Asteroid orbital inversion using a virtual-observation Markov-chain Monte Carlo method

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Introduction

- Asteroid orbit determination is one of the oldest inverse problems
- Paradigm change from deterministic to probabilistic treatment near the turn of the millennium
- Uncertainties in orbital elements, ephemeris uncertainties, collision probabilities, classification
- Identification of asteroids, linkage of asteroid observations
- Incorporation of statistical orbital inversion methods into the Gaia/DPAC data processing pipeline
- Markov-chain Monte Carlo (MCMC, Oszkiewicz et al. 2009)
- OpenOrb open source software (Granvik et al. 2009)

Statistical inversion

- Observation equation

$$\psi = \Psi(\mathbf{P},\mathbf{t}) + \varepsilon$$

- A posteriori probability density function (p.d.f.)

$$p_p(\mathbf{P}) \propto p_{pr}(\mathbf{P})p(\psi|\mathbf{P})$$

- A priori p.d.f., Jeffreys or uniform

$$p_{pr}(\mathbf{P}) \propto \sqrt{\det\Sigma^{-1}(\mathbf{P})}$$

- Observational error p.d.f., multivariate normal

$$p(\varepsilon;\Lambda) = \frac{1}{(2\pi)^{2N}\sqrt{\det\Lambda}}\exp\left[-\frac{1}{2}\varepsilon^T\Lambda^{-1}\varepsilon\right]$$

Statistical inversion

- A posteriori p.d.f. for orbital elements

$$p_p(\mathbf{P}) \propto \sqrt{\det\Sigma^{-1}(\mathbf{P})}\, \exp\left[-\frac{1}{2}\chi^2(\mathbf{P})\right]$$

$$\chi^2(\mathbf{P}) \quad = \quad \Delta\Psi^T(\mathbf{P})\Lambda^{-1}\Delta\Psi(\mathbf{P})$$

- Linearization

$$\Psi(\mathbf{P},t) = \Psi(\mathbf{P}_{ls},t) + \sum_{j=1}^{6}\Delta P_j \frac{\partial\Psi}{\partial P_j}(\mathbf{P}_{ls},t)$$

- A posteriori p.d.f. in the linear approximation

$$p_p(\mathbf{P}) \propto \sqrt{\det\Sigma^{-1}(\mathbf{P}_{ls})} \cdot \exp\left[-\frac{1}{2}\Delta\mathbf{P}^T\Sigma^{-1}(\mathbf{P}_{ls})\Delta\mathbf{P}\right]$$

- Covariance matrix for orbital elements

$$\Sigma(\mathbf{P}_{ls}) = \left(\frac{\partial\mathbf{P}}{\partial\psi}\right)_{ls}^T \Sigma(\psi)\left(\frac{\partial\mathbf{P}}{\partial\psi}\right)_{ls}$$

MCMC ranging

- Initial orbital inversion using exiguous astrometric data (short observational time interval and/or a small number of observations)
- Ranging algorithm
  - Select two observation dates
  - Vary topocentric distances and values of R.A. and Decl.
  - From two Cartesian positions, compute elements and χ^2 against all the observations
- In MC ranging, systematic sampling and weighted sample elements
- How to sample using MCMC?

MCMC ranging

- Gaussian proposal p.d.f. in the space of two Cartesian positions
- Complex proposal p.d.f. in the space of the orbital elements (not needed!)
- Jacobians and cancellation of symmetric proposal p.d.f.s

$$a_r = \frac{p_p(\mathbf{P}')}{p_p(\mathbf{P}_t)} \frac{p_t(\mathbf{Q}_t; \mathbf{Q}') J_t}{p_t(\mathbf{Q}'; \mathbf{Q}_t) J'}$$

$$J = \det \left| \frac{\partial \mathbf{Q}}{\partial \mathbf{P}} \right|$$

$$a_r = \frac{p_p(\mathbf{P}')}{p_p(\mathbf{P}_t)} \frac{J_t}{J'}$$

$$\text{If } a_r \geq 1, \quad \text{then } \mathbf{P}_{t+1} = \mathbf{P}'.$$

$$\text{If } a_r < 1, \quad \text{then } \begin{cases} \mathbf{P}_{t+1} = \mathbf{P}', & \text{with probability } a_r, \\ \mathbf{P}_{t+1} = \mathbf{P}_t, & \text{with probability } 1 - a_r. \end{cases}$$

