Computational Science:
Introduction to Finite-Difference Time-Domain

Formulation of One-Dimensional FDTD

Lecture Outline

- Review of Lecture 4
- Yee grid scheme
- Finite-difference approximation of Maxwell's curl equations
- Governing equations for one-dimensional FDTD
- Derivation of basic update equations
- Implementation of basic update equations for E_y/H_x mode

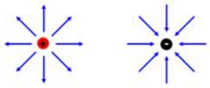
Review of Lecture #4

Maxwell's Equations

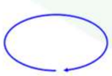
Gauss' Law

$$\boxed{\nabla \bullet \vec{D} = \rho_v} \qquad \nabla \bullet \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Electric fields diverge from positive charges and converge on negative charges.



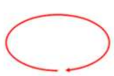
If there are no charges, electric fields must form loops.



Gauss' Law for Magnetic Fields

$$\boxed{\nabla \bullet \vec{B} = 0} \qquad \nabla \bullet \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

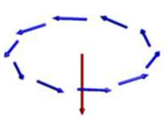
Magnetic fields always form loops.



Faraday'sLaw

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \qquad \nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z$$

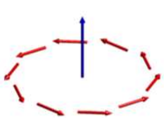
Circulating electric fields induce time varying magnetic fields.
Time varying magnetic fields induce circulating electric fields.



Ampere's Circuit Law

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}} \qquad \nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z$$

Circulating magnetic fields induce currents and/or time varying electric fields.
Currents and/or time varying electric fields induce circulating magnetic fields.



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Summary of Parameter Relations

Permittivity

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\varepsilon_0 = 8.854187817 \times 10^{-12} \quad \text{F/m}$$

Permeability

$$\mu = \mu_0 \mu_r$$

$$\mu_0 = 1.256637061 \times 10^{-6} \quad \text{H/m}$$

Refractive Index

$$n = \sqrt{\mu_r \varepsilon_r}$$

$$n = \sqrt{\varepsilon_r} \quad \text{no magnetic response}$$

Impedance

$$\eta = \eta_0 \sqrt{\mu_r / \varepsilon_r}$$

$$\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 376.73031346177 \quad \Omega$$

Wave Velocity

$$v = \frac{c_0}{n}$$

$$c_0 = 299792458 \quad \text{m/s}$$ ← Exact

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Flow of Maxwell's Equations

$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$$

A circulating $\vec{E}$ field induces a change in the $\vec{B}$ field at the center of circulation.

→

$$\vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

A $\vec{B}$ field induces an $\vec{H}$ field in proportion to the permeability.

↓

$$\nabla \times \vec{H}(t) = \frac{\partial \vec{D}(t)}{\partial t}$$

A circulating $\vec{H}$ field induces a change in the $\vec{D}$ field at the center of circulation.

←

$$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$

A $\vec{D}$ field induces an $\vec{E}$ field in proportion to the permittivity.

↑

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Finite-Difference Approximation of Maxwell's Equations

The time derivatives in Maxwell's equations are approximated with finite-differences.

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{E}\Big|_t = -\mu \frac{\vec{H}\Big|_{t+\Delta t/2} - \vec{H}\Big|_{t-\Delta t/2}}{\Delta t}$$

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{H}\Big|_{t+\Delta t/2} = \varepsilon \frac{\vec{E}\Big|_{t+\Delta t} - \vec{E}\Big|_t}{\Delta t}$$

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The FDTD Update Equation

$$\vec{E}\Big|_{t+\Delta t} = \vec{E}\Big|_t + \frac{\Delta t}{\varepsilon}\left(\nabla \times \vec{H}\Big|_{t+\Delta t/2}\right)$$

Update coefficient

To speed simulation, we calculate these before iteration.

Field at the next time step.

Field at the previous time step.

Curl of the "other" field at an intermediate time step

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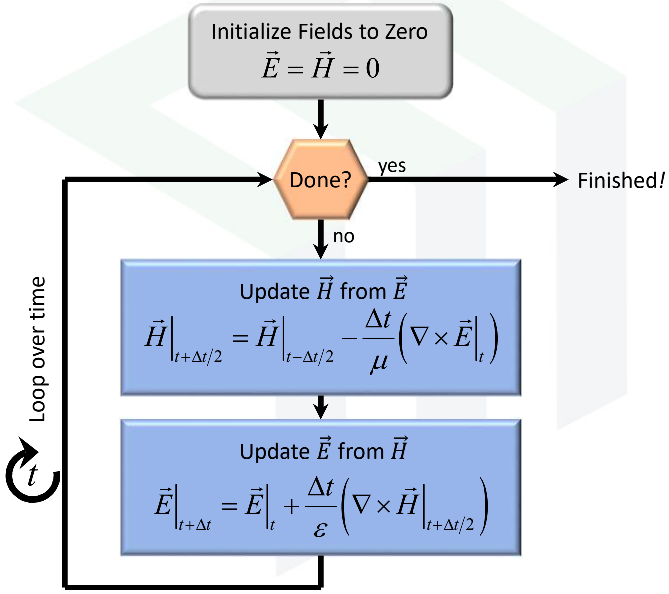
The FDTD Algorithm…for now ☺



$\vec{H}\big|_{t-\frac{\Delta t}{2}} = 0 \quad \vec{E}\big|_t = 0$

$$\vec{H}\big|_{t+\frac{\Delta t}{2}} \Leftarrow \vec{E}\big|_t$$
$$\vec{E}\big|_{t+\Delta t} \Leftarrow \vec{H}\big|_{t+\frac{\Delta t}{2}}$$
$$\vec{H}\big|_{t+\frac{3\Delta t}{2}} \Leftarrow \vec{E}\big|_{t+\Delta t}$$
$$\vec{E}\big|_{t+2\Delta t} \Leftarrow \vec{H}\big|_{t+\frac{3\Delta t}{2}}$$
$$\vec{H}\big|_{t+\frac{5\Delta t}{2}} \Leftarrow \vec{E}\big|_{t+2\Delta t}$$
$$\vec{E}\big|_{t+3\Delta t} \Leftarrow \vec{H}\big|_{t+\frac{5\Delta t}{2}}$$
$$\vdots$$

