>}}{kr_>}$. Then,

$$\frac{e^{ikr'}}{4\pi r'}e^{-ik\frac{\mathbf{x}'}{r'}\cdot\mathbf{x}} = ik\frac{e^{ikr'}}{kr'}\sum_{lm}(-i)^{l+1}j_l(kr)Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

After reorganizing the terms and taking the complex conjugate,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{l=0}^{\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

where $\mathbf{k}$ id the wave vector k, θ', φ'. According to the addition rule for the spherical harmonics,

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \tag{1}$$

where γ is the great-circle angle between (θ,φ) and (θ',φ'). With the help of the addition rule,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\gamma)$$

where γ is now the angle between $\mathbf{k}$ and $\mathbf{x}$. Moreover,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty} i^l\sqrt{4\pi(2l+1)}j_l(kr)Y_{l0}(\gamma)$$

In what follows, we develop the corresponding expansion for a circularly polarized vector plane wave

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)e^{ikz} \\ c\mathbf{B}(\mathbf{x}) &= \hat{\epsilon}_3 \times \mathbf{E} = \mp i\mathbf{E}(\mathbf{x}),\end{aligned}$$

where $\hat{\epsilon}_3 = \hat{e}_z$. Since the plane wave is finite everywhere, we write its multipole expansion using the regular radial functions $j_l(kr)$:

$$\mathbf{E}(\mathbf{x}) = \sum_{lm} \Big[a_\pm(l,m) j_l(kr) \mathbf{X}_{lm} + \frac{i}{k} b_\pm(l,m) \nabla \times j_l(kr) \mathbf{X}_{lm} \Big],$$

$$c\mathbf{B}(\mathbf{x}) = \sum_{lm} \Big[\frac{-i}{k} a_\pm(l,m) \nabla \times j_l(kr) \mathbf{X}_{lm} + b_\pm(l,m) j_l(kr) \mathbf{X}_{lm} \Big].$$

When deriving the coefficients $a_\pm(l,m)$ and $b_\pm(l,m)$, we make use of the orthogonality properties of the vector spherical-harmonics functions $\mathbf{X}_{lm}$ that we summarize in the following:

$$\int_{4\pi} d\Omega [f_l(r) \mathbf{X}_{l'm'}]^* \cdot [g_l(r) \mathbf{X}_{lm}] = f_l^* g_l \delta_{ll'} \delta_{mm'},$$

$$\int_{4\pi} d\Omega [f_l(r) \mathbf{X}_{l'm'}]^* \cdot [\nabla \times g_l(r) \mathbf{X}_{lm}] = 0,$$

$$\frac{1}{k^2} \int_{4\pi} d\Omega [\nabla \times f_l(r) \mathbf{X}_{l'm'}]^* \cdot [\nabla \times g_l(r) \mathbf{X}_{lm}] = \delta_{ll'} \delta_{mm'} \{ f_l^* g_l + \frac{1}{k^2 r^2} \frac{d}{dr} [r f_l^* \frac{d}{dr} (r g_l)] \}.$$

Above, $f_l(r)$ and $g_l(r)$ are, again, linear combinations of spherical Bessel, Neumann, and Hankel functions. The second and third relation follow from the results

$$i \nabla \times \mathbf{L} = \mathbf{r} \nabla^2 - \nabla (1 + r \frac{\partial}{\partial r}),$$

$$\nabla = \frac{\mathbf{r}}{r} \frac{\partial}{\partial r} - \frac{i}{r^2} \mathbf{r} \times \mathbf{L},$$

$$\Big[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} \Big] f_l(r) = 0,$$

and the proof is left for an exercise.

