BASICS OF NON-ELEMENTARY MODEL THEORY

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Introduction

To me, model theory is a study of questions, properties and concepts whose definition(s) do not rely on mathematical structures or classes of structures being of some particular kind. E.g. "is π algebraic?" is not a model theoretic question, it makes sense essentially only in the context of the field of complex numbers, but algebricity is a model theoretic property: There is a general notion of algebricity that in the case of complex numbers is the same as what is intended in the question above but which is such that one can replace complex numbers by, say, some graph and it still makes sense to ask which elements of the graph are algebraic. Let us look another example of a model theoretic property namely pseudo finiteness. In the first-order a.k.a. elementary model theory pseudo finiteness means that every first-order sentence true in the structure is true also in some finite structure or equivalently, there is an elementary embedding of the structure into an ultraproduct of finite structures. So one can think pseudo finiteness as a property that allows one to approximate the structure with finite structures. Obviously the property does not depend on what kind structures we are looking and e.g. many (infinite) fields have it as do vector spaces over a finite field, random graphs etc. Now the reader may ask why in this property we look first-order sentences and/or why finite structures and not e.g. finitely generated ones. The answer is, because the theory of pseudo finiteness is developed in the context of elementary model theory. A critical reader may not be satisfied to this answer. And, as we will see, she is right.

Elementary model theory completely dominated model theory at least from the early seventies to the end of the century and it is still going strong. It has been very successful. Often mentioned example of this is E. Hrushovki's work on Mordell-Lang and Manin-Mumford conjectures from algebraic number theory. However it has limitations. Our example of this comes from metric model theory although in these lectures we will stick to the discrete case (in non-elementary model theory the difference between metric and discrete cases is rather small unless one is interested in concepts like perturbation which appear only in the metric case). We use this metric example because it demonstrates two main problems simultaneously: First-order logic can be too strong and too weak at the same time to be useful in studies of a mathematical structure.

So let us look the standard model from quantum mechanics for a particle in space. This structure comes with a metric which is a problem in elementary model theory. First-order logic is not strong enough to express the basic properties of a metric. One can try to go around this problem by moving to continuous first-order logic. Unfortunately, in addition, the operators in the standard model are not bounded, in particular, they are only partial functions. This forces one to deal with them in first-order logic are predicates. But then continuous first-order logic is not strong enough to express even the functionality of the operators. Now suppose that we are interested in studying possibilities of approximating this structure with finite dimensional substructures (finitely generated Hilbert spaces) - physicists seem to be interested in this. Now, as pointed out above, there is a lot of theory of a question related to this question in the elementary model theory. But if one tries to do the constructions from elementary model theory (heavily modified) in this case, one notices that one cannot find elementary embeddings the way one can in the first-order case. Now the first-order logic is too strong. However, one can find embeddings instead of elementary embeddings and luckily this is enough: In this context, it is the equations that one is interested in, not the quantification.

In these lectures our set up will be that of abstract elementary classes (due to S. Shelah) and we will cover the basic theory as much as the time allows. We will also spend a fair amount of time looking what our concepts mean in examples. This topic can be very misleading if one does not test one's intuition against examples. Of course, this is the case in mathematics in general but in our topic the problem is bigger than usually. Our main examples are the class of right-angled Coxeter groups and its subclasses.

We sometimes refer to the lecture notes of the course Model theory, but, excluding few exercises, only some very basic information from the course is needed here and most of that is already included in the course Introduction to Logic II or the course Elements of set theory and by reading few pages of the lecture notes of the course Model theory, the needed information can be obtained.

And yes I agree, non-elementary model theory is not a good name for our topic but it is what is used of it.

1. Basic definitions and examples

We fix a vocabulary and call it τ. Unless otherwise said, by a model/structure we mean one with vocabulary τ. If $\mathcal{A}$ is a model and $f:\mathcal{A}\to X$ is one-to-one, then by $f(\mathcal{A})$ we mean a model $\mathcal{B}$ such that the universe of $\mathcal{B}$ ($dom(\mathcal{B})$) is $f(dom(\mathcal{A}))$, for all (n-ary) relation symbols $R\in\tau$ and $b_1,...,b_n\in dom(\mathcal{B})$, $(b_1,...,b_n)\in R^{\mathcal{B}}$ iff $(f^{-1}(b_1),...,f^{-1}(b_n))\in R^{\mathcal{A}}$, for all function symbols $F\in\tau$ and $b_1,...,b_n\in dom(\mathcal{B})$, $F^{\mathcal{B}}(b_1,...,b_n)=f(F^{\mathcal{A}}(f^{-1}(b_1),...,f^{-1}(b_n)))$ and for all constant symbols $c\in\tau$, $c^{\mathcal{B}}=f(c^{\mathcal{A}})$ (so f is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$). When there is no risk of confusion, we usually write $\mathcal{A}$ also for $dom(\mathcal{A})$. For models $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\subseteq\mathcal{B}$ means that $\mathcal{A}$ is a substructure of $\mathcal{B}$ (i.e. for all n-ary relation symbols $R\in\tau$, $R^{\mathcal{A}}=R^{\mathcal{B}}\cap dom(\mathcal{A})^n$, for all function symbols $F\in\tau$, $F^{\mathcal{A}}(a_1,...,a_n)=F^{\mathcal{B}}(a_1,...,a_n)$ for $a_1,...,a_n\in\mathcal{A}$ and for all constant symbols $c\in\tau$, $c^{\mathcal{A}}=c^{\mathcal{B}}$, for better understanding on submodels, see the lecture notes of the course Model theory). We write $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ etc. for models and A, B, C, X, Y etc. for sets.

We say that a pair $(X,\leq)$ is a directed system if $\leq$ is a partial order of X (i.e. it is transitive and $a=b$ iff $a\leq b$ and $b\leq a$) and for all $a,b\in X$, there is $c\in X$ such that $a,b\leq c$. Notice that if $\leq$ is a linear order of X, then $(X,\leq)$ is a directed system. Now suppose that X is a set of models and $(X,\leq)$ is a directed system such that $\mathcal{A}\leq\mathcal{B}$ implies $\mathcal{A}\subseteq\mathcal{B}$. The model $\mathcal{A}=\cup X=\cup_{\mathcal{B}\in X}\mathcal{B}$ is defined as follows:

(a) The universe of $\mathcal{A}$ is $\cup_{\mathcal{B}\in X}dom(\mathcal{B})$.

(b) For all $R\in\tau$ and $a_1,...,a_n\in\mathcal{A}$, $(a_1,...,a_n)\in R^{\mathcal{A}}$ if there is $\mathcal{B}\in X$ such that $(a_1,...,a_n)\in R^{\mathcal{B}}$.

(c) For all $F \in \tau$, $a_1, ..., a_{n+1} \in \mathcal{A}$, $F^{\mathcal{A}}(a_1, ..., a_n) = a_{n+1}$, if there is $\mathcal{B} \in X$ such that ($a_1, ..., a_{n+1} \in \mathcal{B}$ and) $F^{\mathcal{B}}(a_1, ..., a_n) = a_{n+1}$.

(d) for all $c \in \tau$, $c^{\mathcal{A}} = a$ if there is $B \in X$ such that $c^{\mathcal{B}} = a$.

1.1 Exercise. *Let $\mathcal{A}$ and X be as above.*

(i) Show that, $F^{\mathcal{A}}$ is a function and that $c^{\mathcal{B}} = c^{\mathcal{C}}$ for all $\mathcal{B}, \mathcal{C} \in X$.

(ii) Show that for all $\mathcal{B} \in X$, $\mathcal{B} \subseteq \mathcal{A}$.

By $a \in \mathcal{A}$ we often mean that a is a finite tuple of elements of $dom(\mathcal{A})$ and if by this we mean that a is an element of $dom(\mathcal{A})$ it is either clear from the context or we say this explicitly. By $A \subseteq \mathcal{A}$ we mean that A is a subset of $dom(\mathcal{A})$ and $|\mathcal{A}|$ means the cardinality of $dom(\mathcal{A})$.

