1 Introduction to scattering theory

Each scattering problem depends on the detailed characteristics of the scattering particle: its size, shape, and refractive index. The size is usually described by the size parameter

$$x = \frac{2\pi a}{\lambda}, \tag{1}$$

where a is a typical radial distance in the particle and and λ is the wavelength of the original electromagnetic field. In the size dependence of scattering, only the ratio a/λ is meaningful. Shape is described by suitable elongation, roughness, or angularity parameters. The constitutive material is characterized by the complex-valued refractive index

$$m = n + \mathrm{i}n', \tag{2}$$

where the real and imaginary parts n and n' are responsible for refraction and absorption of light, respectively. The time dependence of the fields has been chosen to be $\exp(-\mathrm{i}\omega t)$ so that, in physically relevant cases, the imaginary part of the refractive index needs to be non-negative.

1.1 Electromagnetic formulation of the problem

Electromagnetic scattering and absorption is here being assessed from the view point of classical electromagnetics. The natural foundation is provided by the Maxwell equations

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho, \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{H} &= \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}, \end{aligned} \tag{3}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{D}$ is the electric displacement, and $\boldsymbol{H}$ is the magnetic field. ρ and $\boldsymbol{j}$ are, respectively, the densities of free charges and currents. In order for the charge and current densities to determine the fields unambiguously, constitutive relations describing the interaction of matter and fields are introduced,

$$\begin{aligned} \boldsymbol{j} &= \sigma \boldsymbol{E}, \\ \boldsymbol{D} &= \epsilon \boldsymbol{E}, \\ \boldsymbol{B} &= \mu \boldsymbol{H}, \end{aligned} \tag{4}$$

where σ is the electric conductivity, ϵ is the electric permittivity, and μ is the magnetic permeability. In what follows, it is assumed that there are no free charges or currents and that the time dependence of the fields is of the harmonic type $\exp(-\mathrm{i}\omega t)$. The Maxwell equations then reduce to the form

$$\begin{aligned} \nabla \cdot \epsilon \boldsymbol{E} &= 0, \\ \nabla \times \boldsymbol{E} &= \mathrm{i}\omega\mu\boldsymbol{B}, \\ \nabla \cdot \boldsymbol{H} &= 0, \\ \nabla \times \boldsymbol{H} &= -\mathrm{i}\omega\epsilon\boldsymbol{E}, \end{aligned} \tag{5}$$

so that the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ fulfil the vector wave equations

$$\begin{aligned} \nabla^2\boldsymbol{E} + k^2\boldsymbol{E} &= 0, \\ \nabla^2\boldsymbol{H} + k^2\boldsymbol{H} &= 0, \end{aligned} \tag{6}$$

where $k^2 = \omega^2 m^2/c^2$ and m is the relative refractive index of the scatterer, $m^2 = \epsilon\mu/\epsilon_0\mu_0$.

Denote the internal field of the particle by $(\boldsymbol{E}_1, \boldsymbol{H}_1)$. The external field $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ is the superposition of the original field $(\boldsymbol{E}_i, \boldsymbol{H}_i)$ and the scattered field $(\boldsymbol{E}_s, \boldsymbol{H}_s)$,

$$\begin{aligned} \boldsymbol{E}_2 &= \boldsymbol{E}_i + \boldsymbol{E}_s, \\ \boldsymbol{H}_2 &= \boldsymbol{H}_i + \boldsymbol{H}_s. \end{aligned} \tag{7}$$

In what follows, let us assume that the original field is a plane wave,

$$\begin{aligned} \boldsymbol{E}_i &= \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)], \\ \boldsymbol{H}_i &= \boldsymbol{H}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)], \boldsymbol{H}_0 = \frac{1}{\omega\mu_0}\boldsymbol{k}\times\boldsymbol{E}_0, \end{aligned} \tag{8}$$

where $\boldsymbol{k}$ is the wave vector of the medium surrounding the particle. Since there are no free currents according to our hypothesis, the tangential components of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous across the boundary between the particle and the surrounding medium:

$$\begin{aligned} (\boldsymbol{E}_2 - \boldsymbol{E}_1)\times\boldsymbol{n} &= 0, \\ (\boldsymbol{H}_2 - \boldsymbol{H}_1)\times\boldsymbol{n} &= 0, \end{aligned} \tag{9}$$

at the boundary with an outward normal vector $\boldsymbol{n}$. It is our fundamental goal to solve the Maxwell equations everywhere in space with the boundary conditions given.

1.2 Amplitude scattering matrix

Let us place an arbitrary particle in a plane wave field according to the figure (cf. Bohren & Huffman). The propagation directions of the original and scattered fields $\boldsymbol{e}_z$ and $\boldsymbol{e}_r$ define a scattering plane, and the original field is divided into components perpendicular and parallel to that plane,

$$\boldsymbol{E}_i = (E_{0\perp}\boldsymbol{e}_{i\perp} + E_{0\parallel}\boldsymbol{e}_{i\parallel}) \exp[\mathrm{i}(kz - \omega t)] = E_{i\perp}\boldsymbol{e}_{i\perp} + E_{i\parallel}\boldsymbol{e}_{i\parallel}. \quad (10)$$

In the radiation zone, that is, far away from the scattering particle, the scattered field is a transverse spherical wave (cf. Jackson),

$$\boldsymbol{E}_s = \frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\boldsymbol{A}, \boldsymbol{e}_r \cdot \boldsymbol{A} = 0, \quad (11)$$

so that

$$\boldsymbol{E}_s = E_{s\perp}\boldsymbol{e}_{s\perp} + E_{s\parallel}\boldsymbol{e}_{s\parallel}, \quad (12)$$

where

$$\begin{aligned} \boldsymbol{e}_{s\perp} &= -\boldsymbol{e}_\phi, \\ \boldsymbol{e}_{s\parallel} &= \boldsymbol{e}_\theta. \end{aligned} \quad (13)$$

Due to the linearity of the boundary conditions, the amplitude of the scattered field depends linearly on the amplitude of the original field. In a matrix form,

$$\begin{bmatrix} E_{s\perp} \\ E_{s\parallel} \end{bmatrix} = \frac{\exp[\mathrm{i}(kr - kz)]}{-\mathrm{i}kr} \begin{bmatrix} S_1 & S_4 \\ S_3 & S_2 \end{bmatrix} \begin{bmatrix} E_{i\perp} \\ E_{i\parallel} \end{bmatrix}, \quad (14)$$

where the complex-valued amplitude-scattering-matrix elements S_j ($j = 1, 2, 3, 4$) generally depend on the scattering angle θ and the azimuthal angle ϕ. Since only the relative phases are important, the amplitude scattering matrix has seven free parameters.