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Yee Grid Scheme

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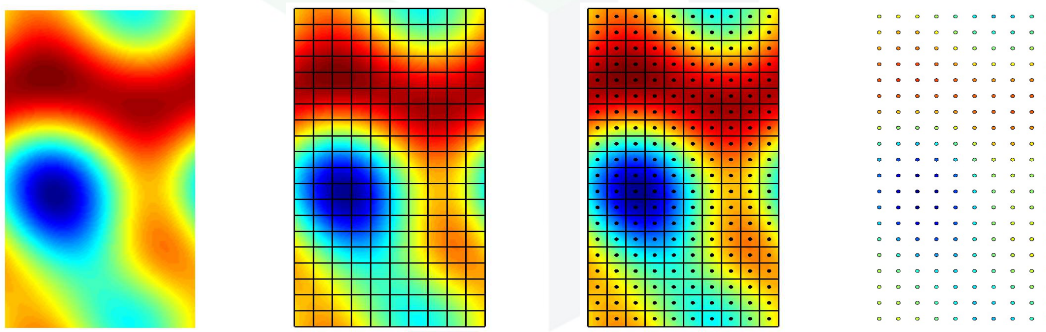
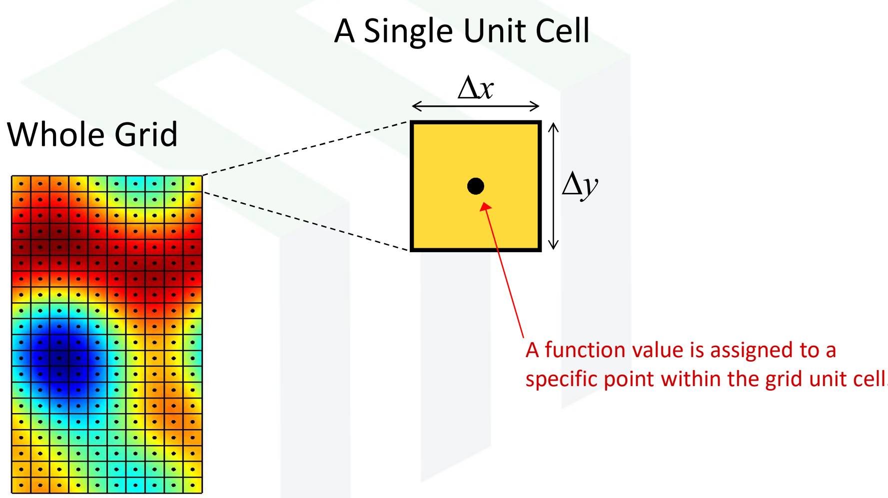
Representing Functions on a Grid

Example physical (continuous) 2D function	A grid is constructed by dividing space into discrete cells	Function is known only at discrete points	Representation of what is actually stored in memory



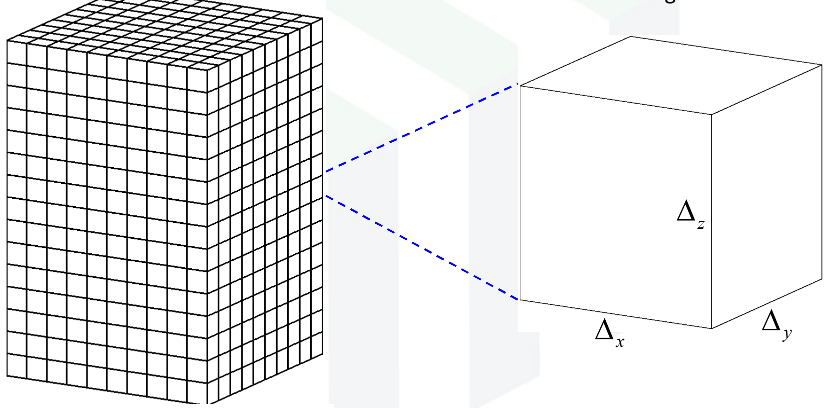
Grid Unit Cell



3D Grids

A three-dimensional grid looks like this:

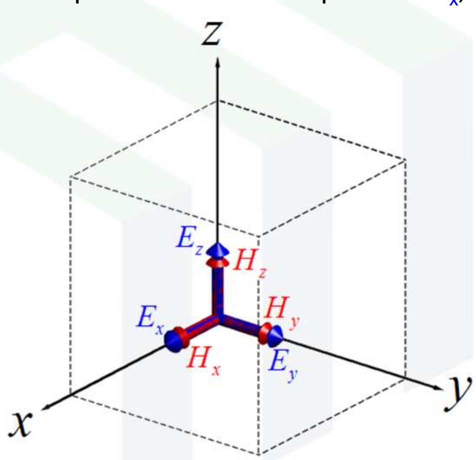
A unit cell from the grid looks like this:



$$N_x = 10,\ N_y = 10,\ N_z = 15$$

Collocated Grid

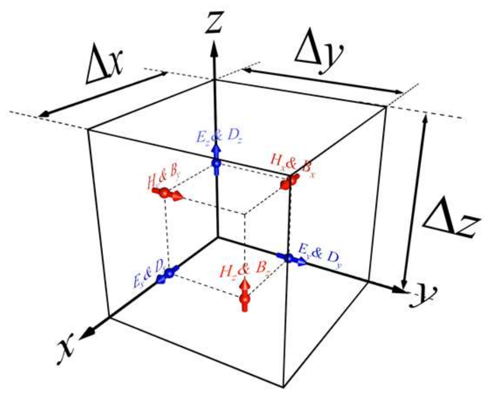
Within the unit cell, we need to place the field components E_x, E_y, E_z, H_x, H_y, and H_z.