Examples

Observations				
Data set	Parameter	2004 HA_{39}	2004 QR	2002 CX_{224}
Full	Nr of observations	57	19	20
	First observation	2004 Apr 25	2004 Aug 16	2001 Oct 21
	Last observation	2004 Sep 16	2004 Sep 23	2003 Oct 24
	Time interval [day]	144.7	38.3	733.2
Used	Nr of observations	5	10	6
	First observation	2004 May 2	2004 Aug 16	2002 Nov 7
	Last observation	2004 May 6	2004 Aug 18	2002 Nov 28
	Time interval [day]	4.75	2.04	20.98

Table 1: The full observational data sets and the data sets used in the computation.

Table 2. Least-squares orbits.

Object	2004 HA_{39}	2004 QR	2002 CX_{224}
a (AU)	2.1493040(34)	2.33155(74)	46.404(57)
e	0.5368719(72)	0.30163(21)	0.1329(49)
i (°)	36.2085(3)	6.3566(36)	16.8447(63)
Ω (°)	204.154337(44)	314.82(1)	42.2592(56)
ω (°)	67.32116(11)	330.701(26)	132.51(1.07)
M_0 (°)	345.02807(39)	28.4223(89)	256.12(1.73)[2]

[2]The 1−σ standard deviations for the orbital elements are given after each element in the units of the last digit shown.