1.3 Stokes parameters and scattering matrix

In the medium surrounding the particle, the time-averaged Poynting vector $\boldsymbol{S}$ can be divided into the Poynting vectors of the original field, scattered field, and that showing the interaction of the original and scattered fields,

$$\boldsymbol{S} = \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_2 \times \boldsymbol{H}_2^*) = \boldsymbol{S}_i + \boldsymbol{S}_s + \boldsymbol{S}_e, \quad (15)$$

where

$$\begin{aligned}\boldsymbol{S}_i &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_i \times \boldsymbol{H}_i^*),\\ \boldsymbol{S}_s &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*),\\ \boldsymbol{S}_e &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_i \times \boldsymbol{H}_s^* + \boldsymbol{E}_s \times \boldsymbol{H}_i^*).\end{aligned} \tag{16}$$

In the radiation zone, the power incident on a surface element ΔA perpendicular to the radial direction is

$$\boldsymbol{S}_s \cdot \boldsymbol{e}_r = \frac{k}{2\omega\mu}\frac{|\boldsymbol{A}|^2}{k^2}\Delta\Omega, \Delta\Omega = \frac{\Delta A}{r^2} \tag{17}$$

and $|\boldsymbol{A}|^2$ can be measures as a function of angles. By placing polarizers in between the scattering particle and the detector, we can measure the Stokes parameters of the scattered field (Bohren & Huffman),

$$\begin{aligned}I_s &= \langle |E_{s\perp}|^2 + |E_{s\parallel}|^2\rangle,\\ Q_s &= \langle -|E_{s\perp}|^2 + |E_{s\parallel}|^2\rangle,\\ U_s &= 2\mathrm{Re}E_{s\perp}^* E_{s\parallel},\\ V_s &= -2\mathrm{Im}E_{s\perp}^* E_{s\parallel}.\end{aligned} \tag{18}$$

Thus, I_s gives the scattered intensity, Q_s gives the difference between the intensities in the scattering plane and perpendicular to the scattering plane, U_s gives the difference between $+\pi/4$ and $-\pi/4$ -polarized intensities and, lastly, V_s gives the difference between right-handed and left-handed circularly polarized intensities. The factor $k/2\omega\mu_0$ has been omitted from the intensities; it is not needed since, in practice, relative intensities are measured instead of absolute ones. The Stokes parameters fully describe the polarization state of an electromagnetic field.

The scattering matrix S interrelates the Stokes parameters of the original field and the scattered field, and can be derived from the amplitude scattering matrix:

$$\boldsymbol{I}_s = \frac{1}{k^2 r^2} S \boldsymbol{I}_i, \tag{19}$$

where the Stokes vectors

$$\begin{aligned}\boldsymbol{I}_s &= (I_s, Q_s, U_s, V_s)^T,\\ \boldsymbol{I}_i &= (I_i, Q_i, U_i, V_i)^T.\end{aligned} \tag{20}$$

The information about the angular dependence of scattering is fully contained in the 16 elements of the scattering matrix. For a single scattering particle, it has seven free parameters whereas, for an ensemble of particles, all 16 elements can be free. Symmetries reduce the number of free parameters: for example, for a spherical particle, there are three free parameters.

For an unpolarized incident field, the Stokes parameters of the scattered field are

$$\begin{aligned} I_s &= \frac{1}{k^2 r^2} S_{11} I_i, \\ Q_s &= \frac{1}{k^2 r^2} S_{21} I_i, \\ U_s &= \frac{1}{k^2 r^2} S_{31} I_i, \\ V_s &= \frac{1}{k^2 r^2} S_{41} I_i. \end{aligned} \tag{21}$$

Thus, S_{11} gives the angular distribution of scattered intensity and the total degree of polarization is

$$P_{\rm tot} = \frac{\sqrt{S_{21}^2 + S_{31}^2 + S_{41}^2}}{S_{11}}. \tag{22}$$

Scattering polarizes light.

1 Introduction to scattering theory

1.1 Electromagnetic formulation of the problem

1.2 Amplitude scattering matrix

1.3 Stokes parameters and scattering matrix

1.4 Extinction, scattering and absorption

Let us assume that medium surrounding the scattering particle is non-absorbing. The total or extinction cross section is then the sum of the absorption and scattering cross sections:

$$\sigma_e = \sigma_s + \sigma_a, \tag{1}$$

where

$$\begin{aligned} \sigma_e &= -\frac{1}{I_i}\int_A dA \boldsymbol{S}_e \cdot \boldsymbol{e}_r, \\ \sigma_s &= \frac{1}{I_i}\int_A dA \boldsymbol{S}_s \cdot \boldsymbol{e}_r, \end{aligned} \tag{2}$$

when A is a spherical envelope of radius r containing the scattering particle.

Let the original field be of $\boldsymbol{e}_x$-polarized form $\boldsymbol{E}_0 = E\boldsymbol{e}_x$. In the radiation zone,

$$\begin{aligned} \boldsymbol{E}_s &\propto \frac{\exp[\mathrm{i}k(r-z)]}{-\mathrm{i}kr}\boldsymbol{X}E, \boldsymbol{e}_r \cdot \boldsymbol{X} = 0, \\ \boldsymbol{H}_s &\propto \frac{k}{\omega\mu}\boldsymbol{e}_r \times \boldsymbol{E}_s, \end{aligned} \tag{3}$$

where the vector scattering amplitude $\boldsymbol{X}$ is related to the amplitude scattering matrix as follows:

$$\boldsymbol{X} = (S_4\cos\phi + S_1\sin\phi)\boldsymbol{e}_{s\perp} + (S_2\cos\phi + S_3\sin\phi)\boldsymbol{e}_{s\parallel}. \tag{4}$$