1.2 Definition. *Let K be a class of structures and $\preceq$ a partial order on K.*

(I) We say that $K = (K, \preceq)$ is an abstract class if the following holds:

(i) K and $\preceq$ are closed under isomorphisms i.e. if $\mathcal{A} \in K$ and $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then $\mathcal{B} \in \mathcal{K}$ and if in addition $\mathcal{C} \preceq \mathcal{A}$, then $f(\mathcal{C}) \preceq \mathcal{B}$.

(ii) $\mathcal{A} \preceq \mathcal{B}$ implies $\mathcal{A} \subseteq \mathcal{B}$.

(II) We say that $K = (K, \preceq)$ is an abstract elementary class (AEC) if the following holds:

(i) $(K, \preceq)$ is an abstract class.

(ii) (Coherence) If $\mathcal{A}, \mathcal{B} \preceq \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \preceq \mathcal{C}$.

(iii) Suppose that $(\mathcal{A}_i)_{i<\alpha}$, $\alpha \in On$, is a sequence of elements of K and it is $\preceq$-increasing i.e. for all $i < j < \alpha$, $\mathcal{A}_i \preceq \mathcal{A}_j$, and continuous i.e. for all limit $\beta < \alpha$, $\mathcal{A}_\beta = \cup_{i<\beta} \mathcal{A}_i$, then the following holds:

(a) $\cup_{i<\alpha} \mathcal{A}_i \in \mathcal{K}$,

(b) for all $j < \alpha$, $\mathcal{A}_j \preceq \cup_{i<\alpha} \mathcal{A}_i$,

(c) (Smoothness) if in addition there is $\mathcal{B} \in K$ such that for all $i < \alpha$, $\mathcal{A}_i \preceq \mathcal{B}$, then $\cup_{i<\alpha} \mathcal{A}_i \preceq \mathcal{B}$.

(iv) (Löwenheim-Skolem property) There is a cardinal $LS(K) \geq \omega$ such that for all $\mathcal{A} \in K$ and $A \subseteq \mathcal{A}$, if $|A| \leq LS(K)$, then there is $\mathcal{B} \preceq \mathcal{A}$ such that $A \subseteq \mathcal{B}$ and $|\mathcal{B}| \leq LS(K)$.

If $(K, \preceq)$ is an AEC, then we refer to $\preceq$ as strong submodel relation. The elements of K are called K-models. We now fix some AEC $K = (K, \preceq)$ i.e. when we talk about K and/or $\preceq$ (outside examples) we always assume that $(K, \preceq)$ is an AEC.

1.3 Exercise.

(i) Suppose that $\mathcal{A}$ is a K-model and $A \subseteq \mathcal{A}$. Show that there is a $\mathcal{K}$-model $\mathcal{B} \preceq \mathcal{A}$ such that $A \subseteq \mathcal{B}$ and $|\mathcal{B}| \leq |A| + LS(K)$ $(= max\{|A|, LS(K)\})$.

(ii) Show that if one drops the continuity assumption from Definition 1.2 (II)(iii), one gets an equivalent condition.

(iii) Suppose that X is a set of K-models and $(X, \leq)$ is a directed system such that $\mathcal{A} \leq \mathcal{B}$ implies that $\mathcal{A} \preceq \mathcal{B}$. Show that $\mathcal{C} = \cup_{\mathcal{A} \in X} \mathcal{A} \in K$, for all $\mathcal{A} \in X$, $\mathcal{A} \preceq \mathcal{C}$ and if there is a K-model $\mathcal{B}$ such that $\mathcal{A} \preceq \mathcal{B}$ for all $\mathcal{A} \in K$, then $\mathcal{C} \preceq \mathcal{B}$.

If $\mathcal{A}$ and $\mathcal{B}$ are K-models and $f : \mathcal{A} \to \mathcal{B}$, then we say that f is a strong embedding if it is one-to-one and $f(\mathcal{A}) \preceq \mathcal{B}$. Notice that $\mathcal{A} \preceq \mathcal{B}$ iff $id : \mathcal{A} \to \mathcal{B}$ is a strong embedding ($id(x) = x$ for all $x \in \mathcal{A}$).

Let us now look at examples. We start with classical but misleading examples.

1.4 Exercise.

(i) Let T be a complete first-order theory, $K = mod(T)$ (the class of all models of T) and let $\preceq$ be the elementary submodel relation. Show that $(K, \preceq)$ is an AEC.

(ii) Let T be an $\forall\exists$-axiomatizable theory (see the lecture notes of the course Model theory) and $\preceq = \subseteq$. Show that $(K, \preceq)$ is an AEC.

(iii) Let ϕ be an $L_{\omega_1\omega}$-sentence and $F = F_\phi$ be the least fragment that contains ϕ (see the lecture notes of the course Model theory). Let $K = mod(\phi)$ and $\preceq$ the following relation: $\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all $\psi(x) \in F$ and $a \in \mathcal{A}$, $\mathcal{A} \models \psi(a)$ iff $\mathcal{B} \models \psi(a)$. Show that $(K, \preceq)$ is an AEC.

Let us then look at some a bit more unusual examples of AEC's/abstract classes and their properties.

1.5 Exercise.

(i) Let K be the class of linear orders $\mathcal{A} = (\mathcal{A}, <)$ such that for some $\alpha \leq \omega_1$, $\mathcal{A}$ is isomorphic with $(\alpha, <)$. Let $\preceq$ be the initial segment relation i.e. $\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, if $b <^{\mathcal{B}} a$, then $b \in \mathcal{A}$. Show that $K = (K, \preceq)$ is an AEC with $LS(K) = \omega$ and notice that K has a largest model i.e. a model $\mathcal{A}$ such that for all K-models $\mathcal{B}$ there is a strong embedding from $\mathcal{B}$ to $\mathcal{A}$ and also a maximal model i.e. a model that has no proper strong extension.

(ii) (Suppose $(K, \preceq)$ is an AEC.) Show that if $\mathcal{A}$ is a largest model, then it is also a maximal model. Conclude that if $\mathcal{A}$ and $\mathcal{B}$ are largest models, then $\mathcal{A} \cong \mathcal{B}$. Show that there is an AEC that has maximal models but not a largest model. Hint for the first claim: Assume not and construct a K-model that is too large.

(iii) Let τ consist of two unary relation symbols P_0 and P_1 and one binary relation symbol R. Let K be the class of all τ-models $\mathcal{A}$ such that

(a) $P_0^{\mathcal{A}} \cup P_1^{\mathcal{A}} = dom(\mathcal{A})$ and $P_0^{\mathcal{A}} \cap P_1^{\mathcal{A}} = \emptyset$,

(b) if $(a, b) \in R^{\mathcal{A}}$, then $a \in P_0^{\mathcal{A}}$ and $b \in P_1^{\mathcal{A}}$,

(c) for all $X \subseteq P_1^{\mathcal{A}}$ there is at most one $a \in P_0^{\mathcal{A}}$ such that

$$R^{\mathcal{A}}(a, \mathcal{A})(= \{b \in \mathcal{A} |\ (a, b) \in R^{\mathcal{A}}\}) = X.$$

Let $\preceq$ be such that $\mathcal{A} \preceq \mathcal{C}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all $a \in P_0^{\mathcal{B}}$, if $R^{\mathcal{B}}(a, \mathcal{B}) \subseteq P_1^{\mathcal{A}}$, then $a \in \mathcal{A}$. Show that $K = (K, \preceq)$ satisfies all the requirements of AEC except coherence, smoothness and Löwenheim-Skolem property and that these three requirements are not satisfied.

Let us then look a real example. This right-angled Coxeter groups example is from geometric/combinatorial group theory and these groups were originally introduced by H.S.M. Coxeter and later popularized by J. Tits. Here we will get only a rough picture of what is going on, for more detailed picture, see [HP].

The right-angled Coxeter groups are by far the best understood ones among all Coxeter groups. Our definition of them is not the original/usual one but it is equivalent by a theorem due to Tits and our definition, unlike the usual one, can be understood without any knowledge from group theory.