A straightforward approach would be to locate all of the field components within in a grid cell at the origin of the cell.

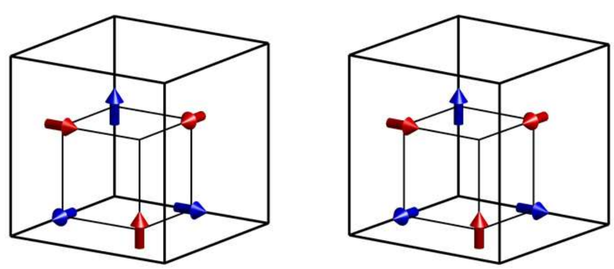
Yee Grid

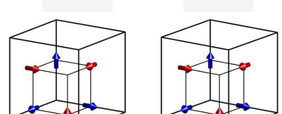
Instead, the position of each field component will be staggered within the grid cells.



K. S. Yee, "Numerical solution of the initial boundary value problems involving Maxwell's equations in isotropic media," IEEE Trans. Microwave Theory and Techniques, vol. 44, pp. 61–69, 1998.

Stereo Picture of Yee Cell





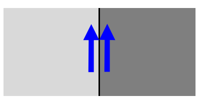
Reasons to Use the Yee Grid Scheme

1. Divergence-free

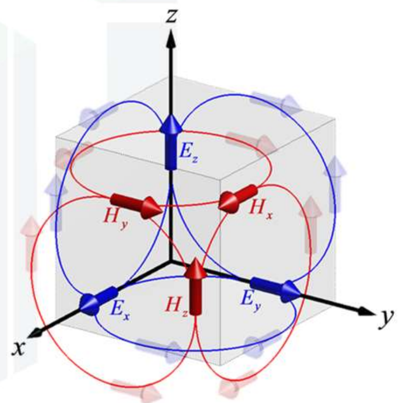
$$\nabla \bullet \left(\varepsilon \vec{E} \right) = 0$$

$$\nabla \bullet \left(\mu \vec{H} \right) = 0$$

2. Physical boundary conditions are naturally satisfied

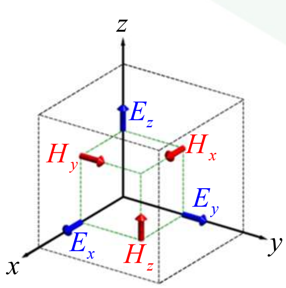


3. Elegant arrangement to approximate Maxwell's curl equations

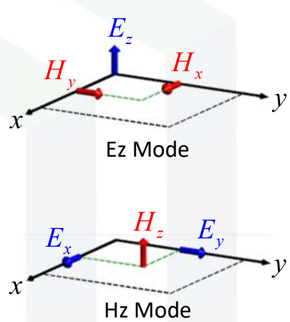


Yee Cell for 1D, 2D, and 3D Grids

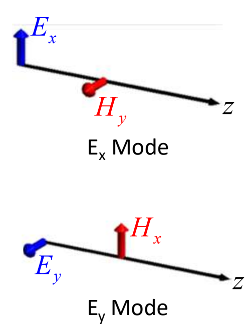
3D Yee Grid



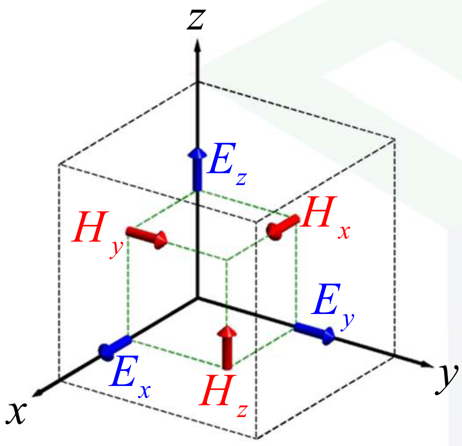
2D Yee Grids



1D Yee Grid



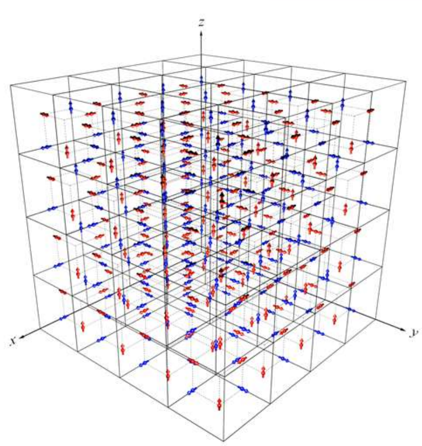
Consequences of the Yee Grid



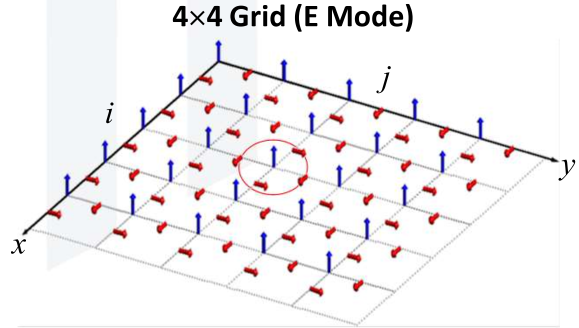
- Field components are in physically different locations
- Field components may reside in different materials even if they are in the same unit cell
- Field components will be out of phase
- Recall the field components are also staggered in time

Visualizing Extended Yee Grids

4×4×4 Grid



4×4 Grid (E Mode)



Finite-Difference Approximation to Maxwell's Equations

Normalize the Magnetic Field

The divergence equations were satisfied by adopting the Yee grid scheme. Now, only the the curl equations have to be dealt with.