By making use of the asymptotic forms of the scattered field shown above and $\boldsymbol{e}_x$-polarized original field, the so-called optical theorem can be derived: extinction depends only on scattering in the exact forward direction,

$$\sigma_e = \frac{4\pi}{k^2}\mathrm{Re}[(\boldsymbol{X}\cdot\boldsymbol{e}_x)_{\theta=0}]. \tag{5}$$

In addition,

$$\sigma_s = \int_{4\pi} d\Omega \frac{d\sigma_s}{d\Omega}, \tag{6}$$

where the differential scattering cross section is

$$\frac{d\sigma_s}{d\Omega} = \frac{|\boldsymbol{X}|^2}{k^2}. \tag{7}$$

The extinction, scattering, and absorption efficiencies are defined as the ratios of the corresponding cross sections to the geometric cross section of the particle $A_\perp$ as projected in the propagation direction of the original field:

$$\begin{aligned} q_e &= \frac{\sigma_e}{A_\perp}, \\ q_s &= \frac{\sigma_s}{A_\perp}, \\ q_a &= \frac{\sigma_a}{A_\perp}. \end{aligned} \tag{8}$$

For an unpolarized original field, the cross sections are

$$\begin{aligned} \sigma_e &= \frac{1}{2}(\sigma_e^{(1)} + \sigma_e^{(2)}), \\ \sigma_s &= \frac{1}{2}(\sigma_s^{(1)} + \sigma_s^{(2)}), \end{aligned} \tag{9}$$

where the indices 1 and 2 refer to two polarization states of the original field perpendicular to one another.

2 Plane waves

The electromagnetic plane wave

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \end{aligned} \tag{10}$$

can, under certain conditions, fulfil Maxwell's equations. The physical fields correspond to the real parts of the complex-valued fields. The vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ above are constant vectors and can be complex-valued. Similarly, the wave vector $\boldsymbol{k}$ can be complex-valued:

$$\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}'', \qquad \boldsymbol{k}', \boldsymbol{k}'' \in \mathbb{R}^n \tag{11}$$

Inserting (11) into equation (10), we obtain

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t} \end{aligned} \tag{12}$$

In Eq. (12), $\boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ and $\boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ are amplitudes and $\boldsymbol{k}'\cdot\boldsymbol{x} - \omega t = \phi$ is the phase of the wave.
An equation of the form $\boldsymbol{k}\cdot\boldsymbol{x}$ =constant defines, in the case of a real-valued vector $\boldsymbol{k}$, a planar surface, whose normal is just the vector $\boldsymbol{k}$. Thus, $\boldsymbol{k}'$ is perpendicular to the planes of constant phase and $\boldsymbol{k}''$ is perpendicular to the planes of constant amplitude. If $\boldsymbol{k}' \parallel \boldsymbol{k}''$, the planes coincide and the wave is *homogeneous*. If $\boldsymbol{k}' \nparallel \boldsymbol{k}''$, the wave is *inhomogeneous*. A plane wave propagating in vacuum is homogeneous.

In the case of plane waves, Maxwell's equaitons can be written as

$$\begin{aligned} \boldsymbol{k}\cdot\boldsymbol{E}_0 &= 0 \\ \boldsymbol{k}\cdot\boldsymbol{H}_0 &= 0 \\ \boldsymbol{k}\times\boldsymbol{E}_0 &= \omega\mu\boldsymbol{H}_0 \\ \boldsymbol{k}\times\boldsymbol{H}_0 &= -\omega\epsilon\boldsymbol{E}_0 \end{aligned} \tag{13}$$

The two upmost equations are conditions for the transverse nature of the waves: $\boldsymbol{k}$ is perpendicular to both $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. The two lowermost equations show that $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are perpendicular to each other. Since $\boldsymbol{k}$, $\boldsymbol{E}_0$, and $\boldsymbol{H}_0$ are complex-valued, the geometric interpretation is not simple unless the waves are homogeneous.
It follows from Maxwell's equations (13) that, on one hand,

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \omega\mu\boldsymbol{k}\times\boldsymbol{H}_0 = -\omega^2\epsilon\mu\boldsymbol{E}_0 \tag{14}$$

and, on the other hand,

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}_0) - \boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{k}) = -\boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{k}), \tag{15}$$

so that

$$\boldsymbol{k}\cdot\boldsymbol{k} = \omega^2\epsilon\mu. \tag{16}$$

Plane waves solutions are in agreement with Maxwell's equations if

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = \boldsymbol{k}\cdot\boldsymbol{H}_0 = \boldsymbol{E}_0\cdot\boldsymbol{H}_0 = 0 \tag{17}$$

and if

$$k'^2 - k''^2 + 2i\boldsymbol{k}' \cdot \boldsymbol{k}'' = \omega^2 \epsilon \mu. \tag{18}$$

Note that ϵ and μ are properties of the medium, whereas $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are properties of the wave. Thus, ϵ and μ do not unambiguously determine the details of wave propagation.
In the case of a homogeneous plane wave ($\boldsymbol{k}' \| \boldsymbol{k}''$),

$$\boldsymbol{k} = (k' + ik'')\hat{\mathbf{e}}, \tag{19}$$

where k' and k'' are non-negative and $\hat{\mathbf{e}}$ is an arbitrary real-valued unit vector.

According to Eq. (16),

$$(k' + ik'')^2 = \omega^2 \epsilon \mu = \frac{\omega^2 m^2}{c^2}, \tag{20}$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum and m is the complex-valued refractive index

$$m = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}} = m_r + i m_i, \qquad m_r, m_i \geq 0. \tag{21}$$

In vacuum, the wave number is $\omega/c = 2\pi/\lambda$, where λ is the wavelength. The general homogeneous plane wave takes the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{2\pi m_i s}{\lambda}} e^{i\frac{2\pi m_r s}{\lambda} - i\omega t} \tag{22}$$

where $s = \boldsymbol{e} \cdot \boldsymbol{x}$. The imaginary and real parts of the refractive index determine the attenuation and phase velocity $v = c/m_r$ of the wave, respectively.

3 Poynting vector

Let us study the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ that is time harmonic. For the physical fields (the real parts of the complex-valued fields), the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{23}$$

describes the direction and amount of energy transfer everywhere in the space.