Let $\Gamma = (\Gamma, E)$ be a graph. In this context the elements of Γ are called also letters. We define a group G_Γ as follows. Let X be the set of all (finite) words made of letters from Γ (i.e. finite sequences of letters). We define two moves that can be used to manipulate words $w = w_0 w_1 ... w_n$:

(M1) If $w_i = w_{i+1}$, then cancel the letters w_i and w_{i+1}.

(M2) If $(w_i, w_{i+1}) \in E$, then exchange w_i and w_{i+1}.

We say that w' is a normal form of w if w' can be got from w by using moves (M1) and (M2) and the length of w' is minimal among these. Notice that these normal forms need not be unique. We define a binary relation E^* to X by wE^*w' if w and w' have identical normal forms. Notice that if Γ' is a substructure (=induced subgraph - subgraph usually means something else) of a graph Γ and w and w' are words made of letters from Γ', then it does not matter whether one calculates their normal forms in Γ or in Γ' and thus they decide whether wE^*w' holds or not the same way. Notice also that in the following fact only one direction is trivial.

1.6 Fact. *wE^*w' iff $w = w'$ or there are words w^i, $i \leq n$, such that $w^0 = w$, $w^n = w'$ and for all $i < n$, either w_{i+1} is got from w_i by applying $(M1)$ or $(M2)$ or w_i is got from w_{i+1} by applying $(M1)$ or $(M2)$. In particular, E^* is an equivalence relation.*

Then we let the set of elements of G_Γ be X/E^* and multiplication is defined to G_Γ by $(w/E^*)(w'/E^*) = ww'/E^*$, where by ww' we mean the concatenation of w and w'. It is easy to check that this definition does not depend of the choice of the representatives of the equivalence classes and that this multiplication makes G_Γ a group ($\emptyset/E^*$ is the neutral element and $(w/E^*)^{-1} = w'/E^*$ where w' is w in reverse order). We write w also for w/E^* and words of length 1 are identified with their only letter. Notice that each letter is an involution and letters a and b commute iff $(a, b) \in E$ (in an appropriate sense, these requirements determine G_Γ).

For the next exercise, we make the following definition: We say that an element a of a group G is divisible if for all $n \in \omega - \{0\}$ there is $b \in G$ such that $b^n = a$. Fact: Excluding the neutral element, (right-angled) Coxeter groups do not contain divisible elements.

1.7 Exercise. *Suppose $\Gamma = (\Gamma, E)$ is a graph that contains a and b such that $(a, b) \notin E$, U is an ultrafilter on ω which does not contain finite sets and for all $i < \omega$, $\mathcal{A}_i = G_\Gamma$. Show that $\Pi_{i<\omega} \mathcal{A}_i / U$ contains a divisible element that is not the neutral element (for the ultraproducts, see the lecture notes of the course Model theory). Conclude that if K is a class of right-angled Coxeter groups and at least one of the groups is not commutative, then K is not first-order axiomatizable. Hint: Keep in mind the original definition of E^*, not Fact 1.6.*

Let K_0 be a class of graphs and let K be the class of groups isomorphic to G_Γ for some $\Gamma \in K_0$. If K_0 is the class of all graphs, then K is the class of all right-angled Coxeter groups. If $\mathcal{A} \in K$, $\Gamma \in K_0$ and f is an isomorphism from $\mathcal{A}$ to G_Γ, then $S = f^{-1}(\Gamma)$ is called a basis of $\mathcal{A}$. Notices that G can have more than one basis. However, unlike with all Coxeter groups in general, the right-angled ones are rigid:

1.8 Fact. *If S and S' are bases of $\mathcal{A}$, then there is $f \in Aut(\mathcal{A})$ such that $f(S) = S'$. In particular, if G_Γ and $G_{\Gamma'}$ are isomorphic, then Γ and Γ' are isomorphic. (Exercise: prove the in particular part using the main fact.)*

Now we look these groups as models. For this we need to choose a vocabulary for groups. We use the one that contain one binary function symbol for the multiplication, one unary function symbol for the inverse and a constant symbol for the neutral element.

The subgroup relation used in geometric group theory for Coxeter groups is the parabolic subgroup relation: $\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A}$ is a subgroup (=submodel in our vocabulary) of $\mathcal{B}$ and $\mathcal{A}$ has a basis that extends to a basis of $\mathcal{B}$.

Notice that by Exercise 1.4 (ii), if K_0 is the class of all graphs, then $(K_0, \subseteq)$ is an AEC.

We start by sketching a proof for the fact that $(K, \preceq)$ need not be an AEC even if $(K_0, \subseteq)$ is. For this we need the following definition: Let G be a group. We say that $f \in Aut(G)$ is an inner automorphism if there is $a \in G$ such that for all $x \in G$, $f(x) = axa^{-1}$. We write $InnAut(G)$ for the set of all inner automorphisms. Notice that for all $a \in G$, $x \mapsto axa^{-1}$ is an automorphism of G.

1.9 Lemma. *If K_0 is (e.g.) the class of all graphs, then smoothness fails for $(K, \preceq)$.*

Proof. (Sketch) We define a graph Γ so that the set of its elements is $A \cup B$ where $A = \{a_i |\ i < \omega\}$ and $B = \{b_i |\ i < \omega_1\}$. The edge relation is defined so that xEy if $x \neq y$ and $x, y \in B$ or $\{x, y\} = \{a_i, b_j\}$ and $j < i$. Let $\mathcal{B} = G_\Gamma$. For all $n < \omega$, we write $c_n = a_0 a_1 ... a_n$ and $e_n = c_n b_n c_n^{-1}$. Notice that for $n < m < \omega$, $e_n = c_m b_n c_m^{-1}$. Thus if we let $\mathcal{B}_n$ be the subgroup (submodel) generated by $\{e_0, ..., e_n\}$, $\mathcal{B}_n \preceq \mathcal{B}$ (if $\mathcal{B}'_n$ is the subgroup generated by $\{b_0, ..., b_n\}$, then $\mathcal{B}'_n \preceq \mathcal{B}$ and $\mathcal{B}_n$ is the image $\mathcal{B}'_n$ in the inner automorphism $x \mapsto c_n x c_n^{-1}$ of $\mathcal{B}$). Let $\mathcal{B}_\omega = \cup_{n<\omega} \mathcal{B}_n$. Notice that $\mathcal{B}_\omega$ is commutative and that a word w is in $\mathcal{B}_\omega$ iff w can be written in a form $c_m w' c_m^{-1}$ where w' contains only letters from the set $\{b_i |\ i \leq m\}$ and $c_m w' c_m^{-1}$ is in normal form if in addition all letters in w' are different and b_m appears in w'. In particular each $c_n b_n c_n^{-1}$ is in normal form. Now it is enough to show that $\mathcal{B}_\omega$ is not a parabolic subgroup of $\mathcal{B}$. For a contradiction, we suppose that it is.

Now pick a basis S for $\mathcal{B}$ so that $S' = S \cap \mathcal{B}_\omega$ is a basis of $\mathcal{B}_\omega$. Then there is an automorphism f of $\mathcal{B}$ such that $f(S) = A \cup B$. Since all elements of S' commute with each other, $f(S') \subseteq B$. Now there is $d \in B - f(S')$ and then it commutes with every element of $f(\mathcal{B}_\omega)$ and $d \notin f(\mathcal{B}_\omega)$. Let $c = f^{-1}(d)$. Then $c \in \mathcal{B} - \mathcal{B}_\omega$ and it commutes with every element of $\mathcal{B}_\omega$. Let w_c be a normal form for c in basis $A \cup B$.

Now we need the following fact:

(*) If $v = v_0...v_n$ is a word in normal form and u is a letter, then u commutes with v iff u commutes with every letter v_i.