$$\nabla \times \vec{E} = -[\mu]\frac{\partial \vec{H}}{\partial t} \qquad \nabla \times \vec{H} = [\varepsilon]\frac{\partial \vec{E}}{\partial t}$$

The $\vec{E}$ and $\vec{H}$ fields are related through the impedance of the material they are in, so they are roughly three orders of magnitude different.

$$\left|\vec{E}\right| \cong \eta \left|\vec{H}\right| \qquad \eta \sim 300\ \Omega$$

This may cause rounding errors in your simulation and it is always good practice to normalize parameters so they are all the same order of magnitude. Here, the magnetic field is normalized.

$$\left|\vec{E}\right| \cong \left|\vec{\tilde{H}}\right| \quad \rightarrow \quad \vec{\tilde{H}} = \eta \vec{H}$$

$$\boxed{\vec{\tilde{H}} = \eta_0 \vec{H}}$$

The actual impedance is not known ahead of time, so the free space impedance is used.

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Curl Equations with Normalized Magnetic Field

Using the normalized magnetic field, the curl equations become

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0}\frac{\partial \vec{\tilde{H}}}{\partial t} \qquad \nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0}\frac{\partial \vec{E}}{\partial t}$$

<u>Proof</u>

$$\vec{\tilde{H}} = \eta_0 \vec{H} \quad \rightarrow \quad \vec{H} = \vec{\tilde{H}}/\eta_0$$

$$\nabla \times \vec{E} = -[\mu]\frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{E} = -[\mu_0 \mu_r]\frac{\partial\left(\vec{\tilde{H}}/\eta_0\right)}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\mu_0}{\eta_0}[\mu_r]\frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0\sqrt{\frac{\varepsilon_0}{\mu_0}}[\mu_r]\frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{E} = -\sqrt{\mu_0\varepsilon_0}[\mu_r]\frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0}\frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{H} = [\varepsilon]\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \left(\vec{\tilde{H}}/\eta_0\right) = [\varepsilon_0\varepsilon_r]\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \varepsilon_0\eta_0[\varepsilon_r]\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \varepsilon_0\sqrt{\frac{\mu_0}{\varepsilon_0}}[\varepsilon_r]\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \sqrt{\mu_0\varepsilon_0}[\varepsilon_r]\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0}\frac{\partial \vec{E}}{\partial t}$$

Note:

$$[\mu] = \mu_0[\mu_r]$$

$$[\varepsilon] = \varepsilon_0[\varepsilon_r]$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

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Expand the Curl Equations

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0}\frac{\partial \vec{\tilde{H}}}{\partial t} \quad \Rightarrow$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c_0}\left(\mu_{xx}\frac{\partial \tilde{H}_x}{\partial t} + \mu_{xy}\frac{\partial \tilde{H}_y}{\partial t} + \mu_{xz}\frac{\partial \tilde{H}_z}{\partial t}\right)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c_0}\left(\mu_{yx}\frac{\partial \tilde{H}_x}{\partial t} + \mu_{yy}\frac{\partial \tilde{H}_y}{\partial t} + \mu_{yz}\frac{\partial \tilde{H}_z}{\partial t}\right)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c_0}\left(\mu_{zx}\frac{\partial \tilde{H}_x}{\partial t} + \mu_{zy}\frac{\partial \tilde{H}_y}{\partial t} + \mu_{zz}\frac{\partial \tilde{H}_z}{\partial t}\right)$$

$$\nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0}\frac{\partial \vec{E}}{\partial t} \quad \Rightarrow$$

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{1}{c_0}\left(\varepsilon_{xx}\frac{\partial E_x}{\partial t} + \varepsilon_{xy}\frac{\partial E_y}{\partial t} + \varepsilon_{xz}\frac{\partial E_z}{\partial t}\right)$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{1}{c_0}\left(\varepsilon_{yx}\frac{\partial E_x}{\partial t} + \varepsilon_{yy}\frac{\partial E_y}{\partial t} + \varepsilon_{yz}\frac{\partial E_z}{\partial t}\right)$$

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{1}{c_0}\left(\varepsilon_{zx}\frac{\partial E_x}{\partial t} + \varepsilon_{zy}\frac{\partial E_y}{\partial t} + \varepsilon_{zz}\frac{\partial E_z}{\partial t}\right)$$

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Assume Only Diagonal Tensors

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0}\frac{\partial \vec{\tilde{H}}}{\partial t} \Rightarrow$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c_0}\left(\mu_{xx}\frac{\partial \tilde{H}_x}{\partial t} + \cancel{\mu_{xy}\frac{\partial \tilde{H}_y}{\partial t}} + \cancel{\mu_{xz}\frac{\partial \tilde{H}_z}{\partial t}} \right)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c_0}\left(\cancel{\mu_{yx}\frac{\partial \tilde{H}_x}{\partial t}} + \mu_{yy}\frac{\partial \tilde{H}_y}{\partial t} + \cancel{\mu_{yz}\frac{\partial \tilde{H}_z}{\partial t}} \right)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c_0}\left(\cancel{\mu_{zx}\frac{\partial \tilde{H}_x}{\partial t}} + \cancel{\mu_{zy}\frac{\partial \tilde{H}_y}{\partial t}} + \mu_{zz}\frac{\partial \tilde{H}_z}{\partial t} \right)$$

$$\nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0}\frac{\partial \vec{E}}{\partial t} \Rightarrow$$

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{1}{c_0}\left(\varepsilon_{xx}\frac{\partial E_x}{\partial t} + \cancel{\varepsilon_{xy}\frac{\partial E_y}{\partial t}} + \cancel{\varepsilon_{xz}\frac{\partial E_z}{\partial t}} \right)$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{1}{c_0}\left(\cancel{\varepsilon_{yx}\frac{\partial E_x}{\partial t}} + \varepsilon_{yy}\frac{\partial E_y}{\partial t} + \cancel{\varepsilon_{yz}\frac{\partial E_z}{\partial t}} \right)$$

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{1}{c_0}\left(\cancel{\varepsilon_{zx}\frac{\partial E_x}{\partial t}} + \cancel{\varepsilon_{zy}\frac{\partial E_y}{\partial t}} + \varepsilon_{zz}\frac{\partial E_z}{\partial t} \right)$$

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Final Analytical Equations

These are the final form of Maxwell's equations from which the FDTD method will be formulated.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t} \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t} \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{\varepsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t} \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{\varepsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

Next, these equations will be approximated with finite-differences in the Yee grid.