Let $\boldsymbol{n}$ be the unit normal vector of the planar surface element A. Electromagnetic energy is transferred through the planar surface with power $\boldsymbol{S} \cdot \boldsymbol{n}\ A$, where $\boldsymbol{S}$ is assumed constant on the surface. For an arbitrary surface and $\boldsymbol{S}$ depending on location, the power is

$$W = -\int_A \boldsymbol{S} \cdot \boldsymbol{n} dA, \tag{24}$$

where $\boldsymbol{n}$ is the outward unit normal vector and the sign has been chosen so that positive W corresponds to absorption in the case of a closed surface.
The time-averaged Poynting vector

$$\langle \boldsymbol{S} \rangle = \frac{1}{\tau} \int_t^{t+\tau} \boldsymbol{S}(t') dt' \qquad \tau >> 1/\omega \tag{25}$$

is more important than the momentary Poynting vector (cf. measurements).

The time-averaged Poynting vector for time-harmonic fields is

$$\langle \boldsymbol{S} \rangle = \frac{1}{2} \mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} \tag{26}$$

and, in what follows, this is the Poynting vector meant even though the averaging is not always shown explicitly.
For a plane wave field, the Poynting vector is

$$\boldsymbol{S} = \frac{1}{2} \mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} = \mathbf{Re}\Big\{ \frac{\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*)}{2\omega\boldsymbol{\mu}^*} \Big\}, \tag{27}$$

where

$$\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*) = \boldsymbol{k}^* (\boldsymbol{E} \cdot \boldsymbol{E}^*) - \boldsymbol{E}^* (\boldsymbol{k}^* \cdot \boldsymbol{E}). \tag{28}$$

For a homogeneous plane wave,

$$\boldsymbol{k} \cdot \boldsymbol{E} = \boldsymbol{k}^* \cdot \boldsymbol{E} = 0 \tag{29}$$

and

$$\boldsymbol{S} = \frac{1}{2} \mathbf{Re}\Big\{ \frac{\sqrt{\epsilon\mu}}{\mu^*} \Big\} |\boldsymbol{E}_0|^2 e^{-\frac{4\pi \mathbf{Im}(m) z}{\lambda}} \hat{\mathbf{e}}_z. \tag{30}$$

4 Stokes parameters

Consider the following experiment for an arbitrary monochromatic light source (see Bohren & Huffman p. 46). In the experiment, we make use of a measuring apparatus and polarizers with ideal performance: the measuring apparatus detects energy flux density independently of the state of polarization and the polarizers do not change the amplitude of the transmitted wave.

Denote

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{ikz-i\omega t}, \qquad \boldsymbol{E}_0 = E_\perp \hat{\mathbf{e}}_\perp + E_\parallel \hat{\mathbf{e}}_\parallel \\ E_\perp &= a_\perp e^{-i\delta_\perp} \\ E_\parallel &= a_\parallel e^{-i\delta_\parallel} \qquad a_\perp, a_\parallel \geq 0, \delta_\perp, \delta_\parallel \in \mathbb{R} \end{aligned} \tag{31}$$

<u>Experiment I</u>
No polarizer: the flux density is proportional to

$$|\boldsymbol{E}_0|^2 = E_\parallel E_\parallel^* + E_\perp E_\perp^* \tag{32}$$

<u>Experiment II</u>
Linear polarizers $\parallel$ and $\perp$:
1) $\parallel$: the amplitude of the transmitted wave is $E_\parallel$ and the flux density is $E_\parallel E_\parallel^*$
2) $\perp$: the amplitude of the transmitted wave is $E_\perp$ and the flux density is $E_\perp E_\perp^*$

The difference of the two measurements is $I_\parallel - I_\perp = E_\parallel E_\parallel^* - E_\perp E_\perp^*$.

<u>Experiment III</u>
Linear polarizers $+45°$ ja $-45°$: The new basis vectors are

$$\begin{cases} \hat{\mathbf{e}}_+ = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + \hat{\mathbf{e}}_\perp) \\ \hat{\mathbf{e}}_- = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - \hat{\mathbf{e}}_\perp) \end{cases}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_+ \hat{\mathbf{e}}_+ + E_- \hat{\mathbf{e}}_- \\ E_+ &= \frac{1}{\sqrt{2}}(E_\parallel + E_\perp) \\ E_- &= \frac{1}{\sqrt{2}}(E_\parallel - E_\perp). \end{aligned}$$

1) $+45°$: the amplitude of the transmitted wave is E_+ and the flux density is $E_+E_+^* = \frac{1}{2}(E_\parallel E_\parallel^* + E_\parallel E_\perp^* + E_\perp E_\parallel^* + E_\perp E_\perp^*)$
2) $-45°$: the amplitude of the transmitted wave is E_- and the flux density is $E_-E_-^* = \frac{1}{2}(E_\parallel E_\parallel^* - E_\parallel E_\perp^* - E_\perp E_\parallel^* + E_\perp E_\perp^*)$

The difference os the measurements is $I_+ - I_- = E_\parallel E_\perp^* + E_\perp E_\parallel^*$.

Experiment IV

Circular polarizers R and L:

$$\begin{aligned}
\hat{\mathbf{e}}_R &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_R^* = 1 \\
\hat{\mathbf{e}}_L &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_L \cdot \hat{\mathbf{e}}_L^* = 1 \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_L^* = 0
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{E}_0 &= E_R\hat{\mathbf{e}}_R + E_L\hat{\mathbf{e}}_L \\
E_R &= \frac{1}{\sqrt{2}}(E_\parallel - iE_\perp) \\
E_L &= \frac{1}{\sqrt{2}}(E_\parallel + iE_\perp).
\end{aligned}$$

1) R: the amplitude of the transmitted wave is E_R and the flux density is $E_RE_R^* = \frac{1}{2}(E_\parallel E_\parallel^* - iE_\parallel^* E_\perp + iE_\perp^* E_\parallel + E_\perp E_\perp^*)$
2) L: the amplitude of the transmitted wave is E_L and the flux density is $E_LE_L^* = \frac{1}{2}(E_\parallel E_\parallel^* + iE_\parallel^* E_\perp - iE_\perp^* E_\parallel + E_\perp E_\perp^*)$

The difference of the measurements is $I_R - I_L = i(E_\perp^* E_\parallel - E_\parallel^* E_\perp)$.