The coefficients $a_\pm(l,m)$ and $b_\pm(l,m)$ are determined via the scalar product between $\mathbf{X}_{lm}^*$ and the multipole expansions of the fields and the integration over the angular variables:

$$a_\pm(l,m) j_l(kr) = \int_{(4\pi)} d\Omega \mathbf{X}_{lm}^* \cdot \mathbf{E}(\mathbf{x}),$$

$$b_\pm(l,m) j_l(kr) = \int_{(4\pi)} d\Omega \mathbf{X}_{lm}^* \cdot \mathbf{B}(\mathbf{x}).$$

Explicitly,

$$a_\pm(l,m) j_l(kr) = \int_{(4\pi)} d\Omega (\hat{\epsilon}_1 \pm i \hat{\epsilon}_2) \cdot \frac{\mathbf{L}^* Y_{lm}^*}{\sqrt{l(l+1)}} e^{ikz} = \int_{(4\pi)} d\Omega \frac{(L_\mp Y_{lm})^*}{\sqrt{l(l+1)}} e^{ikz}$$

where the operators $L_{\mp}$ have been defined earlier. Here we recognize the strength of these operators together with the analyses based on circular polarization, as it follows, in a straightforward way, that

$$a_{\pm}(l,m)j_l(kr) = \frac{\sqrt{(l \pm m)(l \mp m+1)}}{\sqrt{l(l+1)}} \int_{(4\pi)} d\Omega Y_{l,m\mp 1} e^{ikz}$$

and, by incorporating the expansion for e^{ikz},

$$\begin{aligned} a_{\pm}(l,m) &= i^l \sqrt{4\pi(2l+1)}\delta_{m,\pm 1} \\ b_{\pm}(l,m) &= \mp i a_{\pm}(l,m) \end{aligned}$$

The multipole expansion of the circularly polarized vector plane wave is thus

$$\mathbf{E}(\mathbf{x}) = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \Big[j_l(kr)\mathbf{X}_{l,\pm 1} \pm \frac{1}{k} \nabla \times j_l(kr)\mathbf{X}_{l,\pm 1} \Big],$$

$$c\mathbf{B}(\mathbf{x}) = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \Big[-\frac{1}{k} \nabla \times j_l(kr)\mathbf{X}_{l,\pm 1} \mp i j_l(kr)\mathbf{X}_{l,\pm 1} \Big].$$

The expansions for plane waves linearly polarized in the directions of the vectors $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ can be obtained from the previous results,

$$\hat{\epsilon}_1 e^{ikz} = \sum_{l=1}^{\infty} i^l \sqrt{\pi(2l+1)} \Big[j_l(kr)[\mathbf{X}_{l,1} + \mathbf{X}_{l,-1}] + \frac{1}{k} \nabla \times j_l(kr)[\mathbf{X}_{l,1} - \mathbf{X}_{l,-1}] \Big],$$

$$\hat{\epsilon}_2 e^{ikz} = \sum_{l=1}^{\infty} i^{l-1} \sqrt{\pi(2l+1)} \Big[j_l(kr)[\mathbf{X}_{l,1} - \mathbf{X}_{l,-1}] + \frac{1}{k} \nabla \times j_l(kr)[\mathbf{X}_{l,1} + \mathbf{X}_{l,-1}] \Big].$$

2 Scattering by a spherical particle

Outside the spherical particle, the electromagnetic field is a superposition of the original incident field and the scattered field:

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \mathbf{E}_i(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}), \\ \mathbf{B}(\mathbf{x}) &= \mathbf{B}_i(\mathbf{x}) + \mathbf{B}_s(\mathbf{x}). \end{aligned}$$

where the plane-wave fields $\mathbf{E}_i, \mathbf{B}_i$ have been given earlier. Since the scattered fields are, asymptotically at the infinity, outgoing waves, they must be of the form

$$\mathbf{E}_s = \frac{1}{2} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \Big[\alpha_{\pm}(l) h_l^{(1)}(kr)\mathbf{X}_{l,\pm 1} \pm \frac{\beta_{\pm}(l)}{k} \nabla \times h_l^{(1)}(kr)\mathbf{X}_{l,\pm 1} \Big],$$

$$c\mathbf{B}_s = \frac{1}{2}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\Big[\frac{-i\alpha_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm 1} \mp i\beta_\pm(l) h_l^{(1)}(kr)\mathbf{X}_{l,\pm 1}\Big],$$

where the coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ are determined from the boundary conditions on the surface of the particle. Generally, the expansions include a summation over the order m but, in the case of the spherical symmetry, only the multipoles $m = \pm 1$ contribute to the expansion.

