Electromagnetic scattering 1: Finite-element method (FEM)

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Finite-element method (FEM)

Strengths:

- Solid mathematical background
- Applicability
- Sparse matrix
- Simple implementation

Weaknesses:

- Matrix conditioning $\rightarrow$ preconditioning needed
- PML is needed for open region problems

FDTD literature

Introduction to FDTD, FEM, IEM

- Sheng Xin-Qing, Song Wei, Essentials of computational electromagnetics, IEEE,Wiley, 2012.

Some FEM books

- Jin, J., The Finite Element Method in Electromagnetics, John Wiley & Sons, Inc., New York, 2002.
- Monk, P., Finite Element Methods for Maxwell's Equations, Oxford Science Publications, Clarendon Press, Oxford, 2003.

Maxwell's equation

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

Ampères law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \sigma \mathbf{E} + \mathbf{J} \tag{2}$$

Gauss's law for elecric field

$$\nabla \cdot \mathbf{D} = \rho \tag{3}$$

Gauss's law for magnetic field

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

Constitutive relations:

$$\mathbf{D} = \epsilon * \mathbf{E} \tag{5}$$

$$\mathbf{B} = \mu * \mathbf{H} \tag{6}$$

ϵ electric permittivity
μ magnetic permeability

Vector wave equation

Time-harmonic case ($\exp(-i\omega t)$)

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \tag{7}$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} + \mathbf{J}^s \tag{8}$$

Radiation condition

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{r}| \left(\eta\mathbf{H}(\mathbf{r}) \times \frac{\mathbf{r}}{|\mathbf{r}|} - \mathbf{E}(\mathbf{r}) \right) = 0 \tag{9}$$

Vector wave equation

$$\nabla \times \left(\mu^{-1}\nabla \times \mathbf{E}\right) - \epsilon\omega^2\mathbf{E} = i\omega\mathbf{J}^s \tag{10}$$

Function spaces

Spaces for potentials:

$$H^1(\Omega) := \{p \in L^2(\Omega), \nabla p \in L^2(\Omega)^3\} \tag{11}$$

$$H_0^1(\Omega) := \{p \in L^2(\Omega)\,, \nabla p \in L^2(\Omega)^3\,, p|_{\partial\Omega} = 0\} \tag{12}$$

Spaces for fields:

$$H_{curl}(\Omega) := \{\mathbf{f} \in L^2(\Omega)^3\,, \nabla \times \mathbf{f} \in L^2(\Omega)^3\} \tag{13}$$

$$H_{0,curl}(\Omega) := \{\mathbf{f} \in L^2(\Omega)^3\,, \nabla \times \mathbf{f} \in L^2(\Omega)^3, \mathbf{n} \times \mathbf{f}|_{\partial\Omega} = 0\} \tag{14}$$

Spaces for flux densities:

$$H_{div}(\Omega) := \{\mathbf{g} \in L^2(\Omega)^3\,, \nabla \cdot \mathbf{g} \in L^2(\Omega)\} \tag{15}$$

$$H_{0,div}(\Omega) := \{\mathbf{g} \in L^2(\Omega)^3\,, \nabla \cdot \mathbf{g} \in L^2(\Omega), \mathbf{n} \cdot \mathbf{g}|_{\partial\Omega} = 0\} \tag{16}$$

$L^2(\Omega)$ is a space of square integrable functions in Ω (if Ω is unbounded, square integrability is defined locally on each bounded subset of Ω)

Inner products

L^2-inner product

$$< \mathbf{f}, \mathbf{g} >_{L^2,\Omega} = \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\mathbf{r} \tag{17}$$

H^1-inner product

$$< f, g >_{H^1,\Omega} = \int_\Omega fg \, d\mathbf{r} + \int_\Omega \nabla f \cdot \nabla g \, d\mathbf{r} \tag{18}$$

H_{curl}-inner product

$$< \mathbf{f}, \mathbf{g} >_{H_{curl},\Omega} = \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\mathbf{r} + \int_\Omega \nabla \times \mathbf{f} \cdot \nabla \times \mathbf{g} \, d\mathbf{r} \tag{19}$$

H_{div}-inner product

$$< \mathbf{f}, \mathbf{g} >_{H_{div},\Omega} = \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\mathbf{r} + \int_\Omega \nabla \cdot \mathbf{f} \cdot \nabla \cdot \mathbf{g} \, d\mathbf{r} \tag{20}$$