Now if w_c contains only letters from B, then $c_0^{-1}w_cc_0$ does not commute with $b_0 = c_0^{-1}e_0c_0$ by (*) because $c_0^{-1}w_cc_0$ is in normal form, a contradiction. So let $m < \omega$ maximal such that a_m appears in w_c. Now since $c_m^{-1}cc_m$ commutes with $b_m = c_m^{-1}e_mc_m$ and this is not possible by (*), if the normal form of $c_m^{-1}cc_m$ contains letters from A. Thus c can be written in the form $c_mwc_m^{-1}$ where w contains only letters from B. We can choose w also so that it is in normal form i.e. there is no repetition. Then w does not contains letters b_n for any $n > m$, since otherwise $c_{m+1}^{-1}cc_{m+1} = a_{m+1}^{-1}wa_{m+1}$ does not commute with $c_{m+1}^{-1}e_{m+1}c_{m+1} = b_{m+1}$ again by (*) and the fact that $a_{m+1}^{-1}wa_{m+1}$ is in normal form. But then $c \in \mathcal{B}_\omega$, a contradiction. $\square$

When one analyses the difficulties in proving that a class of right-angled Coxeter groups is an AEC (and difficulties in proving other properties for the class, see below), one notices that to behave very well, the groups should be more rigid. By looking the theory of Coxeter groups one finds that exactly the right property has already been isolated and studied:

1.10 Definition. *Let G be a right-angled Coxeter group.*

(i) We say that G is strongly rigid if for all bases S and S', there is $f \in InnAut(G)$ such that $f(S) = S'$

(ii) We say that a graph Γ is strongly rigid if the group G_Γ is strongly rigid.

There is no graph-theoretic characterization for a graph to be strongly rigid (i.e. this is an open question) but there is a necessary condition which is also sufficient in the case the graph is finite. There are also properties that if added to the necessary condition guarantee that the graph is strongly rigid, e.g. the graph is triangle free and contains a 4-circle as a substructure. These show that there are a lot of natural AEC's of strongly rigid graphs.

1.11 Theorem. *Suppose K_0 is an AEC and every element of K_0 is a strongly rigid graph. Then excluding smoothness, K satisfies the requirements of an AEC. If in addition every $\Gamma \in K_0$ is triangle free and contains a 4-cycle as a submodel, then also smoothness is satisfied.*

Proof. As an example we sketch the prove of coherence. So suppose $\mathcal{A}, \mathcal{B} \preceq \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$. We need to show that $\mathcal{A} \preceq \mathcal{B}$. Let S be a basis of $\mathcal{C}$ that witnesses $\mathcal{A} \preceq \mathcal{C}$ and T a basis that witnesses $\mathcal{B} \preceq \mathcal{C}$. Let $S' = \mathcal{A} \cap S$ and $T' = T \cap \mathcal{B}$. Then there is $g \in \mathcal{C}$ such that $S = gTg^{-1}$ i.e. for every $s \in S$ there is $t_s \in T$ such that $s = gt_sg^{-1}$. Let $w = w_0w_1...w_n$ be a normal form of g in the basis T. Since $gt_sg^{-1} \in \mathcal{B}$ it can be written using letters from T'. It follows that $t_s \in T'$ (otherwise we need to get rid of it and this would be impossible since t_s appears in $w_0...w_nt_sw_n...w_0$ odd many times). Also if w' is a word got from w by removing all $w_i \notin T'$, $w't_sw'^{-1} = gt_sg^{-1}$ (using the moves (M1) and (M2) one must be able to get rid of them). But then $w'T'w'^{-1}$ is a basis of $\mathcal{B}$ that contains S'. $\square$

1.12 Exercise. *Suppose $K_0 = (K_0, \subseteq)$ is an AEC and every element of K_0 is a strongly rigid graph.*

(i) Show that K has Löwenheim-Skolem property with $LS(K) = LS(K_0)$.

(ii) Show that K satisfies the first two items from Definition 1.2 (II)(iii).

Hint: Keep in mind the remark immediately after the definition of E^*.

2. Representation theorem and Ehrenfeucht-Mostowski models

From now on, if there is no risk of confusion, we write $F(a_0, ..., a_n)$ for the element $F^{\mathcal{A}}(a_0, ..., a_n)$ and similarly for other terms.

Let $K = (K, \preceq)$ be an AEC and $\kappa = LS(K)$. For all $n < \omega$ and $i < \kappa$, let f_i^n be a new $n+1$-ary function symbol and $\tau_s = \tau \cup \{f_i^n \mid i < \kappa,\ n < \omega\}$. Let T be the theory that says the following:

(a) for all x, $f_0^0(x) = x$,

(b) for all $m \leq n < \omega$ and $i < \kappa$, if $\{y_0, ..., y_m\} = \{x_0, ..., x_n\}$, then $f_i^n(x_0, ..., x_n) = f_i^m(y_0, ..., y_m)$.

Notice that (b) above implies that

(c) for all $n < \omega$ and $i < \kappa$, if π is a permutation of the set $\{0, ..., n\}$, then $f_i^n(x_0, ..., x_n) = f_i^n(x_{\pi(0)}, ..., x_{\pi(n)})$.

Notice also that T can be written in the first-order logic.

Now suppose $\mathcal{A} \models T$, $n < \omega$ and $a_0, ..., a_n \in \mathcal{A}$. We write $qcl(a_0, ..., a_n) = qcl_{\mathcal{A}}(a_0, ..., a_n)$ (or $qcl(\{a_0, ..., a_n\})$, recall (b) above) for the the set

$$\{f_i^n(a_0, ..., a_n) \mid i < \kappa\}.$$

If $qcl_{\mathcal{A}}(a_0, ..., a_n)$ happens to be a submodel of $\mathcal{A} \restriction \tau$ (i.e. contains the interpretations of constant symbols $c \in \tau$ and is closed under the interpretations of function symbols $f \in \tau$), we think it also as a structure in the vocabulary τ (with the induced structure). We write $t_s((a_0, ..., a_n)/\emptyset; \mathcal{A})$ for the set of all formulas $\phi(t_0(v_0, ..., v_n), ..., t_m(v_0, ..., v_n))$, $m < \omega$, such that

$$\mathcal{A} \models \phi(t_0(a_0, ..., a_n), ..., t_m(a_0, ..., a_n)),$$

ϕ is an atomic or negated atomic formula in vocabulary τ and for all $j \leq m$, $t_j(v_0, ..., v_n) = f_i^k(x_0, ..., x_k)$ for some $k \leq n$, $i < \kappa$ and $x_0, ..., x_k \in \{v_0, ..., v_n\}$. We let SK be the set of all $t_s((a_0, ..., a_n)/\emptyset; \mathcal{A})$, for any τ_s-model $\mathcal{A} \models T$ and $a_0, ..., a_n \in \mathcal{A}$.

2.1 Exercise. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of T, $n < \omega$, $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$, $i \leq n$, and $qcl(a_1, ..., a_n)$ is a submodel of $\mathcal{A} \restriction \tau$. Show that if $t_s((a_0, ..., a_n)/\emptyset; \mathcal{A}) = t_s((b_0, ..., b_n)/\emptyset; \mathcal{B})$, then $qcl(b_1, ..., b_n)$ is a submodel of $\mathcal{B} \restriction \tau$ and there is an isomorphism $\pi : qcl(a_0, ..., a_n) \to qcl(b_0, ..., b_n)$ such that $\pi(a_i) = b_i$ for all $i \leq n$.*

2.2 Definition. *We let GSK be the set of all $t_s((a_0,...,a_n)/\emptyset;\mathcal{A})\in SK$ such that*

(i) $\mathcal{A}\models T$ (and $a_0,...,a_n\in\mathcal{A}$),

(ii) $qcl(a_0,...,a_n)$ is a submodel of $\mathcal{A}\restriction\tau$ and belongs to K,

(iii) if $m\le n$ and $b_0,..,b_m\in\{a_0,...,a_n\}$, then $qcl(b_0,...,b_m)\in K$ and

$$qcl(b_0,...,b_m)\preceq qcl(a_0,...,a_n).$$

Notice that from Definition 2.2 (iii) it follows that $qcl(b_0,...,b_m)\subseteq qcl(a_0,...,a_n)$.