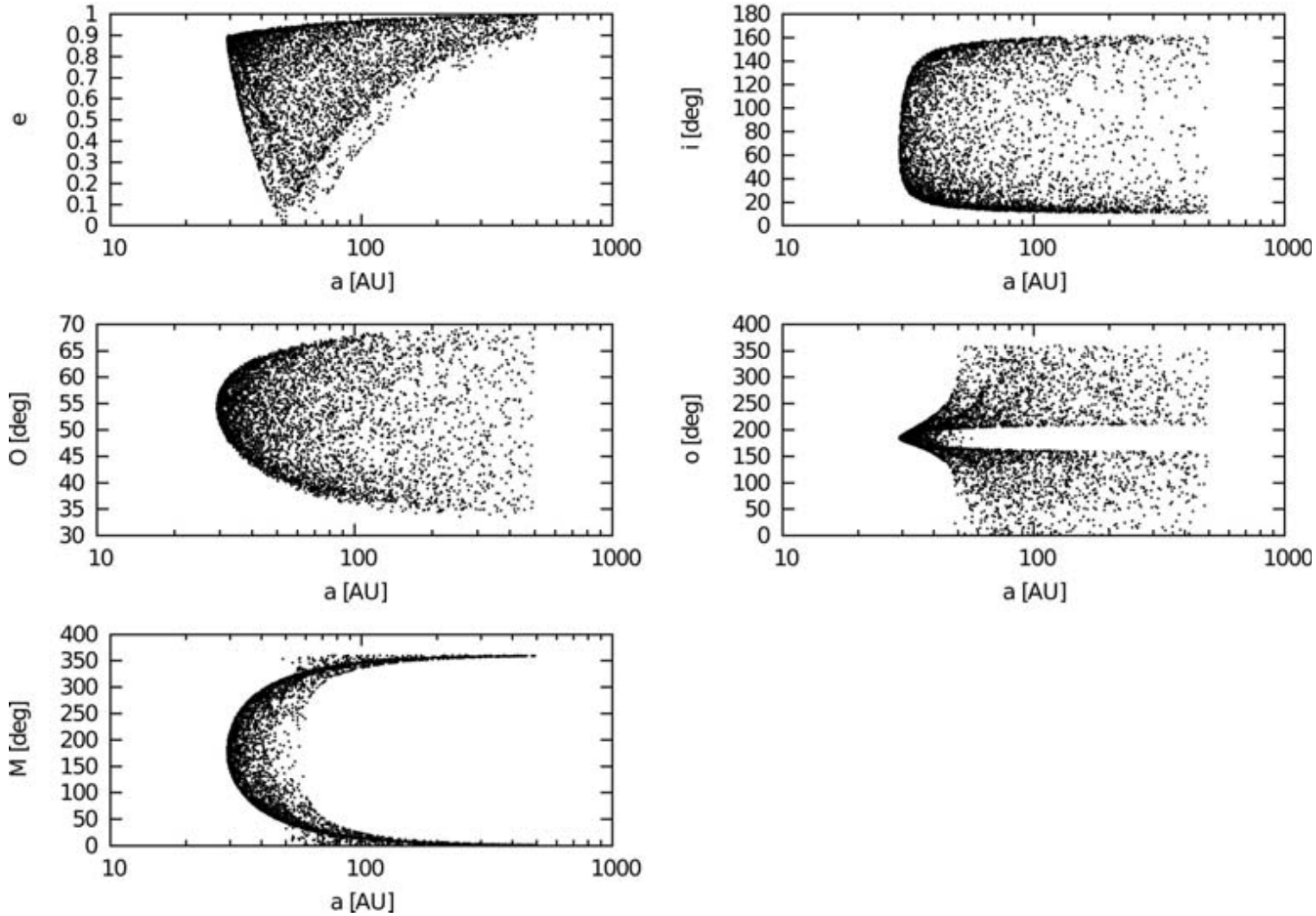


Fig. 1. A sample of 5000 different orbit solutions for TNO 2002 CX_{224} (observational time interval 20.98 d) obtained with MCMC ranging. The following proposal standard deviations were used: $\sigma_{\alpha_A} = \sigma_{\delta_A} = \sigma_{\alpha_B} = \sigma_{\delta_B} = 0.5''$ for the angular coordinates, $\sigma_{\rho_A} = \sigma_{\rho_B} = 2.0$ AU for the ranges, and high correlation $Cor\ (\rho_A, \rho_B) = 0.999$ between the ranges.