Norms

L^2-norm

$$||\mathbf{v}||_0 = \left(\int_\Omega |\mathbf{v}|^2 \, d\Omega \right)^{1/2} \tag{21}$$

H^1-norm

$$||p||_{1,\Omega} = \left(\int_\Omega |\nabla p|^2 \, d\Omega + \int_\Omega |p|^2 \, d\Omega \right)^{1/2} \tag{22}$$

H_{curl}-norm

$$||\mathbf{f}||_{curl,\Omega} = \left(\int_\Omega |\nabla \times \mathbf{f}|^2 \, d\Omega + \int_\Omega |\mathbf{f}|^2 \, d\Omega \right)^{1/2} \tag{23}$$

H_{div}-norm

$$||\mathbf{g}||_{div,\Omega} = \left(\int_\Omega |\nabla \cdot \mathbf{g}|^2 \, d\Omega + \int_\Omega |\mathbf{g}|^2 \, d\Omega \right)^{1/2} \tag{24}$$

Trace operators

Tangential trace operator:

$$\gamma_t \mathbf{F} = -\mathbf{n} \times \mathbf{n} \times \mathbf{F}|_{\partial\Omega} \tag{25}$$

Defines mapping: $H_{curl}(\Omega) \rightarrow H_{Curl}^{-1/2}(\partial\Omega)$
Rotated tangential trace operator:

$$\gamma_r \mathbf{F} = \mathbf{n} \times \mathbf{F}|_{\partial\Omega} \tag{26}$$

Defines mapping: $H_{curl}(\Omega) \rightarrow H_{Div}^{-1/2}(\partial\Omega)$
Normal trace operator:

$$\gamma_n \mathbf{F} = \mathbf{n} \cdot \mathbf{F}|_{\partial\Omega} \tag{27}$$

Defines mapping: $H_{div}(\Omega) \rightarrow H^{-1/2}(\partial\Omega)$

Some useful vector identities

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \tag{28}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \tag{29}$$

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F} \tag{30}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} \tag{31}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{32}$$

$$\nabla \times (\nabla f) = 0 \tag{33}$$

$$\int_V \nabla \cdot \mathbf{F}\, \mathrm{d}V = \int_{\partial V} \mathbf{n} \cdot \mathbf{F}\, \mathrm{d}S \tag{34}$$

$$\int_V \nabla \times \mathbf{F}\, \mathrm{d}V = \int_{\partial V} \mathbf{n} \times \mathbf{F}\, \mathrm{d}S \tag{35}$$

Weak problem

Let Ω be simply connected and bounded domain with the PEC boundary condition $\mathbf{n} \times \mathbf{E} = 0$ on $\partial\Omega$. Find $\mathbf{E} \in H_{0,curl}(\Omega)$ such that

$$< \mathbf{w}, \nabla \times \mu^{-1} \nabla \times \mathbf{E} - \epsilon\omega^2 \mathbf{E} >= i\omega < \mathbf{w}, \mathbf{J}^s >, \tag{36}$$

$\forall\, \mathbf{w} \in H_{0,curl}(\Omega)$. Here $< \cdot, \cdot >$ denotes the $L^2(\Omega)$-inner product.

The above equation can be written as

$$< \nabla \times \mathbf{w}, \mu^{-1} \nabla \times \mathbf{E} > -\epsilon\omega^2 < \mathbf{w}, \mathbf{E} >= i\omega < \mathbf{w}, \mathbf{J}^s > . \tag{37}$$

It is clear that any solution for the wave-equation satisfies (37) but does a solution of (37) satisfy the wave-equation?

Finite-element solution

- In the finite-element method, a weak problem is solved in a finite-dimensional space $\mathcal{B}_h$
- The finite-element space is constructed by dividing the domain into smaller elements e.g. tetrahedral elements T_h.
- Basis functions (associated with elements T_h) should span the finite-dimensional subspace "finite-element space" $\mathcal{B}_h \subset H_{0,curl}(\Omega)$
- Testing functions should span the finite-dimensional subspace $\mathcal{W}_h \subset H_{0,curl}(\Omega)$