2.3 Exercise. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of T, $n<\omega$, $a_i\in\mathcal{A}$ and $b_i\in\mathcal{B}$, $i\le n$, and the sequence $(a_0,...,a_n)$ satisfies Definition 2.2 (ii) and (iii) in $\mathcal{A}$.*

(i) Show that $a_i\in qcl(a_0,...,a_n)$ for all $i\le n$.

(ii) Show that if $t_s((a_0,...,a_n)/\emptyset;\mathcal{A})=t_s((b_0,...,b_n)/\emptyset;\mathcal{B})$, then for all $m\le n$ and $b'_0,...,b'_m\in\{b_0,...,b_n\}$, $qcl(b'_0,...,b'_m)$ is a strong submodel of $qcl(b_0,...,b_n)$.

2.4 Theorem. *(Representation theorem) Let $\mathcal{A}$ be a τ-model. Then the following are equivalent:*

(i) $\mathcal{A}\in K$.

(ii) There is an τ_s-model $\mathcal{B}\models T$ such that $\mathcal{B}\restriction\tau=\mathcal{A}$ and for all $n<\omega$ and $a_0,...,a_n\in\mathcal{B}$, $t_s((a_0,...,a_n)/\emptyset;\mathcal{B})$ belongs to GSK (i.e. $\mathcal{B}$ omits every type in SK-GSK, see the lecture notes of the course Model theory)

Proof. "$\Rightarrow$": We get $\mathcal{B}$ by adding to $\mathcal{A}$ interpretations for the function symbols f^n_i. We do this by recursion on $n<\omega$. So suppose that we have done this for all $m<n$. Let $a_0,...,a_n\in\mathcal{A}$. If these are not distinct, then T tells what the values $f^n_i(a_0,...,a_n)$ are. Also for the same reason, we may assume that we have not determined the values for any permutation of the sequence $(a_0,...,a_n)$. Then we choose $\mathcal{C}\preceq\mathcal{A}$ of power κ such that for all $m<n$, $b_0,...,b_m\in\{a_0,...,a_n\}$ and $i<\kappa$, $f^m_i(b_0,...,b_m)\in\mathcal{C}$. If $n=0$, then we just require that $a_0\in\mathcal{C}$. Then we choose the values $f^n_i(a_0,...,a_n)$ so that

$$\mathcal{C}=\{f^n_i(a_0,...,a_n)|\ i<\kappa\}.$$

If $n=0$, in addition, we choose these so that $f^0_0(a_0)=a_0$. It is easy to see that with these interpretations, $\mathcal{B}$ is as wanted.

"$\Leftarrow$": Let S be the set of all $qcl(a_0,...,a_n)$, $n<\omega$ and $a_0,...,a_n\in\mathcal{B}$. We would like to define a partial order $\le$ to S so that $qcl(a_0,...,a_n)\le qcl(b_0,...,b_m)$ if $\{a_0,...,a_n\}\subseteq\{b_0,...,b_m\}$. Unfortunately this relation is not well-defined. Thus we define (we write A, B etc. for the elements of S although we think them also as τ-models): We let $R\subseteq S^2$ be the relation $(A,B)\in R$, if there are $a_0,...,a_n,b_0,...,b_m\in\mathcal{B}$ such that $A=qcl(a_0,...,a_n)$, $B=qcl(b_0,...,b_m)$ and $\{a_0,...,a_n\}\subseteq\{b_0,...,b_m\}$. Now R may not be transitive. Thus we define $A\le B$ if there are $k<\omega$ and $A_0,...,A_k\in S$ such that $A=A_0$, $B=A_k$ and for all $i<k$, $(A_i,A_{i+1})\in R$, i.e. $\le$ is the transitive closure of R.

2.4.1 Exercise. *Show that $(S,\leq)$ is a directed system of elements of K and for all $A,B\in S$, if $A\leq B$, then $A\preceq B$.*

Now $\mathcal{B}\restriction\tau=\mathcal{A}$ belongs to K by Exercises 2.4.1 and 1.3 (iii). □

2.5 Exercise. *Let $\mathcal{B}\models T$ be such that for all $n<\omega$ and $a_i\in\mathcal{B}$, $i\leq n$, $t_s((a_0,...,a_n)/\emptyset;\mathcal{B})$ belongs to GSK. Suppose that $\mathcal{A}$ is a submodel of $\mathcal{B}$ (in vocabulary τ_s).*

(i) Show that $\mathcal{A}\models T$ and for all $n<\omega$ and $a_0,...,a_n\in\mathcal{A}$, $t_s((a_0,...,a_n)/\emptyset;\mathcal{A})$ belongs to GSK. Conclude that $\mathcal{A}\restriction\tau\in K$.

(ii) Show that $\mathcal{A}\restriction\tau\preceq\mathcal{B}\restriction\tau$.

2.6 Definition. *Let τ_0 be a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ be τ_0-models.*

(i) If $a_0,...,a_n\in\mathcal{A}$, then by $t_{at}((a_0,...,a_n)/\emptyset;\mathcal{A})$ we mean the set of all atomic or negated atomic formulas $\phi(v_0,...,v_n)$ such that $\mathcal{A}\models\phi(a_0,...,a_n)$. We call these diagrams or more precisely τ_0-diagrams in variables $v_0,...,v_n$ (or complete atomic types over $\emptyset$).

(ii) If Φ is a τ_0-diagrams in variables $v_0,...,v_n$ and $b_0,..,b_n\in\mathcal{B}$, we say that $(b_0,...,b_n)$ realizes Φ if $\mathcal{B}\models\phi(b_0,...,b_n)$ for all $\phi\in\Phi$ i.e. $t_{at}((b_0,...,b_n)/\emptyset;\mathcal{B})=\Phi$.

(iii) For $A\subseteq\mathcal{A}$, by $SH_{\mathcal{A}}(A)$ we mean the least substructure of $\mathcal{A}$ that contains A. We write just $SH(A)$ for $SH_{\mathcal{A}}(A)$ if $\mathcal{A}$ is clear from the context.

Notice that the universe of $SH(A)$ is the least subset of $\mathcal{A}$ that contains A, contains all interpretations of constant symbols $c\in\tau_0$ and is closed under interpretations of all function symbols $f\in\tau_0$.

We write $P_\omega(A)$ for the set of all finite subsets of A.

2.7 Exercise.

(i) Show that $SH(A)=\cup_{B\in P_\omega(A)}SH(B)$.

(ii) Suppose $a_i\in\mathcal{A}$ and $b_i\in\mathcal{B}$ for all $i\in I$, where $I=(I,<)$ is a linear ordering and that for all $n<\omega$ and $i_0<...<i_n\in I$, $t_{at}((a_{i_0},...,a_{i_n})/\emptyset;\mathcal{A})=t_{at}((b_{i_0},...,b_{i_n})/\emptyset;\mathcal{B})$. Show that there is an isomorphism $\pi: SH(\{a_i|\ i\in I\})\to SH(\{b_i|\ i\in I\})$ such that for all $i<\alpha$, $\pi(a_i)=b_i$.

Recall that $\kappa=LS(K)$. In the following theorem we will work with elements b^i_j of the models $\mathcal{B}_i$ for simplicity. If needed, one can replace them by finite tuples of fixed length and no changes are needed anywhere. Notice that there are AEC's for which there are no models $\mathcal{B}_i$ as assumed to exist in the following theorem (Exercise 1.5 (i)).

2.8 Theorem. *Let $\lambda=(2^\kappa)^+$. Suppose that $\mathcal{B}_\alpha$, $\alpha<\lambda$, are models of T that omit every type in SK-GSK and for all $\alpha<\lambda$, we have $b^\alpha_j\in\mathcal{B}_\alpha$, $j<\beth_\alpha$, such that for all $j<k<\beth_\alpha$, $b^\alpha_j\neq b^\alpha_k$. Then for all $n<\omega$ there are τ_s-diagrams Φ_n in variables $v_0,...,v_n$ such that the following holds:*

(i) For all $n<\omega$, there is $\alpha<\lambda$ and $j_0<...<j_n<\beth_\alpha$ such that $(b^\alpha_{j_0},...,b^\alpha_{j_n})$ realizes Φ_n.