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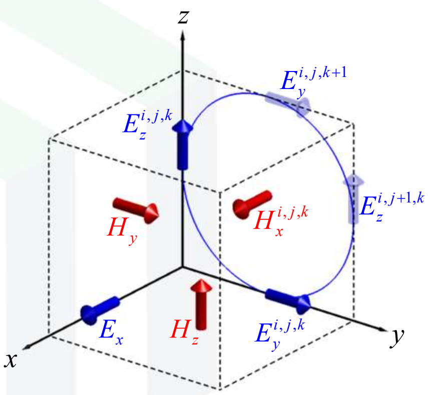
Finite-Difference Equation for H_x

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$



$$\frac{E_z^{i,j+1,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

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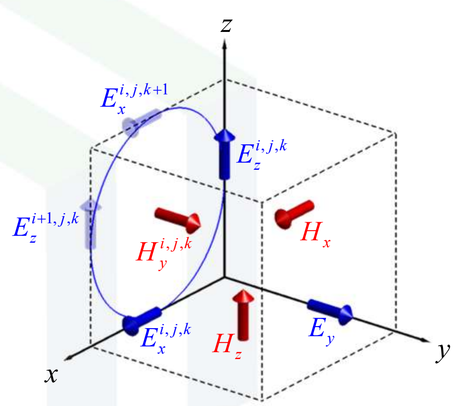
Finite-Difference Equation for H_y

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$



$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} - \frac{E_z^{i+1,j,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta x} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

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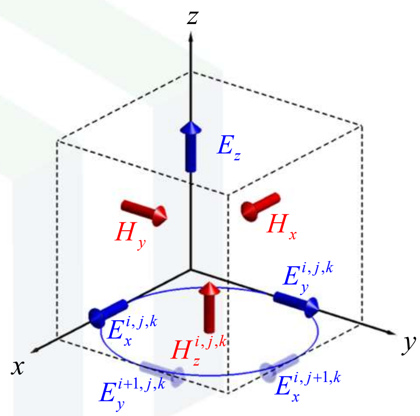
Finite-Difference Equation for H_z

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$



$$\frac{E_y^{i+1,j,k}\big|_t - E_y^{i,j,k}\big|_t}{\Delta x} - \frac{E_x^{i,j+1,k}\big|_t - E_x^{i,j,k}\big|_t}{\Delta y} = -\frac{\mu_{zz}^{i,j,k}}{c_0}\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

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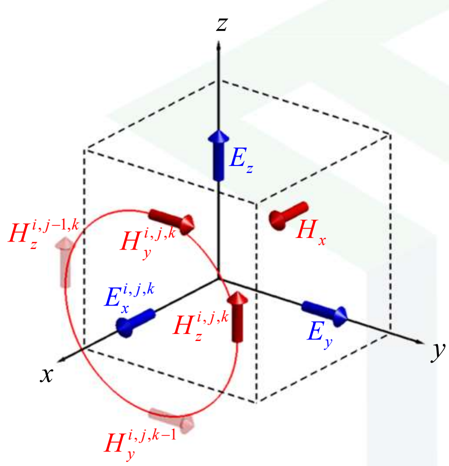
Finite-Difference Equation for E_x



$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t}$$

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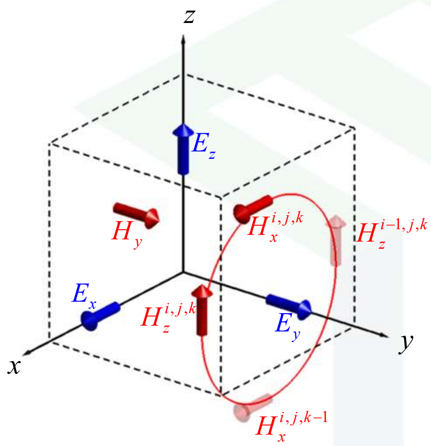
Finite-Difference Equation for E_y



$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{\varepsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$\frac{\tilde{H}_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{\varepsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\Big|_{t+\Delta t} - E_y^{i,j,k}\Big|_{t}}{\Delta t}$$

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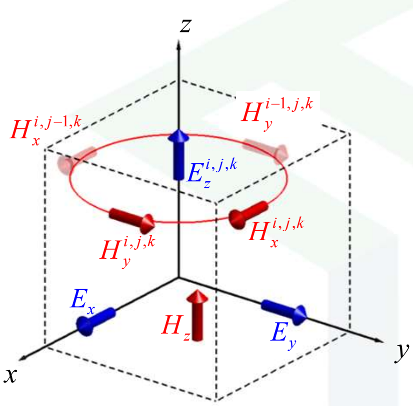
Finite-Difference Equation for E_z



$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{\varepsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$$\frac{\tilde{H}_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{\varepsilon_{zz}^{i,j,k}}{c_0}\frac{E_z^{i,j,k}\Big|_{t+\Delta t} - E_z^{i,j,k}\Big|_{t}}{\Delta t}$$

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Summary of Finite-Difference Equations