Projection method

Solve linear problem of the form

$$Lu = f, \quad L : U \to F, f \in F$$

U and F are some Hilbert spaces, and the unknown $u \in U$
Expand the unknown with a set of basis functions b_n span $B_N \subset U$

$$u \approx \tilde{u}_N = \sum_{n=1}^{N} c_n b_n$$

Require the residual

$$R_N = L\tilde{u}_N - f,$$

to be orthogonal to the space $T_M \subset F$ spanned by testing functions t_m

$$< t_m, R_N >_F = 0, \forall m = 1, 2, ..., M$$

Finite-element solution

Expand the unknown electric field as a linear combination of known basis functions $\mathbf{b}_n$ as

$$\mathbf{E} \approx \sum_{n=1}^{N} c_n \mathbf{b}_n, \tag{38}$$

where c_n are unknown coefficients.
Taking the inner product with testing functions $\mathbf{t}_m$, gives rise to a matrix equation

$$A_{mn} c_n = f_m \tag{39}$$

where

$$A_{mn} = < \nabla \times \mathbf{t}_m, \mu^{-1} \nabla \times \mathbf{b}_n > - \epsilon \omega^2 < \mathbf{t}_m, \mathbf{b}_n >, \tag{40}$$

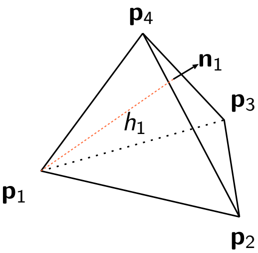
and the force vector read as

$$f_m = i\omega < \mathbf{t}_m, \mathbf{J}^s > . \tag{41}$$

The unknown coefficients can be solve by inverting the matrix

Shape functions

Consider tetrahedral element T_k



Linear shape functions $N_{p_i}(x, y, z)$:

- $N_{p_i}(x, y, z) = 1$ at node $\mathbf{p}_i$
- $N_{p_i}(x, y, z) = 0$ at other nodes
- linear inside the tetrahedron
- $\nabla N_{p_i} = -\frac{\mathbf{n}_i}{h_i}$

Any linear function inside the tetrahedron can be expressed as

$$f(x, y, z) = \sum_{i=1}^{4} a_i N_{p_i}, \qquad (42)$$

where a_i are some coefficients.

Basis functions H^1

Basis functions should have the same differentiability and continuity properties as the original unknown functions e.g. **E**, **H**, **D**, **B** etc.

Let a first consider a scalar space $H^1(\Omega)$, i.e., the space of square integrable functions whose gradients are also square integrable. These functions are **continuous**!

Functions in $H^1(\Omega)$ can be approximated in tetrahedral mesh as a combinations of shape functions

$$\phi \approx \sum_{n=1}^{N} c_n N_n \tag{43}$$

where N is the number of nodes. (Degrees of freedom = number of nodes)

This is suitable space for e.g. Poisson equation (electrostatic)

$$\nabla^2 \phi = \rho \tag{44}$$

Basis functions H_{curl}

H_{curl} is a suitable space for fields.

The lowest order curl-conforming basis function associated into the edge e_{ij}(between nodes ij) and can be expressed as

$$\mathbf{w}_{e_{ij}} = N_i \nabla N_j - N_j \nabla N_i \tag{45}$$

This function has a **continuous tangential component**, and its curl,

$$\nabla \times \mathbf{w}_{e_{ij}} = 2(\nabla N_i \times \nabla N_j), \tag{46}$$

is piecewise constant and square integrable.

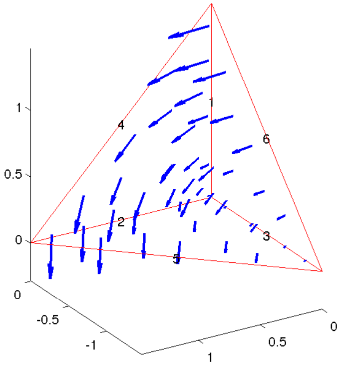
Therefore, the electric or magnetic field can be expanded as

$$\mathbf{E} \approx \sum_{e=1}^{E} c_e \mathbf{w}_e, \tag{47}$$

where E is the number of edges

H_{curl}-function

Edge-element: continuous tangential component



Basis functions H_{div}

H_{div} is a suitable space for flux densities or currents.