With the help of the coefficients $\alpha(l)$ and $\beta(l)$, we obtain the total scattered and absorbed power. The scattered power follows from the integration of the outward directed component of the scattered-field Poynting vector over the spherical surface. The absorbed power follows from the integration of the inward directed Poynting-vector component of the total field. By reorganizing triple scalar products, we obtain

$$\begin{aligned} P_s &= -\frac{a^2}{2\mu_0}\Re\int_{(4\pi)} d\Omega \mathbf{E}_s\cdot(\mathbf{n}\times\mathbf{B}_s) \\ P_a &= \frac{a^2}{2\mu_0}\Re\int_{(4\pi)} d\Omega \mathbf{E}\cdot(\mathbf{n}\times\mathbf{B}) \end{aligned}$$

where $\mathbf{n} = \hat{e}_r$. Only the transverse field components contribute to the values of the integrals.

Explicitly,

$$\begin{aligned} \mathbf{X}_{lm}(\theta,\varphi) &= \frac{-m}{\sqrt{l(l+1)}\sin\theta}\Bigg[\hat{e}_\theta Y_{lm}(\theta,\varphi) - \\ &\quad i\hat{e}_\varphi\Big[\sqrt{\frac{(l^2-m^2)}{(2l-1)(2l+1)}}Y_{l-1,m}(\theta,\varphi) + \sqrt{\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}}Y_{l+1,m}(\theta,\varphi)\Big]\Bigg], \\ \frac{1}{k}\nabla\times Z_l(kr)\mathbf{X}_{lm}(\theta,\varphi) &= \frac{i\hat{e}_r\sqrt{l(l+1)}}{kr}Z_l(kr)Y_{lm}(\theta,\varphi) + \\ &\quad \frac{1}{kr}\frac{d}{d(kr)}[krZ_l(kr)]\hat{e}_r\times\mathbf{X}_{lm}(\theta,\varphi), \end{aligned} \tag{2}$$

where we see that $\mathbf{X}_{lm}$ is transverse and that, in the latter term, the transverse component is proportional to $\hat{e}_r\times\mathbf{X}_{lm}$. Upon inserting the multipole expansions of the fields into the expressions for the power, we obtain a double summation over l and l' of relations that are of the form $\mathbf{X}^*_{lm}\cdot\mathbf{X}_{l'm'}$, $\mathbf{X}^*_{lm}\cdot(\hat{e}_r\times\mathbf{X}_{l'm'})$, $(\hat{e}_r\times\mathbf{X}_{lm})\cdot(\hat{e}_r\times\mathbf{X}_{l'm'})$. Integration over the angles removes the other summation. Each remaining term in the sum contains a product of spherical Bessel's functions and/or their derivatives—these products can be eliminated with the help of the Wronskian determinants. The cross sections of scattering and absorption are finally (exercise)

$$\begin{aligned} \sigma_s &= \frac{\pi}{2k^2}\sum_l(2l+1)[|\alpha(l)|^2 + |\beta(l)|^2], \\ \sigma_a &= \frac{\pi}{2k^2}\sum_l(2l+1)[2 - |\alpha(l)+1|^2 - |\beta(l)+1|^2], \end{aligned}$$

and the exticntion cross section is the sum of the two above,

$$\sigma_t = -\frac{\pi}{k^2}\sum_l (2l+1)\Re[\alpha(l)+\beta(l)]$$

The differential scattering cross section is

$$\frac{d\sigma_s}{d\Omega} = \frac{\pi}{2k^2}|\sum_l \sqrt{2l+1}[\alpha_\pm(l)\mathbf{X}_{l,\pm1} \pm i\beta_\pm(l)\hat{e}_r \times \mathbf{X}_{l,\pm1}]|^2,$$

for the original polarization state $\hat{\epsilon}_1 \pm i\hat{\epsilon}_2$. Thereby, scattered radiation is in general elliptically polarized.