(ii) For all $m < n < \omega$, $\Phi_m \subseteq \Phi_n$ i.e. if (for some $a_0, ..., a_n \in \mathcal{A}$, $\mathcal{A}$ a τ_s-model,) $(a_0, ..., a_n)$ realizes Φ_n, then $(a_0, ..., a_m)$ realizes Φ_m.

(iii) For all linear orderings $I = (I, <)$, there is a τ_s-model $\mathcal{B}$ and $b_i \in \mathcal{B}$, $i \in I$, such that

(a) $\mathcal{B} = SH(\{b_i | \ i \in I\})$,

(b) for all $n < \omega$ and $i_0, ..., i_n \in I$, if $i_0 < ... < i_n$, then $(b_{i_0}, ..., b_{i_n})$ realizes Φ_n.

Proof. We skip the proof in this course. The proof is identical to the proof of Theorem 12.7 (see also the beginning of the proof of Corollary 12.8) in the lecture notes of the course Model theory. □

For the rest of this section we assume that for all $\alpha < (2^\kappa)^+$ there is $\mathcal{A}_\alpha \in K$ such that $|\mathcal{A}| \geq \beth_\alpha$. Then the required models $\mathcal{B}_\alpha$ in Theorem 2.8 exist. Let Φ_n, $n < \omega$, I, $\mathcal{B}$ and b_i, $i \in I$, be as in Theorem 2.8. Then we write $\Phi = (\Phi_n)_{n<\omega}$, $\mathcal{B} = EM_{\tau_s}(I, \Phi)$ and call the set $\{b_i | \ i \in I\}$ the skeleton of $\mathcal{B}$.

Remark: Notice that there may be $\pi \in Aut(\mathcal{B})$ such that even $\{\pi(b_i) | \ i \in I\} \neq \{b_i | \ i \in I\}$ and thus $\mathcal{B}$ may have more than one skeleton. However, when we talk about EM-models we always assume that we know what the skeleton is.

We write $EM(I, \Phi)$ for $\mathcal{B} \restriction \tau$.

2.9 Exercise. *Suppose $\mathcal{B} = EM_{\tau_s}(I, \Phi)$ and $\{b_i | \ i \in I\})$ is the skeleton of it.*

(i) Show that $\mathcal{B} \models T$ and it omits every type in SK-GSK. Conclude $\mathcal{B} \restriction \tau \in K$ and that K has arbitrarily large models.

(ii) Suppose $J \subseteq I$. Show that $SH(\{b_i | \ i \in J\}) \restriction \tau \preceq \mathcal{B} \restriction \tau$.

(iii) Suppose $J = (J, <)$ is a linear ordering, $\mathcal{A} = EM_{\tau_s}(J, \Phi)$ with skeleton $\{a_j | \ j \in J\}$ and $\pi_0 : (I, <) \to (J, <)$ is an isomorphism. Show that there is an isomorphism $\pi : \mathcal{B} \to \mathcal{A}$ such that for all $i \in I$, $\pi(a_i) = b_{\pi_0(i)}$.

3. Amalgamation and Galois types

In this section we will look what is the right notion of a type in AEC's.

3.1 Definition.

(i) We define a binary relation ET^* to the class of all pairs $(a, \mathcal{A})$, where $\mathcal{A}$ is a K-model and $a = (a_i)_{i<\alpha} \in \mathcal{A}^\alpha$ (i.e. a is a tuple of elements of $\mathcal{A}$ of length ordinal α and α may be infinite), the following way: $((a_i)_{i<\alpha}, \mathcal{A}) ET^* ((b_i)_{i<\beta}, \mathcal{B})$ if $\alpha = \beta$ and there are a K-model $\mathcal{C}$ and strong embeddings $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ such that for all $i < \alpha$, $f(a_i) = g(b_i)$.

(ii) We let ET be the transitive closure of ET^* (i.e. $(a, \mathcal{A}) ET (b, \mathcal{B})$ if there are $n < \omega$ and $(a_i, \mathcal{A}_i)$, $i \leq n$, such that $(a_0, \mathcal{A}_0) = (a, \mathcal{A})$, $(a_n, \mathcal{A}_n) = (b, \mathcal{B})$ and for all $i < n$, $(a_i, \mathcal{A}_i) ET^* (a_{i+1}, \mathcal{A}_{i+1})$). We write $t^g(a/\emptyset; \mathcal{A})$ for the ET-equivalence class of $(a, \mathcal{A})$ and call it Galois type (of a over $\emptyset$ in $\mathcal{A}$).

3.2 Exercise.

(i) Suppose $\mathcal{A} \preceq \mathcal{B}$ and $a \in \mathcal{A}^\alpha$. Show that $t^g(a/\emptyset; \mathcal{A}) = t^g(a/\emptyset; \mathcal{B})$.

(ii) Let $n \geq 2$ be a natural number and K_n be the class of all graphs $\mathcal{A} = (\mathcal{A}, E)$ with the following property:

(*) For all $a \in \mathcal{A}$, the set $\{b \in \mathcal{A} |\ (a,b) \in E\}$ has cardinality at most n.
Then $(K, \subseteq)$ is an AEC as one can easily see (Exercise 1.4 (ii)). Show that in this AEC, ET^* is not transitive. Hint: Notice that if $\mathcal{A}, \mathcal{B} \in K_n$ and $\mathcal{A}$ contains just one element, then every function $f : \mathcal{A} \to \mathcal{B}$ is a strong embedding.

While these Galois types are natural objects they do not behave very well in general. However, by adding some assumptions on the class K, they start to behave as one hopes.

3.3 Definition.

(i) We say that an AEC $K = (K, \preceq)$ has the amalgamation property (AP) if for all K-models $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and strong embeddings $f_0 : \mathcal{A} \to \mathcal{B}$ and $g_0 : \mathcal{A} \to \mathcal{C}$, there are a K-model $\mathcal{D}$ and strong embeddings $f_1 : \mathcal{B} \to \mathcal{D}$ and $g_1 : \mathcal{C} \to \mathcal{D}$ such that for all $a \in \mathcal{A}$, $f_1(f_0(a)) = g_1(g_0(a))$.

(ii) We say that an AEC $K = (K, \preceq)$ has the joint embedding property (JEP) if for all K-models $\mathcal{A}$ and $\mathcal{B}$ there are a K-model $\mathcal{C}$ and strong embeddings $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$.

3.4 Exercise.

(i) Show that the following are equivalent:

(a) K has AP.

(b) For all K-models $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{A} \preceq \mathcal{C}$, then there are a $\mathcal{K}$-model $\mathcal{D}$ such that $\mathcal{C} \preceq \mathcal{D}$ and a strong embedding $f : \mathcal{B} \to \mathcal{D}$ such that $f \restriction \mathcal{A}$ is identity.

(ii) Show that if K has AP, then ET^* is a transitive relation (and thus $ET = ET^*$).

3.5 Exercise. *Let us return to Coxeter groups. Suppose K_0 is a(n abstract elementary) class of strongly rigid graphs such that it has AP and let $K = (K, \preceq)$ be the related class of Coxeter groups as in Section 1. Show that K has AP.*

3.6 Theorem. *Suppose K has AP and JEP and λ is an infinite cardinal. There is a K-model M such that*

(i) (λ-universality) For all K-models $\mathcal{A}$ of power $\leq \lambda$, there is a strong embedding $f : \mathcal{A} \to M$.

(ii) (λ-model homogeneity) If $\mathcal{A}, \mathcal{B} \preceq M$ are of power $< \lambda$ and $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then there is an automorphism g of M such that $f \subseteq g$ (i.e. $g \restriction \mathcal{A} = f$).