The lowest order div-conforming basis function associated into the face f_{ijk}(between nodes ijk) and can be expressed as

$$\mathbf{v}_{f_{ijk}} = N_i(\nabla N_j \times \nabla N_k) + N_j(\nabla N_k \times \nabla N_i) + N_k(\nabla N_i \times \nabla N_j) \tag{48}$$

This function has a **continuous normal component**, and its div,

$$\nabla \cdot \mathbf{v}_{e_{ijk}} = 1/V, \tag{49}$$

where V is the volume of tetrahedron.

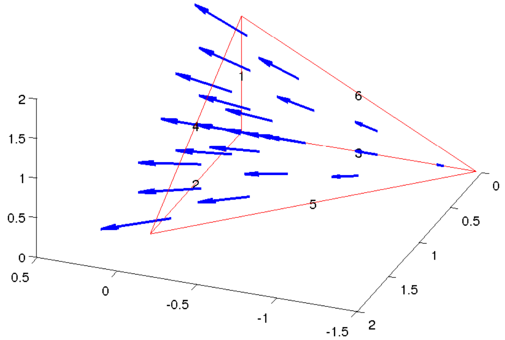
Therefore, the electric or magnetic flux can be expanded as

$$\mathbf{D} \approx \sum_{f=1}^{F} c_f \mathbf{v}_f, \tag{50}$$

where F is the number of faces

H_{div}-function

Face-element: continuous normal component



Basis functions L^2

Scalar L^2-functions for the charge density ρ

Scalar case:

$$p = \sum_{t=1}^{T} c_t \frac{1}{\sqrt{V}}, \tag{51}$$

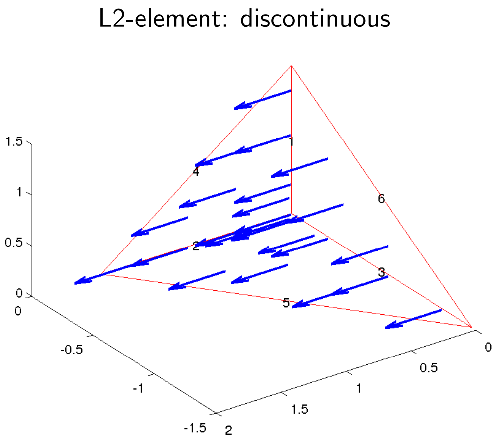
where T is the number of tetrahedra
Vector case (equivalent volumetric currents):

$$\mathbf{p} = \sum_{t=1}^{T} c_t^x \frac{1}{\sqrt{V_t}} \hat{\mathbf{e}}_x + c_t^y \frac{1}{\sqrt{V_t}} \hat{\mathbf{e}}_y + c_t^z \frac{1}{\sqrt{V_t}} \hat{\mathbf{e}}_z, \tag{52}$$

where T is the number of tetrahedra

These functions are **discontinuous**!

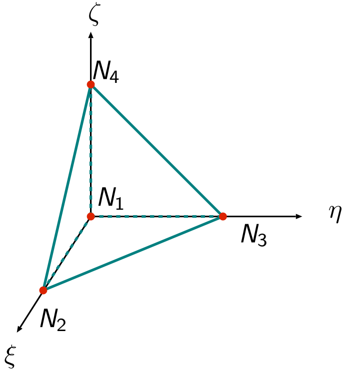
H_{div}-function



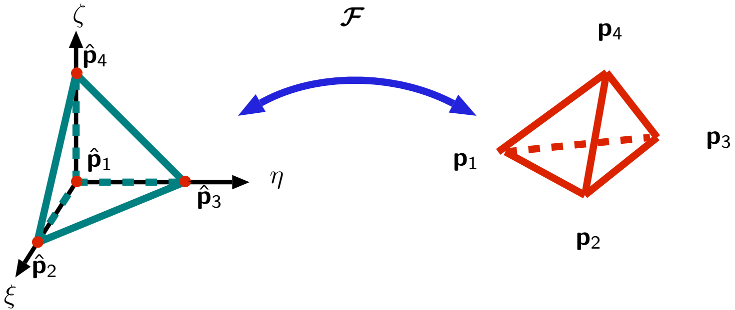
Reference element

In practise, all integrals are evaluate in a reference element.

$$
\begin{aligned}
N_1 &= 1 - \xi - \eta - \zeta \\
N_2 &= \xi \\
N_3 &= \eta \\
N_4 &= \zeta \\
\\
\nabla N_1 &= [-1, -1, -1] \\
\nabla N_2 &= [1, 0, 0] \\
\nabla N_3 &= [0, 1, 0] \\
\nabla N_4 &= [0, 0, 1]
\end{aligned} \tag{53}
$$