Let us study the solution for the coefficients $\alpha(l)$ and $\beta(l)$ based on the boundary conditions and start by defining the fields. The original plane-wave field is $\mathbf{E}_i = (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)e^{ikz}$:

$$\begin{aligned}
\mathbf{E}_i(\mathbf{x}) &= \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\Big[j_l(kr)\mathbf{X}_{l,\pm1} \pm \frac{1}{k}\nabla\times j_l(kr)\mathbf{X}_{l,\pm1}\Big],\\
\mathbf{H}_i(\mathbf{x}) &= \frac{1}{\mu_0}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\Big[\frac{-i}{k}\nabla\times j_l(kr)\mathbf{X}_{l,\pm1} \mp i j_l(kr)\mathbf{X}_{l,\pm1}\Big].
\end{aligned}$$

The scattered field takes the form

$$\mathbf{E}_s(\mathbf{x}) = \frac{1}{2}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\Big[\alpha_\pm(l)h_l^{(1)}(kr)\mathbf{X}_{l,\pm1} \pm \frac{\beta_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\Big],$$

$$\mathbf{H}_s(\mathbf{x}) = \frac{1}{2\mu_0 c}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\Big[\frac{-i\alpha_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm1} \mp i\beta_\pm(l)h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\Big].$$

The internal field is

$$\mathbf{E}_t(\mathbf{x}) = \frac{1}{2}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\Big[\eta_\pm(l)j_l(k_t r)\mathbf{X}_{l,\pm1} \pm \frac{\zeta_\pm(l)}{k_t}\nabla\times j_l(k_t r)\mathbf{X}_{l,\pm1}\Big],$$

$$\mathbf{H}_t(\mathbf{x}) = \frac{1}{2\mu c}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\Big[\frac{-i\eta_\pm(l)}{k_t}\nabla\times j_l(k_t r)\mathbf{X}_{l,\pm1} \mp i\zeta_\pm(l)j_l(k_t r)\mathbf{X}_{l,\pm1}\Big].$$

The boundary conditions on the surface of the spherical particle (radius a) are

$$\begin{aligned}
\hat{e}_r \times [\mathbf{E}_i + \mathbf{E}_s - \mathbf{E}_t]|_{r=a} &= 0,\\
\hat{e}_r \times [\mathbf{H}_i + \mathbf{H}_s - \mathbf{H}_t]|_{r=a} &= 0.
\end{aligned}$$

As found earlier,

$$\hat{e}_r \times \Big[\frac{1}{k}\nabla\times Z_l(kr)\mathbf{X}_{lm}(\theta,\varphi)\Big] = -\frac{1}{kr}[krZ_l(kr)]'\mathbf{X}_{lm}(\theta,\varphi),$$

so that, due to the orthogonality of the functions $\mathbf{X}_{lm}$ and $\hat{e}_r \times \mathbf{X}_{lm}$, the boundary conditions simplify into the form,

$$
\begin{aligned}
j_l(ka) + \frac{1}{2}\alpha_\pm(l)h_l^{(1)}(ka) - \frac{1}{2}\eta_\pm(l)j_l(k_t a) = 0, \\
\pm\frac{1}{ka}[kaj_l(ka)]' \mp \frac{1}{2}\beta_\pm(l)\cdot\frac{1}{ka}[kah_l^{(1)}(ka)]' \mp \zeta_\pm(l)\frac{1}{k_t a}[k_t a j_l(k_t a)]' = 0, \\
\frac{1}{ka}[kaj_l(ka)]' + \frac{1}{2}\alpha_\pm(l)\frac{1}{ka}[kah_l^{(1)}(ka)]' - \frac{\mu_0}{2\mu}\cdot\eta_\pm(l)\frac{1}{k_t a}[k_t a j_l(k_t a)]' = 0, \\
\mp j_l(ka) \mp \frac{1}{2}\beta_\pm(l)h_l^{(1)}(ka) \pm \frac{1}{2}\frac{\mu_0}{\mu}\zeta_\pm(l)j_l(k_t a) = 0.
\end{aligned}
$$