Proof. We skip the proof in this course. The proof is identical to the proof of Theorem 5.11 in the lecture notes of the course Model theory (overlook the claim of the model being existentially closed). □

3.7 Exercise. *Suppose that M and λ are as in Theorem 3.6 and $\lambda > LS(K)$.*

(i) Suppose that $\mathcal{A} \preceq M$ and $\mathcal{B}$ is a K-model of power $< \lambda$ such that $\mathcal{A} \preceq \mathcal{B}$. Show that there is a strong embedding $f : \mathcal{B} \to M$ such that $f \restriction \mathcal{A}$ is the identity.

(ii) Suppose that $\alpha < \lambda$, $A = \{a_i | \ i < \alpha\}$ is a subset of M, $M \preceq \mathcal{A}$ and $b = (b_0, ..., b_n)$ is a tuple of elements of $\mathcal{A}$. Show that there is a tuple $c = (c_0, ..., c_n)$ of elements of M such that $t^g(ca/\emptyset; M) = t^g(ba/\emptyset; \mathcal{A})$, where ca is the concatenation of c and $a = (a_i)_{i<\alpha}$ and ba similarly.

From now on in these lectures we assume that K has AP and JEP and also to avoid potential anomalies we assume also that K has arbitrarily large models. Then when λ is large enough, the model M from Theorem 3.5 is called a monster model. And from now on we will be using the monster model technique: Unless otherwise stated, whenever we talk about a K-model, we assume that it is a strong submodel of M of size $< \lambda$. By Theorem 3.5 (i), this is w.o.l.g. if the size of the model is $< \lambda$ (we can always move to an isomorphic copy) and since we can choose λ freely we can always assume that the size is $< \lambda$. Strictly speaking here is a possibility of making errors in arguments, so one needs to be a bit careful when one applies this technique. But in these lectures we do not look the kind of arguments in which this possibility exists. It will turn out that this technique simplifies not only notations and definitions but also arguments. So from now on, by M we mean a monster model. Notice that with this convention, if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \preceq \mathcal{B}$ by Coherence, see Definition 1.2.

Let us start by looking an example of how this technique works.

3.8 Lemma. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are K-models, $a = (a_i)_{i<\alpha} \in \mathcal{A}^\alpha$ and $b = (b_i)_{i<\alpha} \in \mathcal{B}^\alpha$. Then $t^g(a/\emptyset; \mathcal{A}) = t^g(b/\emptyset; \mathcal{B})$ iff there is $h \in Aut(M)$ (automorphism of M) such that $h(a) = b$ (i.e. for all $i < \alpha$, $h(a_i) = b_i$).*

Proof. $\Rightarrow$: By Exercise 3.4 (ii), there are a K-model $\mathcal{C}$ and strong embeddings $f_0 : \mathcal{A} \to \mathcal{C}$ and $g_0 : \mathcal{B} \to \mathcal{C}$ such that $f_0(a) = g_0(b)$. By theorem 3.6 (ii), f_0 and g_0 can be extended to f and g that belong to $Aut(M)$. Then $h = g^{-1} \circ f$ is as wanted.

$\Leftarrow$: M, h and id_M (identity function on M) show that $(a, M)ET^*(b, M)$. This suffices, since by Exercise 3.4 (i), $t^g(a/\emptyset; \mathcal{A}) = t^g(a/\emptyset; M)$ and similarly for b and $\mathcal{B}$. □

We want to make one more assumption before we start developing stability theory for our classes.

3.9 Definition. *We say that K is homogeneous if (in addition to AP, JEP and arbitrary large models) it satisfies: For all K-models $\mathcal{A}$ and $\mathcal{B}$, ordinals α and $a = (a_i)_{i<\alpha} \in \mathcal{A}^\alpha$ and $b = (b_i)_{i<\alpha} \in \mathcal{B}^\alpha$, if for all finite $X \subseteq \alpha$, $t^g((a_i)_{i\in X}/\emptyset; \mathcal{A}) = t^g((b_i)_{i\in X}/\emptyset; \mathcal{B})$, then $t^g(a/\emptyset; \mathcal{A}) = t^g(b/\emptyset; \mathcal{B})$.*

There are many other properties that improve the behaviour of Galois types. One such is finite character:

3.10 Lemma. *If K is homogeneous the it has finite character i.e. the following holds: Suppose $\mathcal{A}$ is a submodel of a K-model $\mathcal{B}$ such that it belongs to $\mathcal{K}$ (we do* *not assume that $\mathcal{A} \preceq M$) and for all finite tuples $a \in \mathcal{A}^n$, $t^g(a/\emptyset;\mathcal{A}) = t^g(a/\emptyset;\mathcal{B})$. Then $\mathcal{A} \preceq \mathcal{B}$.*

Proof. As above, w.o.l.g. $\mathcal{B} \preceq M$. By Coherence it is enough to show that $\mathcal{A} \preceq M$. By λ-universality of M, there is a strong embedding f of $\mathcal{A}$ to M. Now for all finite tuples $a \in \mathcal{A}^n$,

$$t^g(a/\emptyset;M) = t^g(a/\emptyset;\mathcal{B}) = t^g(a/\emptyset;\mathcal{A}) = t^g(f(a)/\emptyset;f(\mathcal{A})) = t^g(f(a)/\emptyset;M).$$

Thus by Lemma 3.8 and homogeneity of K, there is $g \in Aut(M)$ such that $f \subseteq g$. But then by Definition 1.2 (II) (i) (i.e. Definition 1.2 (I) (i) applied to g^{-1}), $\mathcal{A} \preceq M$. $\square$

3.11 Exercise. *Let T be a complete first-order theory, $K = mod(T)$ and $\preceq$ be the elementary submodel relation. Show that $(K,\preceq)$ is a homogeneous class. Hint: Remember Exercise 1.4 (i) and show that two tuples have the same Galois type iff they have the same first-order type.*

The classes of Coxeter groups are typically not homogeneous (even when they are AEC's) but often they have finite character. The following is an example of a homogeneous class that is not elementary: Let K^* be the class of all fields together with two automorphisms that commute with each other (i.e. the second is an automorphism of the field with the first automorphism). Then K^* with submodel relation is an AEC. Now let $\mathcal{A}_0 \in K^*$ be an existentially closed model (see the lecture notes of the course Model theory) and let $\mathcal{A} \in K^*$ be the algebraic closure (in the sense of field theory) of $\emptyset$ in $\mathcal{A}_0$. Then we let $\mathcal{K}$ be class of all existentially closed models of $\mathcal{K}^*$ which contain a copy of $\mathcal{A}$ as a substructure. Then $(K,\subseteq)$ is a homogeneous AEC. At least for some choices of $\mathcal{A}$, it is not elementary, which is a bit surprising since with just one automorphism, these classes are elementary (these are known as difference fields and they play an important role in Hrushovski's work that we mentioned in the introduction). For details, see [HK2].

From now on, unless stated otherwise, by elements we mean elements of M, by a set we mean a subset of M of power $< \lambda$, by tuples we mean tuples of elements of M and, again unless stated otherwise, the tuples are assumed to be finite. Also from now on, $a \in A$ means that a is a finite tuple of elements of A (and A is a subset of M of power $< \lambda$). We will write AB for the union of A and B and ab for the concatenation $a^\frown b$ of tuples a and b. Aa, $a = (a_i)_{i<\alpha}$, means the set $A \cup \{a_i \mid i < \alpha\}$. Finally, for tuples $a_i = (a^i_j)_{j<\alpha_i}$, $i < \alpha$, $\{a_i \mid i < \alpha\}$ means the set $\{a^i_j \mid i < \alpha,\ j < \alpha_i\}$. Also for all, possibly infinite, tuples a we write $t^g(a/\emptyset)$ for $t^g(a/\emptyset;M)$, which is the same as $t^g(a/\emptyset;\mathcal{A})$ for any $\mathcal{A}$ that contains a. For $A \subseteq M$, we write $Aut(M/A)$ for the set of all automorphisms f of M such that $f \restriction A$ is identity. We write $t^g(a/A) = t^g(b/A)$ if there is $f \in Aut(M/A)$ such that $f(a) = b$ and we think $t^g(a/A)$ as the orbit of a under the (natural) action of $Aut(M/A)$ on M e.g. when we talk about sets of Galois types over A.