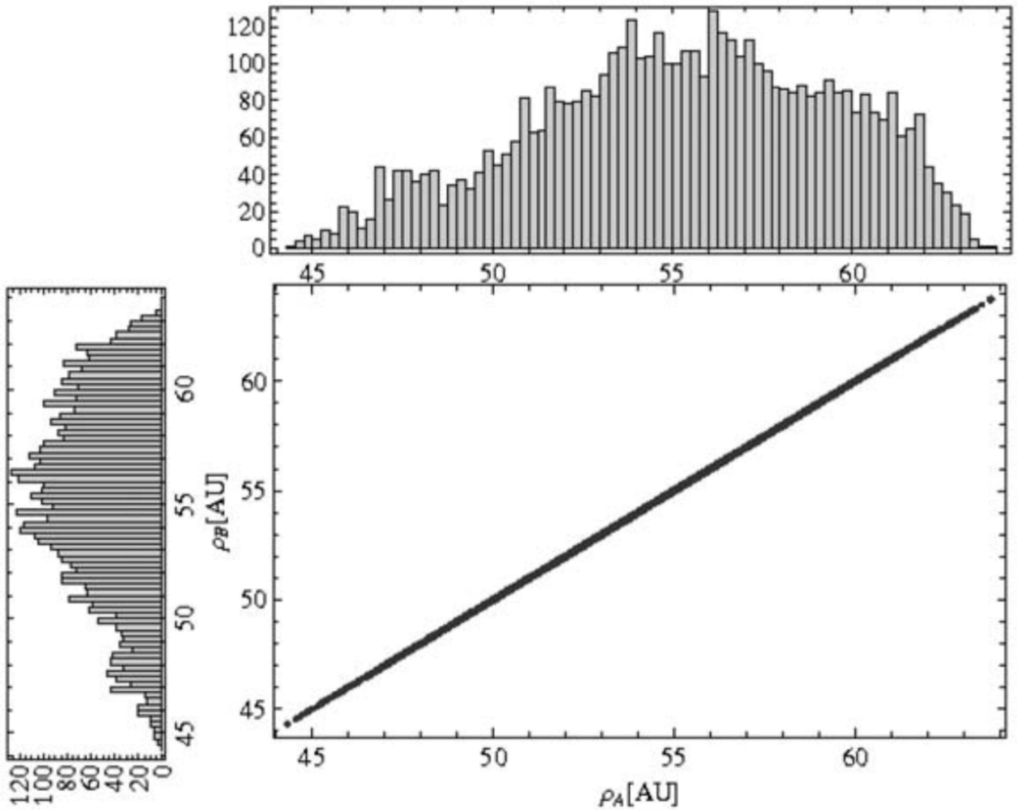
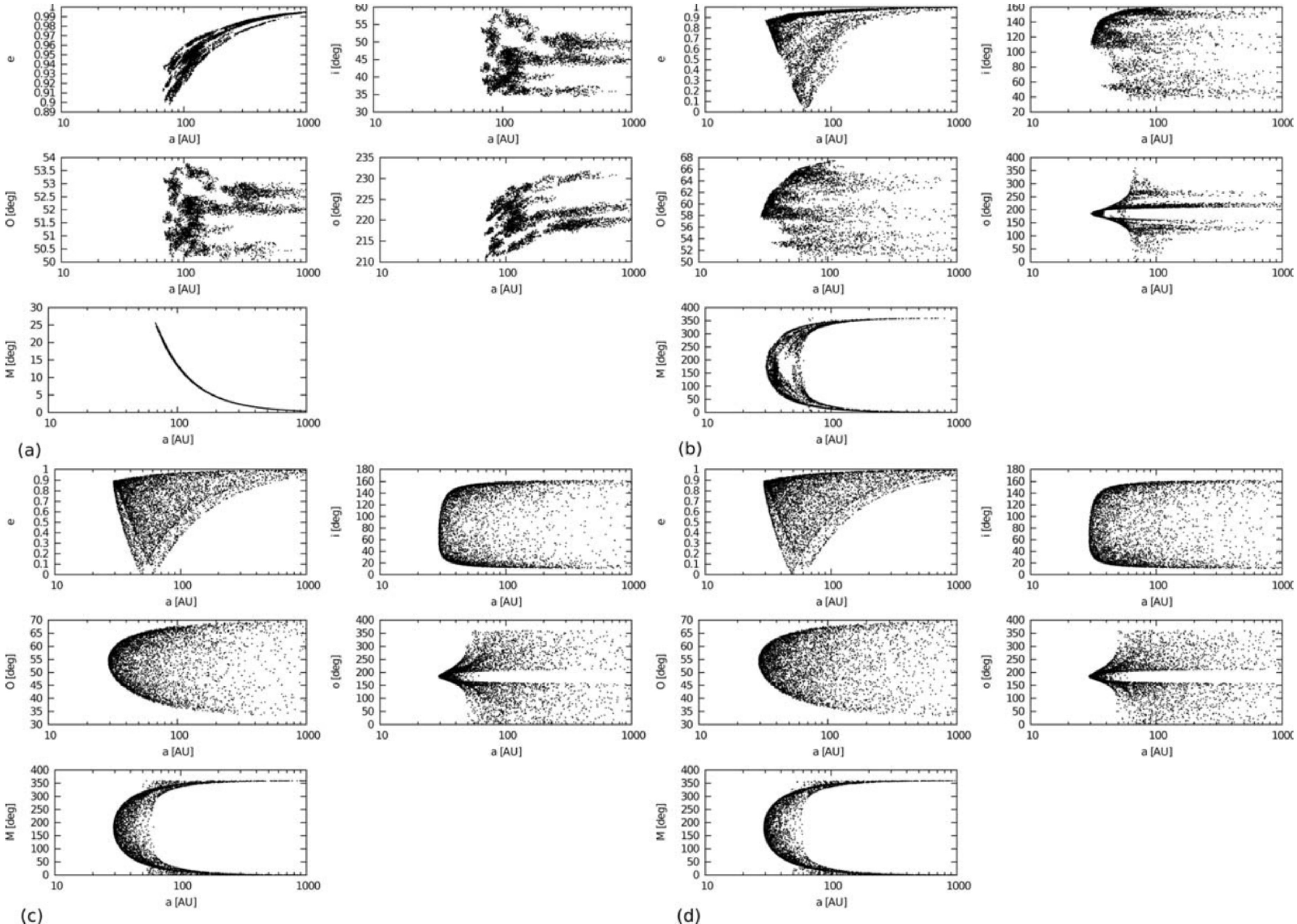


Fig. 2. Distribution of ranges for the observation dates A and B for TNO 2002 CX_{224} (observational time interval 20.98 d) obtained with MCMC ranging (Parameters as in Fig. 1).