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$
$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

$$\frac{E_z^{i,j+1,k}\big|_t-E_z^{i,j,k}\big|_t}{\Delta y}-\frac{E_y^{i,j,k+1}\big|_t-E_y^{i,j,k}\big|_t}{\Delta z}=-\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$\frac{E_x^{i,j,k+1}\big|_t-E_x^{i,j,k}\big|_t}{\Delta z}-\frac{E_z^{i+1,j,k}\big|_t-E_z^{i,j,k}\big|_t}{\Delta x}=-\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$\frac{E_y^{i+1,j,k}\big|_t-E_y^{i,j,k}\big|_t}{\Delta x}-\frac{E_x^{i,j+1,k}\big|_t-E_x^{i,j,k}\big|_t}{\Delta y}=-\frac{\mu_{zz}^{i,j,k}}{c_0}\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_z^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{\partial \tilde{H}_z}{\partial y}-\frac{\partial \tilde{H}_y}{\partial z}=\frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$
$$\frac{\partial \tilde{H}_x}{\partial z}-\frac{\partial \tilde{H}_z}{\partial x}=\frac{\varepsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$
$$\frac{\partial \tilde{H}_y}{\partial x}-\frac{\partial \tilde{H}_x}{\partial y}=\frac{\varepsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$$\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_z^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y}-\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}=\frac{\varepsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\big|_{t+\Delta t}-E_x^{i,j,k}\big|_t}{\Delta t}$$
$$\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}-\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_z^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x}=\frac{\varepsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\big|_{t+\Delta t}-E_y^{i,j,k}\big|_t}{\Delta t}$$
$$\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_y^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x}-\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y}=\frac{\varepsilon_{zz}^{i,j,k}}{c_0}\frac{E_z^{i,j,k}\big|_{t+\Delta t}-E_z^{i,j,k}\big|_t}{\Delta t}$$

Each equation is enforced separately for each cell in the grid. This is repeated for each time step until the simulation is finished. These equations get repeated a lot*!!*

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Governing Equations for One-Dimensional FDTD

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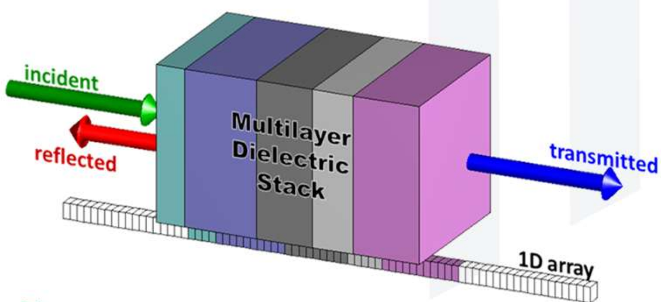
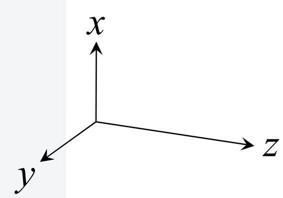
Reduction to One Dimension

In previous lectures, it was observed that some problems composed of dielectric slabs can be described in just one dimension. In this case, the materials and the fields are uniform in two directions. Derivatives in these uniform directions will be zero. The uniform directions will be defined to be the x and y axes.

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial y}=0$$





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x and y Derivatives are Zero

$$\cancel{\frac{\partial E_z}{\partial y}}-\frac{\partial E_y}{\partial z}=-\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$
$$\frac{\partial E_x}{\partial z}-\cancel{\frac{\partial E_z}{\partial x}}=-\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$
$$\cancel{\frac{\partial E_y}{\partial x}}-\cancel{\frac{\partial E_x}{\partial y}}=-\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

$$\cancel{\frac{\partial \tilde{H}_z}{\partial y}}-\frac{\partial \tilde{H}_y}{\partial z}=\frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$
$$\frac{\partial \tilde{H}_x}{\partial z}-\cancel{\frac{\partial \tilde{H}_z}{\partial x}}=\frac{\varepsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$
$$\cancel{\frac{\partial \tilde{H}_y}{\partial x}}-\cancel{\frac{\partial \tilde{H}_x}{\partial y}}=\frac{\varepsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

➡

$$\cancel{\frac{E_z^{i,j+1,k}\big|_t-E_z^{i,j,k}\big|_t}{\Delta y}}-\frac{E_y^{i,j,k+1}\big|_t-E_y^{i,j,k}\big|_t}{\Delta z}=-\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$\frac{E_x^{i,j,k+1}\big|_t-E_x^{i,j,k}\big|_t}{\Delta z}-\cancel{\frac{E_z^{i+1,j,k}\big|_t-E_z^{i,j,k}\big|_t}{\Delta x}}=-\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$\cancel{\frac{E_y^{i+1,j,k}\big|_t-E_y^{i,j,k}\big|_t}{\Delta x}}-\cancel{\frac{E_x^{i,j+1,k}\big|_t-E_x^{i,j,k}\big|_t}{\Delta y}}=-\frac{\mu_{zz}^{i,j,k}}{c_0}\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_z^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\cancel{\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_z^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y}}-\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}=\frac{\varepsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\big|_{t+\Delta t}-E_x^{i,j,k}\big|_t}{\Delta t}$$
$$\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}-\cancel{\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_z^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x}}=\frac{\varepsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\big|_{t+\Delta t}-E_y^{i,j,k}\big|_t}{\Delta t}$$
$$\cancel{\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_y^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x}}-\cancel{\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y}}=\frac{\varepsilon_{zz}^{i,j,k}}{c_0}\frac{E_z^{i,j,k}\big|_{t+\Delta t}-E_z^{i,j,k}\big|_t}{\Delta t}$$

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Maxwell's Equations Decouple Into Two Independent Modes

The longitudinal field components E_z and H_z are always zero.
Maxwell's equations have decoupled into two sets of two equations.