With the help of Experiments I-IV, we have determined the Stokes parameters I, Q, U, and V:

$$\begin{aligned}
I &= E_\parallel E_\parallel^* + E_\perp E_\perp^* = a_\parallel^2 + a_\perp^2 \\
Q &= E_\parallel E_\parallel^* - E_\perp E_\perp^* = a_\parallel^2 - a_\perp^2 \\
U &= E_\parallel E_\perp^* + E_\perp E_\parallel^* = 2a_\parallel a_\perp \cos\delta \\
V &= i(E_\parallel E_\perp^* - E_\perp E_\parallel^*) = 2a_\parallel a_\perp \sin\delta \qquad \delta = \delta_\parallel - \delta_\perp
\end{aligned} \tag{33}$$

1 Scattering at the plane interface between two media

Two kinds of features can be distinguished in the reflection and refraction of waves at the plane interface between two media:
i) Kinematical properties:
a) the angle of reflection coincides with the angle of incidence
b) the angle of refraction relates to the angle of incidence and the refractive indices of the media via Snel's law
ii) Dynamical properties:
a) the intensitities of reflected and refracted radiation
b) phase shifts and polarization

The kinematical properties follow from the wave nature of the phenomena and the existence of the boundary conditions. The dynamical properties depend fully on the characteristics of the waves and their boundary conditions.
The coordinate systems and symbols are defined in Fig. 1. The original plane wave (wave vector $\mathbf{k}$, angular frequency ω) is incident on the interface from the medium μ, ϵ (refractive index $m = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$). The refracted plane wave propagates in the medium μ', ϵ' ($m' = \sqrt{\epsilon'\mu'/\epsilon_0\mu_0}$) with wave vector $\mathbf{k}_t$ and the reflected plane wave in the medium μ, ϵ with wave vector $\mathbf{k}_r$.

The kinematics are described by the angles of incidence θ_i, reflection θ_r, and refraction θ_t. Assume first that μ, ϵ, μ', ϵ' and therefor also m and m' are real-valued.

Based on what has already been described before, we can write the incident, reflected, and refracted fields as follows:

$$\begin{aligned} \mathbf{E}_i &= \mathbf{E}_{0i} e^{i\mathbf{k}_i\cdot\mathbf{x}-i\omega t} \\ \mathbf{B}_i &= \sqrt{\epsilon\mu}\frac{\mathbf{k}_i \times \mathbf{E}_i}{k_i} \end{aligned} \tag{1}$$

$$\begin{aligned} \mathbf{E}_r &= \mathbf{E}_{0r} e^{i\mathbf{k}_r\cdot\mathbf{x}-i\omega t} \\ \mathbf{B}_r &= \sqrt{\epsilon\mu}\frac{\mathbf{k}_r \times \mathbf{E}_r}{k_r} \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbf{E}_t &= \mathbf{E}_{0t} e^{i\mathbf{k}_t\cdot\mathbf{x}-i\omega t} \\ \mathbf{B}_t &= \sqrt{\epsilon'\mu'}\frac{\mathbf{k}_t \times \mathbf{E}_t}{k_t} \end{aligned} \tag{3}$$

The lengths of the wave vectors are

$$
\begin{aligned}
|\mathbf{k}_i| &= |\mathbf{k}_r| = k_i = k_r = \omega\sqrt{\epsilon\mu} \\
|\mathbf{k}_t| &= k_t = \omega\sqrt{\epsilon'\mu'}
\end{aligned} \tag{4}
$$

The boundary conditions are to be valid at the interface $z = 0$ at all times. Therefore, the spatial dependences of the fields need to coincide at the interface and, in particular, the arguments of the phase factors

$$
(\mathbf{k}_i \cdot \mathbf{x})_{z=0} = (\mathbf{k}_r \cdot \mathbf{x})_{z=0} = (\mathbf{k}_t \cdot \mathbf{x})_{z=0} \tag{5}
$$

independently of the detailed properties of the boundary conditions. It follows, first, that the wave vectors must be confined to a single plane. Second, it follows that $\theta_i = \theta_r$ and, third, we obtain Snel's law

$$
\begin{aligned}
& k_i \sin\theta_i = k_t \sin\theta_t \\
\Leftrightarrow\quad & m \sin\theta_i = m' \sin\theta_t .
\end{aligned} \tag{6}
$$

According to the boundary conditions of electromagnetic fields, the normal components of **D** and **B** and the tangential components of **E** and **H** must be continuous across the boundary. Then, at the interface $z = 0$, we have

$$
\begin{aligned}
\hat{\mathbf{n}} &\cdot [\epsilon(\mathbf{E}_{0i} + \mathbf{E}_{0r}) - \epsilon'\mathbf{E}_{0t}] = 0 \\
\hat{\mathbf{n}} &\cdot [\mathbf{k}_i \times \mathbf{E}_{0i} + \mathbf{k}_r \times \mathbf{E}_{0r} - \mathbf{k}_t \times \mathbf{E}_{0t}] = 0 \\
\hat{\mathbf{n}} &\times [\mathbf{E}_{0i} + \mathbf{E}_{0r} - \mathbf{E}_{0t}] = 0 \\
\hat{\mathbf{n}} &\times [\frac{1}{\mu}(\mathbf{k}_i \times \mathbf{E}_{0i} + \mathbf{k}_r \times \mathbf{E}_{0r}) - \frac{1}{\mu'}(\mathbf{k}_t \times \mathbf{E}_{0t})] = 0
\end{aligned} \tag{7}
$$

Let us divide the scattering problem into two cases: first, the incident field is linearly polarized so that the electric field is perpendicular to the plane defined by $\mathbf{k}_i$ and $\hat{\mathbf{n}}$; second, the electric field is within that plane. An arbitrary elliptic polarization can be treated as a linear sum of the results following for the two cases defined above.
First, let the electric field be perpendicular to the plane of incidence (see Fig. 2). The choice of **B**-vectors guarantees a positive flow of energy in the direction of the wave vectors. With the help of the third and fourth boundary conditions above, we obtain

$$
\begin{aligned}
E_{0i} + E_{0r} - E_{0t} = 0 \\
\sqrt{\frac{\epsilon}{\mu}}(E_{0i} - E_{0r}) \cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}} E_{0t} \cos\theta_t = 0
\end{aligned} \tag{8}
$$