(a)
(b)
(c)
(d)
e
i [deg]
O [deg]
o [deg]
M [deg]
a [AU]

Table 3. Methods comparison.

Object	Parameter	MCMC ranging	MC ranging
2004 HA_{39}	Computation time	0 min 19.417 s	0 min 48.663 s
	Acceptance ratio	0.17	0.05
2004 QR	Computation time	0 min 6.344 s	1 min 19.170 s
	Acceptance ratio	0.54	0.03
2002 CX_{224}	Computation time	0 min 4.525 s	0 min 14.155 s
	Acceptance ratio	0.45	0.09[3]

[3]Comparison of MCMC ranging and MC ranging methods.

Gaia/DPAC DU456 demonstration

- DU456, development unit entitled “Orbital inversion”
- MCMC ranging as standalone Java software including GaiaTools
- Potential for a future online computational tool with a friendly interface

Intermediate conclusions

- MCMC ranging more efficient than MC ranging
- Operational within the Gaia/DPAC pipeline and as stand-alone software
- How to improve the efficiency of MCMC ranging?

Virtual-observation MCMC

- Random errors are simulated for all observations, resulting in a set of virtual observations
- Virtual least-squares orbital elements are derived using the Nelder-Mead downhill simplex method
- Repeating the procedure two times allows for a computation of a difference for two sets of virtual orbital elements
- This orbital-element difference constitutes a symmetric proposal in a random-walk Metropolis-Hastings algorithm

- Mathematics of virtual-observation MCMC

$$\boldsymbol{\psi}_{\mathrm{v}} = \boldsymbol{\psi} + \boldsymbol{\epsilon}_{\mathrm{v}}$$

$$\chi^2_{\mathrm{v}}(\boldsymbol{P}) = (\Psi(\boldsymbol{P}) - \boldsymbol{\psi}_{\mathrm{v}})^{\mathrm{T}}(\Lambda + \Lambda_{\mathrm{v}})^{-1}(\Psi(\boldsymbol{P}) - \boldsymbol{\psi}_{\mathrm{v}})$$

$$p(\boldsymbol{P}_{\mathrm{v}}) = \int d\boldsymbol{\psi}_{\mathrm{v}} \delta(\boldsymbol{P}_{\mathrm{v}} - \boldsymbol{P}_{\mathrm{v}}(\boldsymbol{\psi}_{\mathrm{v}}))p(\boldsymbol{\psi}_{\mathrm{v}})$$

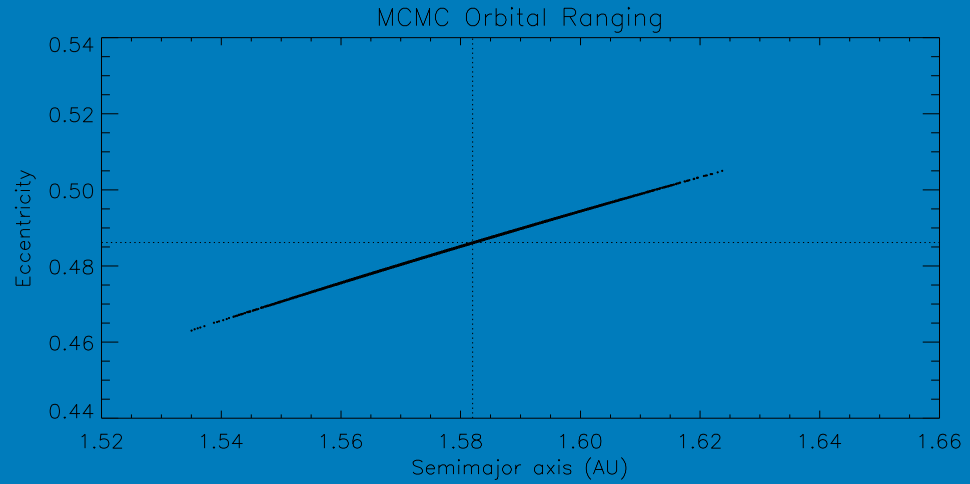
$$p_{\mathrm{t}}(\Delta\boldsymbol{P}) = \int\int d\boldsymbol{P}_{\mathrm{v}1} d\boldsymbol{P}_{\mathrm{v}2}\ \delta(\Delta\boldsymbol{P} - (\boldsymbol{P}_{\mathrm{v}1} - \boldsymbol{P}_{\mathrm{v}2}))p(\boldsymbol{P}_{\mathrm{v}1})p(\boldsymbol{P}_{\mathrm{v}2})$$

