8th exercises for SIM'2020

Ex. 1

Show that when i.i.d. measurements y_i follow exponential distribution and the prior for λ is Gamma (see Eq. 1.44) with hyperparameters $(a, a/b)$ the posterior is also Gamma. Use Eq. (7.2) and note that you only need to show that the posterior is proportional to something which is essentially a Gamma distribution. What are the posterior parameters of the resulting Gamma distribution?

Ex. 2

a) You have 10 i.i.d. measurements from true/false Bernoulli experiment, let's say $\boldsymbol{y} = (0, 1, 1, 0, 0, 0, 1, 0, 0, 0)$. Show that the likelihood of $\boldsymbol{y}$ has the form of Binomial distribution (Eq. 1.35). Your prior distribution for the probability of success π is Beta distribution (see the bottom of the page). Show that Beta is a conjugate prior to binomial.

b) Let's say that you have reason to believe that the success probability π is 0.5, so the mean of the prior Beta(α_0, β_0) should be $\alpha_0/(\alpha_0 + \beta_0) = 0.5$. This means that there is actually only one hyperparameter because $\beta_0 = \alpha_0$. The variance of Beta(α_0, α_0), which is $1/(4+8\alpha_0)$, should describe the uncertainty in your prior knowledge.

Try two values for prior variance, 0.01 and 0.055. Solve the value for α_0 with these variances. Plot the prior distributions and the posteriors with both the prior variances. If posteriors are improper (without normalization) they probably have different scales than priors, in that case do not put priors and posteriors in the same figure.

(c) Find the point-estimates to the success probability π and the 5% equal tail intervals (numerically) with both the prior variances.

Beta distribution:

$$\mathrm{f}(y; \alpha, \beta) = \frac{y^{\alpha-1}\,(1-y)^{\beta-1}}{\mathrm{B}(\alpha, \beta)},$$

where $\mathrm{B}(\cdot, \cdot)$ is the Euler beta function. With Euler gamma functions it can be expressed as $\mathrm{B}(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$