$$-\frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$
$$\frac{\partial E_x}{\partial z} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$
$$0 = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$
$$-\frac{\partial \tilde{H}_y}{\partial z} = \frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$
$$\frac{\partial \tilde{H}_x}{\partial z} = \frac{\varepsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$
$$0 = \frac{\varepsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$\Rightarrow$

$$-\frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$\tilde{H}_z^{i,j,k} = 0$$
$$-\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t}$$
$$\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t}$$
$$E_z^{i,j,k} = 0$$

E_x/H_y Mode

E_y/H_x Mode

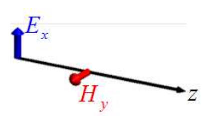
Two Remaining Modes are the Same

The longitudinal field components E_z and H_z are always zero.
Maxwell's equations have decoupled into two sets of two equations.

$$\frac{\partial E_x}{\partial z} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$
$$-\frac{\partial \tilde{H}_y}{\partial z} = \frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

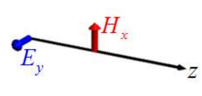
$\Rightarrow$

$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$
$$-\frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t}$$



$$\frac{\partial E_y}{\partial z} = \frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$
$$\frac{\partial \tilde{H}_x}{\partial z} = \frac{\varepsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t}$$
$$\frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = \frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$



While these modes are physical and would propagate independently, they are numerically the same and will exhibit the same electromagnetic behavior in isotropic media. Therefore, it is only necessary to solve one. We will proceed with the E_y/H_x mode.

No Longer Need i and j Array Indices

E_x/H_y Mode

$$-\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{xx}^k}{c_0}\frac{E_x^k\big|_{t+\Delta t} - E_x^k\big|_t}{\Delta t}$$

$$\frac{E_x^{k+1}\big|_t - E_x^k\big|_t}{\Delta z} = -\frac{\mu_{yy}^k}{c_0}\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

E_y/H_x Mode

$$-\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z} = -\frac{\mu_{xx}^k}{c_0}\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{yy}^k}{c_0}\frac{E_y^k\big|_{t+\Delta t} - E_y^k\big|_t}{\Delta t}$$

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Derivation of the Basic Update Equations

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Update Equation for E_x

Start with the finite-difference equation which has E_x in the time-derivative:

$$-\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{xx}^k}{c_0}\frac{E_x^k\big|_{t+\Delta t} - E_x^k\big|_t}{\Delta t}$$

Solve this for E_x at the future time value.

$$\frac{\varepsilon_{xx}^k}{c_0}\frac{E_x^k\big|_{t+\Delta t} - E_x^k\big|_t}{\Delta t} = -\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$E_x^k\big|_{t+\Delta t} - E_x^k\big|_t = -\frac{c_0 \Delta t}{\varepsilon_{xx}^k}\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$\boxed{E_x^k\big|_{t+\Delta t} = E_x^k\big|_t + \left(-\frac{c_0 \Delta t}{\varepsilon_{xx}^k}\right)\left(\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)}$$

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Update Equation for E_y

Start with the finite-difference equation which has E_y in the time-derivative:

$$\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{yy}^k}{c_0}\frac{E_y^k\big|_{t+\Delta t} - E_y^k\big|_t}{\Delta t}$$

Solve this for E_y at the future time value.

$$\frac{\varepsilon_{yy}^k}{c_0}\frac{E_y^k\big|_{t+\Delta t} - E_y^k\big|_t}{\Delta t} = \frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$E_y^k\big|_{t+\Delta t} - E_y^k\big|_t = \frac{c_0 \Delta t}{\varepsilon_{yy}^k}\left(\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)$$

$$\boxed{E_y^k\big|_{t+\Delta t} = E_y^k\big|_t + \left(\frac{c_0 \Delta t}{\varepsilon_{yy}^k}\right)\left(\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)}$$

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Update Equation for H_x

Start with the finite-difference equation which has H_x in the time-derivative:

$$\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z} = \frac{\mu_{xx}^k}{c_0}\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Solve this for H_x at the future time value.

$$\frac{\mu_{xx}^k}{c_0}\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k\big|_{t-\frac{\Delta t}{2}}}{\Delta t} = \frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z}$$

$$\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k\big|_{t-\frac{\Delta t}{2}} = \frac{c_0 \Delta t}{\mu_{xx}^k}\left(\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z}\right)$$

$$\boxed{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\big|_{t-\frac{\Delta t}{2}} + \frac{c_0 \Delta t}{\mu_{xx}^k}\left(\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z}\right)}$$

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Update Equation for H_y

Start with the finite-difference equation which has H_y in the time-derivative:

$$\frac{E_x^{k+1}\big|_t - E_x^k\big|_t}{\Delta z} = -\frac{\mu_{yy}^k}{c_0}\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Solve this for H_y at the future time value.

$$-\frac{\mu_{yy}^k}{c_0}\frac{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k\big|_{t-\frac{\Delta t}{2}}}{\Delta t} = \frac{E_x^{k+1}\big|_t - E_x^k\big|_t}{\Delta z}$$

$$\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k\big|_{t-\frac{\Delta t}{2}} = -\frac{c_0 \Delta t}{\mu_{yy}^k}\frac{E_x^{k+1}\big|_t - E_x^k\big|_t}{\Delta z}$$

$$\boxed{\tilde{H}_y^k\big|_{t+\frac{\Delta t}{2}} = \tilde{H}_y^k\big|_{t-\frac{\Delta t}{2}} - \frac{c_0 \Delta t}{\mu_{yy}^k}\frac{E_x^{k+1}\big|_t - E_x^k\big|_t}{\Delta z}}$$

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Implementation of the Basic Update Equations for the E_y/H_x Mode

Efficient Implementation of the Update Equations

The update coefficients do not change their value during the simulation. They should be computed only once before the main FDTD loop and not at each iteration inside the loop.