Linear mapping



$\mathcal{F}(\xi,\eta,\zeta) =$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (\mathbf{p}_2-\mathbf{p}_1)_x & (\mathbf{p}_3-\mathbf{p}_1)_x & (\mathbf{p}_4-\mathbf{p}_1)_x \\ (\mathbf{p}_2-\mathbf{p}_1)_y & (\mathbf{p}_3-\mathbf{p}_1)_y & (\mathbf{p}_4-\mathbf{p}_1)_y \\ (\mathbf{p}_2-\mathbf{p}_1)_z & (\mathbf{p}_3-\mathbf{p}_1)_z & (\mathbf{p}_4-\mathbf{p}_1)_z \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} + \begin{pmatrix} (\mathbf{p}_1)_x \\ (\mathbf{p}_1)_y \\ (\mathbf{p}_1)_z \end{pmatrix} \tag{54}$$

Integration on the reference element

Shape function on element T_k: $N_{n_i^k}(x,y,z) = \hat{N}_i(\mathcal{F}_k^{-1}(x,y,z))$

$$\begin{aligned} &\int_{T_k} N_{n_i^k}(x,y,z) N_{n_j^k}(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \\ &\int_{\hat{T}} \hat{N}_i(\xi,\eta,\zeta)\hat{N}_j(\xi,\eta,\zeta)|\det(J_{\mathcal{F}_k})|\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \end{aligned} \tag{55}$$

where $J_{\mathcal{F}_k}$ is the Jacobian of the mapping $\mathcal{F}_k$

$$J_{\mathcal{F}_k} = \left[\frac{\partial \mathcal{F}_k}{\partial \xi}, \frac{\partial \mathcal{F}_k}{\partial \eta}, \frac{\partial \mathcal{F}_k}{\partial \zeta}\right] = [\mathbf{p}_2^k - \mathbf{p}_1^k, \mathbf{p}_3^k - \mathbf{p}_1^k, \mathbf{p}_4^k - \mathbf{p}_1^k] \tag{56}$$

The above integral can be numerically evaluated as

$$I_{ij} = |\det(J_{\mathcal{F}_k})| \sum_{m=1}^{M} \hat{N}_i(\xi_m,\eta_m,\zeta_m)\hat{N}_j(\xi_m,\eta_m,\zeta_m) w_m \tag{57}$$

where (ξ_m,η_m,ζ_m) are the Gaussian quadrature points for the reference tetrahedron and w_m are the corresponding weights.

Mapping of derivatives

Functions and their derivatives transforms differently!

$$\hat{N}_i(\xi,\eta,\zeta) = N_{n_i^k}(\boldsymbol{\mathcal{F}}_k(\xi,\eta,\zeta)) \tag{58}$$

Chain rule:

$$\begin{gathered}\frac{\partial \hat{N}_i}{\partial \xi} = \frac{\partial N_{n_i^k}}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial z}\frac{\partial z}{\partial \xi} = \\ \frac{\partial N_{n_i^k}}{\partial x}\frac{\partial \mathcal{F}_k^x}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial y}\frac{\partial \mathcal{F}_k^y}{\partial \xi} + \frac{\partial N_{n_i^k}}{\partial z}\frac{\partial \mathcal{F}_k^z}{\partial \xi}\end{gathered} \tag{59}$$

$$\begin{gathered}\frac{\partial \hat{N}_i}{\partial \eta} = \frac{\partial N_{n_i^k}}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial z}\frac{\partial z}{\partial \eta} = \\ \frac{\partial N_{n_i^k}}{\partial x}\frac{\partial \mathcal{F}_k^x}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial y}\frac{\partial \mathcal{F}_k^y}{\partial \eta} + \frac{\partial N_{n_i^k}}{\partial z}\frac{\partial \mathcal{F}_k^z}{\partial \eta}\end{gathered} \tag{60}$$

Mapping of derivatives

$$\frac{\partial \hat{N}_i}{\partial \zeta} = \frac{\partial N_{n_i^k}}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial z}\frac{\partial z}{\partial \zeta} = \frac{\partial N_{n_i^k}}{\partial x}\frac{\partial \mathcal{F}_k^x}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial y}\frac{\partial \mathcal{F}_k^y}{\partial \zeta} + \frac{\partial N_{n_i^k}}{\partial z}\frac{\partial \mathcal{F}_k^z}{\partial \zeta} \tag{61}$$