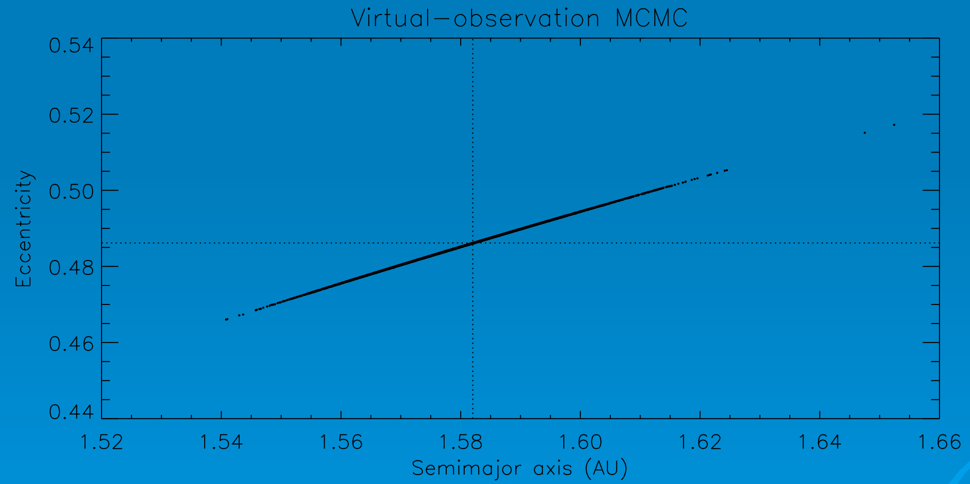
$$p_{\mathrm{t}}(\Delta\boldsymbol{P}) = \int d\boldsymbol{P}_{\mathrm{v}}\ p(\boldsymbol{P}_{\mathrm{v}})p(\boldsymbol{P}_{\mathrm{v}} - \Delta\boldsymbol{P})$$