Denote the Fresnel coefficients by

$$
r_\perp = \frac{E_{0r}}{E_{0i}}, \qquad t_\perp = \frac{E_{0t}}{E_{0i}}.
$$

Then,

$$
\begin{aligned}
1 + r_\perp - t_\perp = 0 \\
\sqrt{\frac{\epsilon}{\mu}}(1 - r_\perp) \cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}} t_\perp \cos\theta_t = 0
\end{aligned} \tag{9}
$$

and it follows that

$$
\begin{aligned}
t_\perp &= 1 + r_\perp \\
\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_t &= (\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_t) r_\perp
\end{aligned} \tag{10}
$$

and, furthermore, we obtain, for the Fresnel coefficients,

$$
\begin{aligned}
r_\perp &= \frac{\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_t}{\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_t} \\
t_\perp &= \frac{2\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i}{\sqrt{\frac{\epsilon}{\mu}} \cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}} \cos\theta_t}
\end{aligned} \tag{11}
$$

Second, let the electric field be within the plane of incidence (see Fig. 3). Again, based on the third and fourth boundary conditions above, we have

$$
\begin{aligned}
(E_{0i} - E_{0r}) \cos\theta_i - E_{0t} \cos\theta_t &= 0 \\
\sqrt{\frac{\epsilon}{\mu}}(E_{0i} + E_{0r}) - \sqrt{\frac{\epsilon'}{\mu'}} E_{0t} &= 0
\end{aligned} \tag{12}
$$

Denote the Fresnel coefficients by

$$r_\parallel = \frac{E_{0r}}{E_{0i}}, \qquad t_\parallel = \frac{E_{0t}}{E_{0i}}.$$

Then,

$$\begin{aligned} (1 - r_\parallel)\cos\theta_i - t_\parallel \cos\theta_t &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(1 + r_\parallel) - \sqrt{\frac{\epsilon'}{\mu'}} t_\parallel &= 0 \end{aligned} \tag{13}$$

and we obtain the following pair of equations,

$$\begin{aligned} t_\parallel &= \frac{\cos\theta_i}{\cos\theta_t}(1 - r_\parallel) \\ \sqrt{\frac{\epsilon}{\mu}} - \sqrt{\frac{\epsilon'}{\mu'}}\frac{\cos\theta_i}{\cos\theta_t} &= -\Big(\sqrt{\frac{\epsilon}{\mu}} + \sqrt{\frac{\epsilon'}{\mu'}}\frac{\cos\theta_i}{\cos\theta_t}\Big) r_\parallel \end{aligned} \tag{14}$$

allowing for the Fresnel coefficients to be explicitly solved for:

$$\begin{aligned} r_\parallel &= \frac{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i - \sqrt{\frac{\epsilon}{\mu}}\cos\theta_t}{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i + \sqrt{\frac{\epsilon}{\mu}}\cos\theta_t} \\ t_\parallel &= \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i}{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i + \sqrt{\frac{\epsilon}{\mu}}\cos\theta_t} \end{aligned} \tag{15}$$

In the case of a plane wave normally incident on the interface ($\theta_i = 0$), we obtain

$$\begin{aligned} r_\parallel &= -r_\perp = \frac{\sqrt{\frac{\epsilon'}{\mu'}} - \sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon'}{\mu'}} + \sqrt{\frac{\epsilon}{\mu}}} \to \frac{m' - m}{m' + m}, \mu = \mu' \\ t_\parallel &= t_\perp = \frac{2\sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon'}{\mu'}} + \sqrt{\frac{\epsilon}{\mu}}} \to \frac{2m}{m' + m}, \mu = \mu' \end{aligned} \tag{16}$$

The Fresnel coefficients derived above are also valid for complex-valued ϵ, μ, ϵ', and μ'. Usually, for visible light, $\mu = \mu' = \mu_0$. The generalization of Snel's law for complex m' is left for an exercise. In addition, the derivation of the 4×4 reflection and refraction matrices relating the Stokes parameters of incident, reflected, and refracted light is left for an exercise. In the case of incident electric field polarized in the plane of incidence, we can find the so-called Brewster angle, at which there is no reflected wave. Let $\mu = \mu'$. At the Brewster angle,

$$\begin{aligned}
m' \cos\theta_{iB} &= m\sqrt{1 - \frac{m^2}{m'^2}\sin^2\theta_{iB}} \\
(\frac{m'}{m})^2 \cos^2\theta_{iB} &= 1 - (\frac{m}{m'})^2 \sin^2\theta_{iB} \\
(\frac{m'}{m})^2 &= 1 + \tan^2\theta_{iB} - (\frac{m}{m'})^2 \tan^2\theta_{iB} \\
\tan^2\theta_{iB} &= \frac{(\frac{m'}{m})^2 - 1}{1 - (\frac{m}{m'})^2} = (\frac{m'}{m})^2
\end{aligned}$$

The physical solution is

$$\theta_{iB} = \arctan(\frac{m'}{m}) \tag{17}$$

As a rule for other angles of incidence, too, the reflected light tends to be polarized perpendicular to the plane of incidence.

Total internal reflection can occur when $m > m'$ (the incident wave is "internal"). If $m > m'$, $\theta_t > \theta_{i0}$ according to Snel's law and

$$\theta_{i0} = \arcsin\frac{m'}{m} \tag{18}$$

When the angle of incidence is θ_{i0}, the refracted wave is propagating parallel to the interface and there is no energy flow across the interface. Thus, all the incident energy is reflected back. When $\theta_i > \theta_{i0}$, $\sin\theta_t > 1$ and θ_t must be a complex-valued angle that has a purely imaginary cosine,

$$\cos\theta_t = i\sqrt{(\frac{\sin\theta_i}{\sin\theta_{i0}})^2 - 1} \tag{19}$$

The refracted wave is of the form

$$
\begin{aligned}
e^{i\mathbf{k}_t\cdot\mathbf{x}} &= e^{ik_t(x\sin\theta_t - z\cos\theta_t)} \\
&= e^{-k_t\sqrt{(\frac{\sin\theta_i}{\sin\theta_{i0}})^2-1}|\mathbf{z}|}e^{ik_t(\frac{\sin\theta_i}{\sin\theta_{i0}})x}
\end{aligned} \qquad (20)
$$

and, thus, attenuates exponentially in the medium m' and propagates only in the direction of the interface.