In other words

$$\hat{\nabla}\hat{N}_i = J_{\boldsymbol{\mathcal{F}}_k}^T \nabla N_{n_i^k} \tag{62}$$

So we can write an expression for gradients in a general tetrahedron as

$$\nabla N_{n_i^k} = (J_{\boldsymbol{\mathcal{F}}_k}^T)^{-1}\hat{\nabla}\hat{N}_i \tag{63}$$

Data structures

3D tetrahedral mesh can be stored as follows:

Coordinates of nodes ($3 \times N_n$):

$$mesh.coord = \begin{bmatrix} p_x^1 & p_x^2 & p_x^3 & \dots & p_x^{N_n} \\ p_y^1 & p_y^2 & p_y^3 & \dots & p_x^{N_n} \\ p_z^1 & p_z^2 & p_z^3 & \dots & p_x^{N_n} \end{bmatrix}$$

Nodes for each tetrahedron ($4 \times N_t$)

$$mesh.etopol = \begin{bmatrix} n_1^1 & n_1^2 & n_1^3 & \dots & n_1^{N_t} \\ n_2^1 & n_2^2 & n_2^3 & \dots & n_2^{N_t} \\ n_3^1 & n_3^2 & n_3^3 & \dots & n_3^{N_t} \\ n_4^1 & n_4^2 & n_4^3 & \dots & n_4^{N_t} \end{bmatrix}$$

$$mesh.param = \begin{bmatrix} \epsilon_r^1 & \epsilon_r^2 & \epsilon_r^3 & \dots & \epsilon_r^{N_t} \end{bmatrix} \tag{64}$$

N_t is the number of tetrahedra
N_n is the number of nodes

Data structures

Some additional data structures:

Nodes of edges + boundary($3 \times N_e$) (last line: 1 if boundary edge and 0 otherwise):

$$mesh.edges = \begin{bmatrix} n_1^1 & n_1^2 & n_1^3 & \dots & n_1^{N_e} \\ n_2^1 & n_2^2 & n_2^3 & \dots & n_2^{N_e} \\ 0 & 0 & 1 & \dots & 0 \end{bmatrix}$$

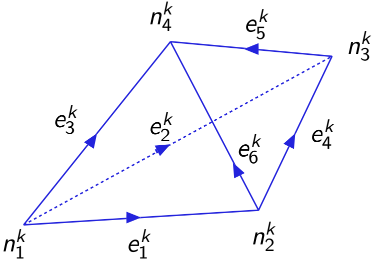
Edges for each tetrahedron ($6 \times N_t$)

$$mesh.etopol2 = \begin{bmatrix} e_1^1 & e_1^2 & e_1^3 & \dots & e_1^{N_t} \\ e_2^1 & e_2^2 & e_2^3 & \dots & e_2^{N_t} \\ e_3^1 & e_3^2 & e_3^3 & \dots & e_3^{N_t} \\ e_4^1 & e_4^2 & e_4^3 & \dots & e_4^{N_t} \\ e_5^1 & e_5^2 & e_5^3 & \dots & e_5^{N_t} \\ e_6^1 & e_6^2 & e_6^3 & \dots & e_6^{N_t} \end{bmatrix}$$

N_e is the number of edges

Data structures

Local edges in tetrahedron k



e^k	n^k	n^k
1	1	2
2	1	3
3	1	4
4	2	3
5	3	4
6	2	4

Edge orientation may be different in global and local notations
$\Rightarrow$ we need to calculate a correct sign for each basis function

e.g. Global edge 19
$node1 = mesh.edges(1, 19)$
$node2 = mesh.edges(2, 19)$

Calculation of matrix elements

Recall:

$$A_{mn} =< \nabla \times \mathbf{t}_m, \mu^{-1} \nabla \times \mathbf{b}_n > - \epsilon\omega^2 < \mathbf{t}_m, \mathbf{b}_n >, \tag{65}$$

where $\mathbf{t}_m$ and $\mathbf{b}_n \in H_{0,curl}(\Omega)^3$
We use lowest order edge elements (edge m between nodes n_j andn_i):

$$\mathbf{t}_m = \mathbf{b}_m = N_{n_i^k} \nabla N_{n_j^k} - N_{n_j^k} \nabla N_{n_i^k} \tag{66}$$

$$\nabla \times \mathbf{t}_m = \nabla \times \mathbf{b}_m = 2(\nabla N_{n_i^k} \times \nabla N_{n_j^k}) \tag{67}$$