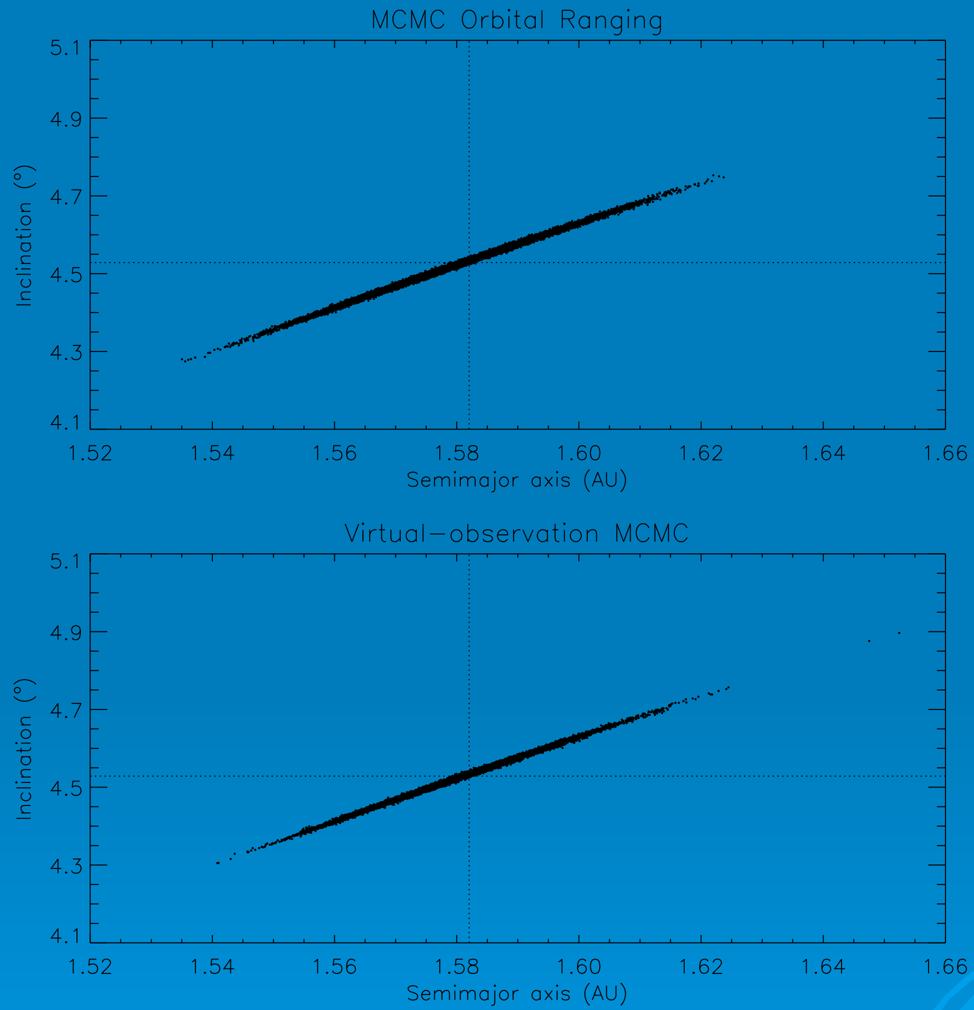
$$\Delta\boldsymbol{P}^{(jk)} = \boldsymbol{P}_{\mathrm{v}}^{(j)} - \boldsymbol{P}_{\mathrm{v}}^{(k)},\ j,k = 1,2,3\ldots,N_{\mathrm{v}};\ j \neq k.$$