1 Electromagnetic field by a localized source

Consider the electromagnetic fields caused by time-dependent charge and current densities localized in a constrained region of space. Here we will mainly study the fields by an electric dipole. Later, the analysis is extended to the full multipole expansion.
Assume harmonic time dependence $e^{-i\omega t}$—arbitrary time dependences can be dealt with using Fourier analysis of their components. The charge density ρ and current density $\mathbf{j}$ are

$$\rho(\mathbf{x},t) = \rho(\mathbf{x})e^{-i\omega t}$$
$$\mathbf{j}(\mathbf{x},t) = \mathbf{j}(\mathbf{x})e^{-i\omega t}$$

and the physical quantities correspond to the real parts of the complex quantities. The electromagnetic potentials and fields are also time-harmonic and the sources are assumed to be located in an otherwise empty space.

Let us start from the vector potential $\mathbf{A}$ in Lorentz gauge,

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}' \int dt' \frac{\mathbf{j}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta(t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c} - t) \tag{1}$$

and, by writing $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \qquad k = \frac{\omega}{c} \tag{2}$$

The magnetic field is, according to definitions, $\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A}$ and, outside the source region, the electric field equals $\mathbf{E} = \frac{i\zeta_0}{k}\nabla\times\mathbf{H}$, where $\zeta_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. When the current density $\mathbf{j}(\mathbf{x}')$ is given, the electromagnetic field can be calculated from the integral above, at least in principle. Let us study the case where the source region (size d) is much smaller than the wavelength: $d \ll \lambda = 2\pi c/\omega$. We can distinguish three regimes of interest:

(i) Near zone (static regime): $d \ll r \ll \lambda$
(ii) Intermediate zone (induction regime): $d \ll r \sim \lambda$
(iii) Far zone (radiation regime): $d \ll \lambda \ll r$

In the near zone (i) $kr \ll 1$ and the exponential part of the integrand for the vector potential can be set to unity, and the inverse distance can be presented using series of spherical harmonics Y_{lm}:

$$\lim_{kr\to 0}\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\frac{\mathrm{Y}_{lm}(\theta,\varphi)}{r^{l+1}}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(r')^l\mathrm{Y}^*_{lm}(\theta',\varphi') \tag{3}$$

We can see that the near fields vary harmonically in time but are static in their character: no wave solution follows for the spatial dependence. Above, we have made use of the relation

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \mathrm{Y}_{lm}^*(\theta', \varphi') \mathrm{Y}_{lm}(\theta, \varphi) \tag{4}$$

In the far zone (iii), $kr \gg 1$ and the exponential part of the vector potential varies strongly and dictates the character of the vector potential. We can approximate

$$|\mathbf{x}-\mathbf{x}'| \approx r - \hat{\mathbf{n}} \cdot \mathbf{x}', \qquad \hat{\mathbf{n}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{r} \tag{5}$$

When the leading term is desired in kr, the inverse distance can be replaced by r. The vector potential is of the form

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}') e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'} \tag{6}$$

Therefore, the vector potential behaves like an outgoing spherical wave (e^{ikr}/r) with angular dependence. It can be shown that the electromagnetic field is also of the form of a spherical wave and thus is a radiation field. (Note that this part of the analysis is valid for localized source regions of arbitrary size.)

Now that $kd \ll 1$ the integral can further be developed into series:

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_n \frac{(-ik)^n}{n!} \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}') (\hat{\mathbf{n}} \cdot \mathbf{x}')^n \tag{7}$$

where the magnitude for the nth term is $(1/n!) \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}')(k\hat{\mathbf{n}} \cdot \mathbf{x}')^n$ and thus becomes rapidly smaller with increasing n. In this case, the main contribution to radiation comes from the first non-vanishing term in the sum.

In the intermediate zone (ii), all powers of kr need to be accounted for, and no simple limits can be taken. The vector potential is then written with the help of the expansion for the exact Green's function in the form

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr) \mathrm{Y}_{lm}(\theta, \varphi) \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}') j_l(kr') \mathrm{Y}_{lm}^*(\theta', \varphi') \tag{8}$$

where we have made use of the expansion

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik \sum_{l=0}^{\infty} j_l(kr_<) h_l^{(1)}(kr_>) \sum_{m=-l}^{l} \mathrm{Y}_{lm}^*(\theta', \varphi') \mathrm{Y}_{lm}(\theta, \varphi) \tag{9}$$

where $r_< = \min(r, r')$, $r_> = \max(r, r')$, and j_l and $h_l^{(1)}$ are the spherical Bessel and Hankel functions.

Again when $kd \ll 1$, the j_l-functions can be approximated and the result is of the same form as the near zone result, when the following replacement is carried out:

$$\frac{1}{r^{l+1}} \rightarrow \frac{e^{ikr}}{r^{l+1}}[1 + a_1(ikr) + a_2(ikr)^2 + \ldots + a_l(ikr)^l] \tag{10}$$

The coefficients a_i derive from the explicit expansions of the Hankel functions. This end result allows us to see the transition from the near-zone $kr \ll 1$ static field to the far-zone $kr \gg 1$ radiation field.

2 Electromagnetic field of an electric dipole

If only the first term in kd is kept in the expansion of the vector potential, one obtains

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}') \tag{11}$$

which holds everywhere outside the source region (this follows from the intermediate-zone results above). With the help of partial integration,

$$\int d^3\mathbf{x}'\mathbf{j} = -\int d^3\mathbf{x}'\mathbf{x}'(\nabla \cdot \mathbf{j}) = -i\omega \int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}') \tag{12}$$

where the substitution term disappears (the source region is constrained) and, according to the continuity equation, $i\omega\rho(\mathbf{x}') = \nabla \cdot \mathbf{j}(\mathbf{x}')$. The vector potential is thus

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}, \tag{13}$$

where $\mathbf{p}$ is the electric dipole moment $\mathbf{p} = \int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}')$.