The finite-difference equations in terms of the update coefficients are:

$$\tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k\left(\frac{E_y^{k+1}\Big|_t - E_y^k\Big|_t}{\Delta z}\right)$$

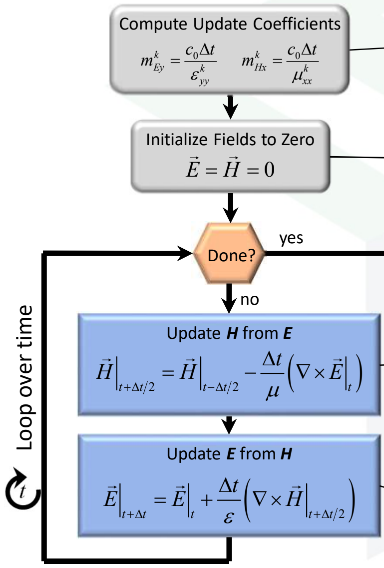
$$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + m_{Ey}^k\left(\frac{\tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)$$

$$m_{Ey}^k = \frac{c_0 \Delta t}{\varepsilon_{yy}^k} \qquad m_{Hx}^k = \frac{c_0 \Delta t}{\mu_{xx}^k}$$

H_x, E_y, ε_{yy}, μ_{xx}, m_{Hx}, and m_{Ey} are all stored in 1D arrays of length N_z.

c_0, Δt, and Δz are single scalar numbers, not arrays.

The Basic 1D-FDTD Algorithm



Calculate the update coefficients before the main loop to save computations.

$\tilde{H}_x^k = E_y^k = 0 \quad \text{for all } k$

Finished!

Loop over all values of z:

$$\tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k\left(\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z}\right)$$

Loop over all values of z:

$$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + m_{Ey}^k\left(\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)$$

Includes:
- Basic FDTD engine

Excludes:
- Dirichlet BC's
- Calculate source parameters
- Simple soft source
- Perfectly absorbing BC's
- TF/SF source
- Fourier transforms
- Reflectance/Transmittance
- Calculate grid parameters
- Incorporate device

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Equations → MATLAB Code

Update Coefficients

$$m_{Ey}^{k}=\frac{c_0\Delta t}{\varepsilon_{yy}^{k}} \qquad m_{Hx}^{k}=\frac{c_0\Delta t}{\mu_{xx}^{k}}$$

```
% INITIALIZE MATERIALS TO FREE SPACE
ER = ones(1,Nz);
UR = ones(1,Nz);

% COMPUTE UPDATE COEFFICIENTS
mEy = (c0*dt) ./ ER;
mHx = (c0*dt) ./ UR;
```

Update Equations

You will need to update the fields at every point in the grid so these equations are placed inside a loop from `1` to `Nz`.

$$\tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k\left(\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z}\right)$$

$$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + m_{Ey}^k\left(\frac{\tilde{H}_x^k\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)$$

```
% MAIN FDTD LOOP
for T = 1 : STEPS

    % Update H from E
    for nz = 1 : Nz
        Hx(nz) = Hx(nz) + mHx(nz)*(Ey(nz+1) - Ey(nz))/dz;
    end

    % Update E from H
    for nz = 1 : Nz
        Ey(nz) = Ey(nz) + mEy(nz)*(Hx(nz) - Hx(nz-1))/dz;
    end

end
```

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Each Cell Has Its Own Set of Update Equations

Each cell has its own update equation and its own update coefficients. They are implemented separately for each cell. All of these equations have the same general form so it is more efficient to implement them using a loop. For a 1D grid with 10 cells, think of it this way…

```
% Update H from E
Hx(1)  = Hx(1) + mHx(1)*(Ey(2) - Ey(1))/dz;
Hx(2)  = Hx(2) + mHx(2)*(Ey(3) - Ey(2))/dz;
Hx(3)  = Hx(3) + mHx(3)*(Ey(4) - Ey(3))/dz;
Hx(4)  = Hx(4) + mHx(4)*(Ey(5) - Ey(4))/dz;
Hx(5)  = Hx(5) + mHx(5)*(Ey(6) - Ey(5))/dz;
Hx(6)  = Hx(6) + mHx(6)*(Ey(7) - Ey(6))/dz;
Hx(7)  = Hx(7) + mHx(7)*(Ey(8) - Ey(7))/dz;
Hx(8)  = Hx(8) + mHx(8)*(Ey(9) - Ey(8))/dz;
Hx(9)  = Hx(9) + mHx(9)*(Ey(10) - Ey(9))/dz;
Hx(10) = Hx(10) + mHx(10)*(Ey(11) - Ey(10))/dz;
```

```
% Update H from E
for nz = 1 : 10
    Hx(nz) = Hx(nz) + mHx(nz)*(Ey(nz+1) - Ey(nz))/dz;
end
```

```
% Update E from H
Ey(1)  = Ey(1) + mEy(1)*(Hx(1) - Hx(0))/dz;
Ey(2)  = Ey(2) + mEy(2)*(Hx(2) - Hx(1))/dz;
Ey(3)  = Ey(3) + mEy(3)*(Hx(3) - Hx(2))/dz;
Ey(4)  = Ey(4) + mEy(4)*(Hx(4) - Hx(3))/dz;
Ey(5)  = Ey(5) + mEy(5)*(Hx(5) - Hx(4))/dz;
Ey(6)  = Ey(6) + mEy(6)*(Hx(6) - Hx(5))/dz;
Ey(7)  = Ey(7) + mEy(7)*(Hx(7) - Hx(6))/dz;
Ey(8)  = Ey(8) + mEy(8)*(Hx(8) - Hx(7))/dz;
Ey(9)  = Ey(9) + mEy(9)*(Hx(9) - Hx(8))/dz;
Ey(10) = Ey(10) + mEy(10)*(Hx(10) - Hx(9))/dz;
```

```
% Update E from H
for nz = 1 : 10
    Ey(nz) = Ey(nz) + mEy(nz)*(Hx(nz) - Hx(nz-1))/dz;
end
```

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