Example: 1998 OX_4

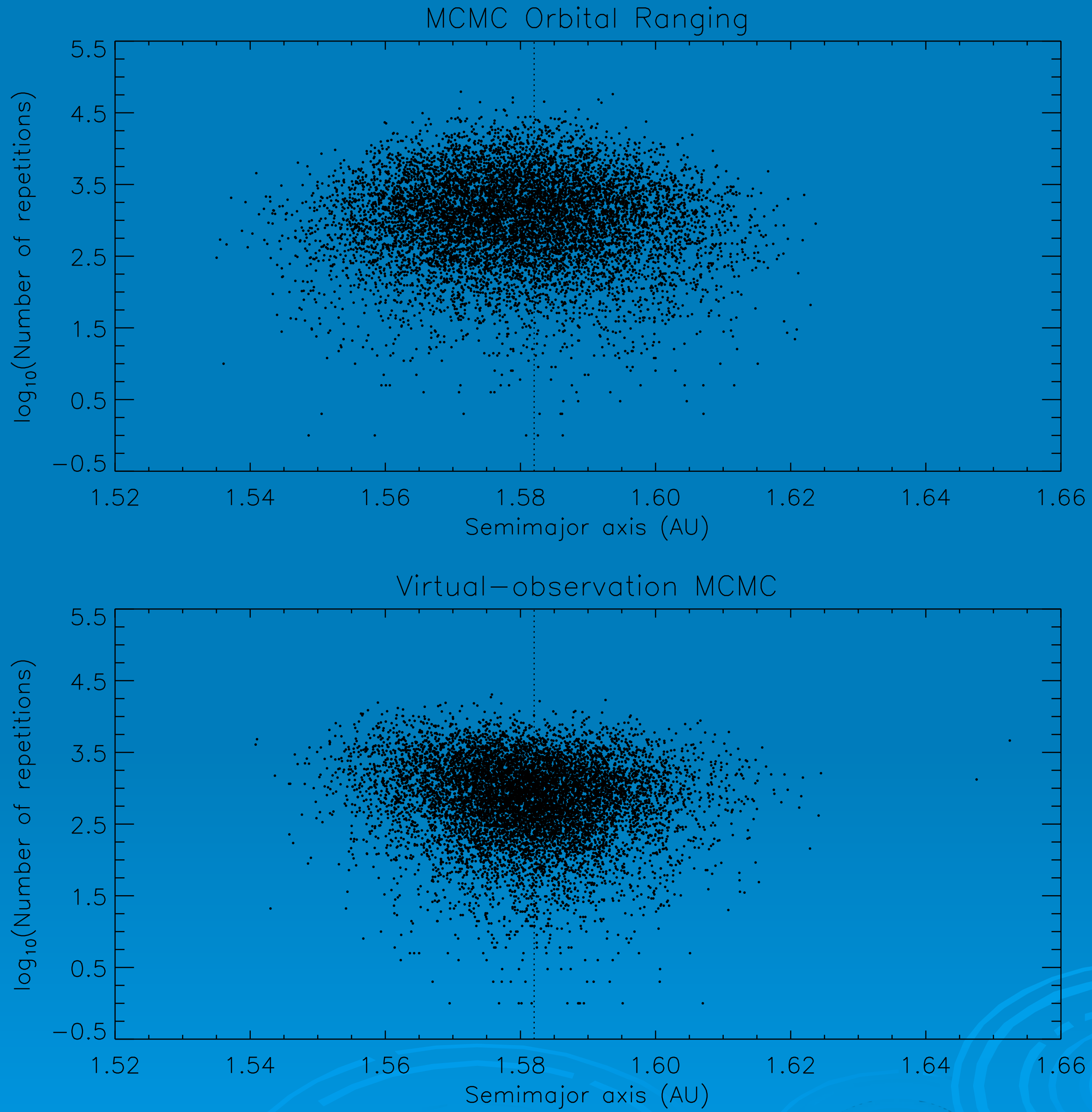
- Discovery apparition only: 21 observations spanning 9.1 days in July-August 1998
- Single outlier observation omitted
- Standard deviations for R.A. and Dec.: 0.57 arcsec and 0.34 arcsec
- See Virtanen et al. (Icarus, 2001) and Muinonen et al. (CMDA, 2001)
- 10,000 sample elements using MCMC ranging and virtual-observation MCMC in 17 min 54 s and 16 min 15 s, respectively


MCMC Orbital Ranging
Eccentricity
0.54
0.52
0.50
0.48
0.46
0.44
1.52
1.54
1.56
1.58
1.60
1.62
1.64
1.66
Semimajor axis (AU)


Virtual−observation MCMC
Eccentricity
0.54
0.52
0.50
0.48
0.46
0.44
1.52
1.54
1.56
1.58
1.60
1.62
1.64
1.66
Semimajor axis (AU)



MCMC Orbital Ranging
Inclination (°)
Semimajor axis (AU)
5.1
4.9
4.7
4.5
4.3
4.1
1.52
1.54
1.56
1.58
1.60
1.62
1.64
1.66
Virtual−observation MCMC
Inclination (°)
Semimajor axis (AU)


MCMC Orbital Ranging
log10(Number of repetitions)
5.5
4.5
3.5
2.5
1.5
0.5
−0.5
1.52
1.54
1.56
1.58
1.60
1.62
1.64
1.66
Semimajor axis (AU)
Virtual−observation MCMC
log10(Number of repetitions)
5.5
4.5
3.5
2.5
1.5
0.5
−0.5
1.52
1.54
1.56
1.58
1.60
1.62
1.64
1.66
Semimajor axis (AU)

Conclusions

- Virtual-observation MCMC is a promising new random-walk Metropolis-Hastings algorithm
- Concept to be proven for sampling well-known distributions (Gaussian, curved bivariate Gaussian)
- Concept to be studied for cases with large numbers of unknowns