With the help of the Riccati-Bessel functions and by writing $x = ka$, $k_t a = mx$, we obtain

$$
\begin{aligned}
\psi_l(x) + \frac{1}{2}\alpha_\pm(l)\xi_l(x) - \frac{1}{2}\eta_\pm(l)\frac{1}{m}\psi_l(mx) = 0, \\
\mp\psi_l'(x) \mp \frac{1}{2}\beta_\pm(l)\xi_l'(x) \mp \frac{1}{2}\zeta_\pm(l)\frac{1}{m}\psi_l'(mx) = 0, \\
\psi_l'(x) + \frac{1}{2}\alpha_\pm(l)\xi_l'(x) - \frac{\mu_0}{2\mu}\eta_\pm(l)\frac{1}{m}\psi_l'(mx) = 0, \\
\mp\psi_l(x) \mp \frac{1}{2}\beta_\pm(l)\xi_l(x) \pm \frac{\mu_0}{2\mu}\zeta_\pm(l)\frac{1}{m}\psi_l(mx) = 0,
\end{aligned}
$$

that leads to

$$
\begin{aligned}
\alpha_\pm(l) &= -2\frac{\frac{\mu_0}{\mu}\psi_l(x)\psi_l'(mx) - \psi_l'(x)\psi_l(mx)}{\frac{\mu_0}{\mu}\xi_l(x)\psi_l'(mx) - \xi_l'(x)\psi_l(mx)}, \\
\beta_\pm(l) &= -2\frac{\mp\frac{\mu_0}{\mu}\psi_l'(x)\psi_l(mx) \mp \psi_l(x)\psi_l'(mx)}{\mp\frac{\mu_0}{\mu}\xi_l'(x)\psi_l(mx) \mp \xi_l(x)\psi_l'(mx)}, \\
\eta_\pm(l) &= 2\frac{\psi_l(x)\xi_l'(x) - \psi_l'(x)\xi_l(x)}{\frac{1}{m}\psi_l(mx)\xi_l'(x) + \frac{1}{m}\frac{\mu_0}{\mu}\psi_l'(mx)\xi_l(x)}, \\
\zeta_\pm(l) &= 2\frac{\mp\psi_l'(x)\xi_l(x)pm\psi_l(x)\xi_l'(x)}{\frac{1}{m}\psi_l'(mx)\xi_l(x) \mp \frac{1}{m}\frac{\mu_0}{\mu}\psi_l(mx)\xi_l'(x)}.
\end{aligned}
$$

We have solved for the scattered field completely for two circular polarization states of the original field $\epsilon_1 \pm i\epsilon_2$. We thus know the elements $S_j^{(c)}$ (j=1,2,3,4) of the amplitude scattering matrix that relates the original field to the scattered field in the circular-polarization representation.

$$
\begin{pmatrix} E_{s-} \\ E_{s+} \end{pmatrix} = \frac{e^{ik(r-z)}}{-ikr}\begin{pmatrix} S_2^{(c)} & S_3^{(c)} \\ S_4^{(c)} & S_1^{(c)} \end{pmatrix}\begin{pmatrix} E_{i-} \\ E_{i+} \end{pmatrix} \tag{3}
$$

The amplitude scattering matrix elements of the circular-polarization representation relate linearly to the commonly used ones of the linear-polarization representation.

$$\begin{pmatrix} S_1^{(c)} \\ S_2^{(c)} \\ S_3^{(c)} \\ S_4^{(c)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & -i & i \\ -1 & 1 & i & i \\ -1 & 1 & -i & -i \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}, \tag{4}$$

where, in the case of Lorenz-Mie scattering ($S_3 = S_4 = 0$),

$$\left.\begin{array}{c} S_1^{(c)} = (S_1 + S_2)/2 \\ S_2^{(c)} = (S_1 + S_2)/2 = S_1^{(c)} \\ S_3^{(c)} = (-S_1 + S_2)/2 \\ S_4^{(c)} = S_3^{(c)} \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} S_1 = S_1^{(c)} - S_3^{(c)} \\ S_2 = S_1^{(c)} + S_3^{(c)} \end{array}\right. \tag{5}$$

The complete scattering matrix follows now from the elements S_1, S_2 in a standard manner. The elements $S_1^{(c)}$, $S_2^{(c)}$, $S_3^{(c)}$, $S_4^{(c)}$ follow from the far-zone expressions for the scattered fields, for which the vector spherical harmonics expansions reduce to the form utlized earlier in the expression for the differential scattering cross section. A more detailed assessment is left for an exercise.