Loop over elements, compute local inner products elementwise and add local contributions to the global matrix

Matrix assembly

First term:

$$\begin{aligned} &< \nabla \times \mathbf{t}_m, \mu^{-1} \nabla \times \mathbf{b}_n >_{T_k} \\ &= \int_{T_k} 2(\nabla N_{n_i^k} \times \nabla N_{n_j^k}) \cdot \mu_k^{-1} 2(\nabla N_{n_p^k} \times \nabla N_{n_l^k}) \, \mathrm{d}V \\ &= 4(\nabla N_{n_i^k} \times \nabla N_{n_j^k}) \cdot \mu_k^{-1} (\nabla N_{n_p^k} \times \nabla N_{n_l^k}) V_k \end{aligned} \tag{68}$$

Second term:

$$< \mathbf{t}_m, \mathbf{b}_n >_{T_k} = \int_{T_k} (N_{n_i^k} \nabla N_{n_j^k} - N_{n_j^k} \nabla N_{n_i^k}) \cdot (N_{n_p^k} \nabla N_{n_l^k} - N_{n_l^k} \nabla N_{n_p^k}) \, \mathrm{d}V$$

$$\begin{aligned} &= \nabla N_{n_j^k} \cdot \nabla N_{n_l^k} \int_{T_k} N_{n_i^k} N_{n_p^k} \mathrm{d}V \\ &- \nabla N_{n_j^k} \cdot \nabla N_{n_p^k} \int_{T_k} N_{n_i^k} N_{n_l^k} \mathrm{d}V \\ &- \nabla N_{n_i^k} \cdot \nabla N_{n_l^k} \int_{T_k} N_{n_j^k} N_{n_p^k} \mathrm{d}V \\ &+ \nabla N_{n_i^k} \cdot \nabla N_{n_p^k} \int_{T_k} N_{n_j^k} N_{n_l^k} \mathrm{d}V \end{aligned} \tag{69}$$

Example: Computation of resonant frequencies

Consider the eigenvalue problem

$$\begin{aligned} \nabla \times \mu^{-1} \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} &= 0, \ \text{in}\Omega \\ \mathbf{n} \times \mathbf{E} &= 0, \ \text{on}\partial\Omega \end{aligned} \quad (70)$$

Corresponding weak problem: Find $\mathbf{E} \in H_{0,curl}$ and $\omega \in \mathbb{R} \backslash \{0\}$ such that

$$< \nabla \times \mathbf{t}, \mu^{-1} \nabla \times \mathbf{E} >_{L^2(\Omega)} - \omega^2 < \mathbf{t}, \epsilon \mathbf{E} >_{L^2(\Omega)} = 0 \quad (71)$$

for all $\mathbf{t} \in H_{0.curl}$.

Use curl conforming edge-elements as basis and testing functions (boundary edges removed), and solve the generalised eigenvalue problem

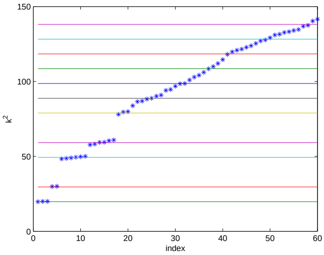
$$SE = \omega^2 ME \quad (72)$$

where S is the "stiffness" matrix arising from $< \nabla \times \mathbf{t}, \mu^{-1} \nabla \times \mathbf{E} >$, and S is the "mass" matrix arising from $< \mathbf{t}, \epsilon \mathbf{E} >$.

Example: Computation of resonant frequencies

Analytical solution (solid lines) $k^2 = \pi^2(l^2 + m^2 + n^2), \quad l, m, n = 0, 1, \ldots$

Numerical solution (stars) FEM with lowest order edge-elements



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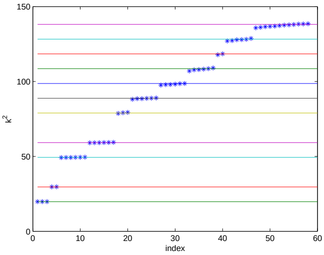

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