The electromagnetic fields are

$$\begin{aligned}
\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r}(1 - \frac{1}{ikr}) \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\Big(k^2(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}\frac{e^{ikr}}{r} + (3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p})(\frac{1}{r^2} - \frac{ik}{r})\frac{e^{ikr}}{r}\Big)
\end{aligned}$$

We note that the magnetic field is always transverse but that the electric field has both longitudinal and transverse components.

In the far zone,

$$\begin{aligned}\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r} \\ \mathbf{E} &= \zeta_0 \mathbf{H} \times \hat{\mathbf{n}}\end{aligned}$$

which shows the typical form of a spherical wave.

In the near zone,

$$\begin{aligned}\mathbf{H} &= \frac{i\omega}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{1}{r^2} \\ \mathbf{E} &= \frac{1}{4\pi\epsilon_0}(3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p})\frac{1}{r^3}\end{aligned}$$

The electric field is, except for the harmonic time dependence, that of a static electric dipole. The field $\zeta_0 \mathbf{H}$ is smaller, by a factor of kr, than the field $\mathbf{E}$ so, in the near zone, the field is electric in its nature. In the static limit $k \to 0$, the magnetic field disappears and the near zone extends to infinity.

The power radiated by the vibrating dipole moment $\mathbf{p}$ as per solid angle is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2} Re(r^2 \hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^*) \\ &= \frac{c^2\zeta_0}{32\pi^2} k^4 |(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}|^2,\end{aligned}$$

where $\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}$ gives the polarization state. If all components of $\mathbf{p}$ are in the same phase,

$$\frac{dP}{d\Omega} = \frac{c^2\zeta_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta \tag{14}$$

which is the typical radiation pattern of an electric dipole (θ is here measured from the direction of $\mathbf{p}$). Independently of the phases of the components for $\mathbf{p}$:n, the total radiated power is

$$P = \frac{c^2\zeta_0 k^4}{12\pi} |\mathbf{p}|^2 \tag{15}$$

3 Scattering by small spherical particles in the electric dipole approximation

Light scattering by particles clearly smaller than the wavelength can be studied in the approximation, where the incident field induces an electric dipole moment to the particle. The dipole fluctuates in a certain phase with the incident field and thus scatters radiation in directions differing from the propagation direction of the incident field. In this case, the dipole moments can be computed using electrostatic methods.

Assume that a monochromatic plane wave is incident on a small scatterer located in free space. Let the propagation direction and polarization vector of the incident field be $\hat{\mathbf{n}}_0$ and $\hat{\epsilon}_0$:

$$\begin{aligned} \mathbf{E}_i &= \hat{\epsilon}_0 E_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}} \\ \mathbf{H}_i &= \hat{\mathbf{n}}_0 \times \mathbf{E}_i/\zeta_0 \end{aligned}$$

where $k = \omega/c$ and the time dependence has been assumed harmonic ($e^{-i\omega t}$). These fields induce a dipole momentn $\mathbf{p}$ in the small particle and the particle radiates energy in (almost) all directions. In the far zone, the scattered fields are of the form

$$\begin{aligned} \mathbf{E}_s &= \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r}((\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}) \\ \mathbf{H}_s &= \hat{\mathbf{n}} \times \mathbf{E}_s/\zeta_0 \end{aligned}$$

where $\hat{\mathbf{n}}$ is the dirction of the observer and r the distance from the scatterer. The power scattered in direction $\hat{\mathbf{n}}$ with polarization $\hat{\epsilon}$ per unit solid angle divided by the incident flux density is the so-called differential cross section

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \hat{\epsilon}, \hat{\mathbf{n}}_0, \hat{\epsilon}_0) = \frac{r^2 \frac{1}{2\zeta_0}|\hat{\epsilon}^* \cdot \mathbf{E}_s|^2}{\frac{1}{2\zeta_0}|\hat{\epsilon}_0^* \cdot \mathbf{E}_i|^2} \tag{16}$$

where the complex conjugation of the polarization vectors is important for proper treatment of circular polarization. Furthermore,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \hat{\epsilon}, \hat{\mathbf{n}}_0, \hat{\epsilon}_0) = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\hat{\epsilon}^* \cdot \mathbf{p}|^2, \tag{17}$$

where the $\hat{\mathbf{n}}_0, \hat{\epsilon}_0$ -dependence is implicit in $\mathbf{p}$. We can see that the differential and total cross sections of the dipole scatterer are both proportional to k^4 and λ^{-4} (Rayleigh's law).
Assume that the scatterer is a small sphere (radius a) with the relative permittivity $\epsilon_r = \epsilon/\epsilon_0$. According to electrostatics, the dipole moment of the sphere is

$$\mathbf{p} = 4\pi\epsilon_0 \Big(\frac{\epsilon_r - 1}{\epsilon_r + 2}\Big) a^3 \mathbf{E}_i \tag{18}$$

so that

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \tag{19}$$

The polarization dependence is purely that of electric dipole scattering. The scattered radiation is polarized in the plane defined by the dipole moment $\hat{\epsilon}_0$ and the vector $\hat{\mathbf{n}}$.

For unpolarized incident radiation, the differential cross sections in different polarization states of the scattered field are

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{k^4a^6}{2}|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \cos^2\theta$$
$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{k^4a^6}{2}|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2$$

where θ is now the scattering angle.

The degree of polarization is

$$P(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta} = -\frac{S_{21}(\theta)}{S_{11}(\theta)} \tag{20}$$

and the differential cross section summed over the polarization states of the scattered field is

$$\frac{d\sigma}{d\Omega} = k^4a^6|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2\frac{1}{2}(1+\cos^2\theta) \propto S_{11}(\theta) \tag{21}$$

where $S_{11}(\theta)$ and $S_{21}(\theta)$ are elements of the scattering matrix. The total scattering cross section is

$$\sigma = \int_{(4\pi)} \frac{d\sigma}{d\Omega}d\Omega = \frac{8\pi}{3}k^4a^6|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \tag{22}$$

The scattered radiation is 100% positively polarized at the scattering angle $\theta = 90°$. It was the polarization characteristics of the blue sky that got Rayleigh interested in scattering by small particles.