3.12 Exercise. *Let $A = \{a_i|\ i < \alpha\}$. Show that $t^g(a/A) = t^g(b/A)$ iff $t^g(a^\frown(a_i)_{i<\alpha}/\emptyset) = t^g(b^\frown(a_i)_{i<\alpha}/\emptyset)$. Conclude that if K is homogeneous, then $t^g(a/A) = t^g(b/A)$ iff for all finite $B \subseteq A$, $t^g(a/B) = t^g(b/B)$.*

3.13 Lemma. *Assume K is homogeneous. Suppose I is any set and for all finite subsets $X \subseteq I$, there are elements a_i^X, $i \in X$, such that if $Y \subseteq X$, then $t^g((a_i^X)_{i\in Y}/\emptyset) = t^g((a_i^Y)_{i\in Y}/\emptyset)$. Then there are elements a_i, $i \in I$ such that for all finite $X \subseteq I$, $t^g((a_i)_{i\in X}/\emptyset) = t^g((a_i^X)_{i\in X}/\emptyset)$.*

Proof. We prove the claim by induction on $|I|$. The cases when I is finite are clear. So suppose $|I| = \xi$ is infinite. By re-indexing we may assume that $I = \xi$. By the induction assumption, for all $\alpha < \xi$, there are b_i^α, $i \leq \alpha$, such that for all finite $X \subseteq \alpha + 1$, $t^g((b_i^\alpha)_{i\in X}/\emptyset) = t^g((a_i^X)_{i\in X}/\emptyset)$. We find the elements a_α, $\alpha < \xi$, by recursion on $\alpha < \xi$: We let $a_0 = b_0^0$ and for $\alpha > 0$ we do the following. We notice that by Lemma 3.8, homogeneity and the assumption that we have already found a_i, $i < \alpha$, as wanted, there is $f \in Aut(M)$ such that for all $i < \alpha$, $f(b_i^\alpha) = a_i$. Then we let $a_\alpha = f(b_\alpha^\alpha)$. Clearly these are as wanted. □

3.14 Exercise. *Assume that K is homogeneous and A ($\subseteq M$) is a set. Suppose that for all finite $C \subseteq A$ there is a (possibly infinite) tuple a_C such that the following holds: If $C \subseteq D \subseteq A$ are finite then $t^g(a_D/C) = t^g(a_C/C)$. Show that there is a tuple a such that for all finite $C \subseteq A$, $t^g(a/C) = t^g(a_C/C)$.*

4. Independence and ω-stability

In this section we assume that $K = (K, \preceq)$ is a homogeneous AEC. In addition we assume that K is ω-stable:

4.1 Definition. *Let κ be an infinite cardinal. We say that a (homogeneous) class K is κ-stable if $LS(K) \leq \kappa$ and for all A of size $\leq \kappa$, the set $\{t^g(a/A)|\ a \in M\}$ is of size $\leq \kappa$.*

Usually stability theory is developed under the assumption that the class is stable (or even less is assumed). However we make this stronger ω-stability assumption because it simplifies the part of the theory of independence that is needed in these lectures (we can use splitting instead of strong splitting or dividing or Lascar splitting and Lascar types are not needed). For the general theory, see [HS] (in this paper the approach is not that of AEC's but the proofs and results translate easily to our context).

4.2 Definition.

(i) Let κ be an infinite cardinal. We say that a model $\mathcal{A}$ is κ-saturated if for all $A \subseteq \mathcal{A}$ of power $< \kappa$ and (finite) tuples a, there is $b \in \mathcal{A}$ such that $t^g(b/A) = t^g(a/A)$. We say that $\mathcal{A}$ is saturated if it is $|\mathcal{A}|$-saturated.

(ii) Suppose $A \subseteq B$ and $a \in M$. We say that $t^g(a/B)$ splits over A if there are $b, c \in B$ such that $t^g(b/A) = t^g(c/A)$ but $t^g(ab/\emptyset) \neq t^g(ac/\emptyset)$.

(iii) We write $a \downarrow^s_C B$ (a is independent from B over C) if there is finite $A \subseteq C$ such that $t^g(a/CB)$ does not split over A. We write $C \downarrow^s_A B$ if for all (finite tuples) $a \in C$, $a \downarrow^s_A B$.

As pointed out above there are other ways of defining independence and they do not give the same notion of independence. However, in our ω-stable case, they all give the same notion over ω-saturated models.

Notice that by Exercise 3.12, in Definition 4.2 (ii), we could replace $t^g(ab/\emptyset) \neq t^g(ac/\emptyset)$ by $t^g(ab/A) \neq t^g(ac/A)$ (or by $t^g(b/Aa) \neq t^g(c/Aa)$ or $t^g(b/a) \neq t^g(c/a)$) and get an equivalent definition. We will use this freely below. In more general cases one needs to be more careful here.

4.3 Exercise.

(i) Show that for all infinite cardinals κ and A of power κ there is an ω-saturated model $\mathcal{A}$ of power κ such that $A \subseteq \mathcal{A}$ and that for all A of power ω_1 there is ω_1-saturated $\mathcal{A}$ of power ω_1 such that $A \subseteq \mathcal{A}$.

(ii) Suppose $\mathcal{A}$ and $\mathcal{B}$ are κ-saturated models of power κ. Show that $\mathcal{A} \cong \mathcal{B}$.

(iii) Suppose $\mathcal{A}$ is a κ-saturated model of power κ and $a, b \in \mathcal{A}$ are tuples of length $< \kappa$ such that $t^g(a/\emptyset) = t^g(b/\emptyset)$. Show that there is $f \in Aut(\mathcal{A})$ such that $f(a) = b$.

(iv) Suppose that for all $A \subseteq \mathcal{A}$ of power $< \kappa$ and elements a, there is $b \in \mathcal{A}$ such that $t^g(b/A) = t^g(a/A)$. Show that $\mathcal{A}$ is κ-saturated.

(v) Let K be the class of all vector spaces over the rational numbers. Then $(K, \subseteq)$ is a homogeneous AEC. Suppose that $a = (a_0, ..., a_n)$ is linearly independent (for linear independence, see any book on linear algebra or wikipedia). We say that a is linearly independent from $\mathcal{A}$ if for all linearly independent $b = (b_0, ..., b_m) \in \mathcal{A}$, ab is linearly independent. Show that if $\mathcal{A}$ has infinite dimension, then $a \downarrow_\emptyset \mathcal{A}$ iff a is linearly independent from $\mathcal{A}$.

4.4 Theorem.

(i) For all a and A, $a \downarrow^s_A \emptyset$. Furthermore, if $a \in A$, then $a \downarrow^s_A B$ for all B.

(ii) (Existence) For all a, A and ω-saturated $\mathcal{A}$ there is b such that $t^g(b/\mathcal{A}) = t^g(a/\mathcal{A})$ and $b \downarrow^s_\mathcal{A} A$.

(iii) (Stationarity) If $\mathcal{A}$ is ω-saturated, $a \downarrow^s_\mathcal{A} A$, $b \downarrow^s_\mathcal{A} A$ and $t^g(a/\mathcal{A}) = t^g(b/\mathcal{A})$, then $t^g(a/\mathcal{A}A) = t^g(b/\mathcal{A}A)$.

(iv) (Local character) If $a \not\downarrow^s_\mathcal{A} A$ and $\mathcal{A}$ is ω-saturated, then there is $b \in A$ such that $a \not\downarrow^s_\mathcal{A} b$.

(v) If $\mathcal{A}$ is ω-saturated, $a \notin \mathcal{A}$ and $a \downarrow^s_\mathcal{A} A$, then $a \notin \mathcal{A}A$.

(vi) If $\mathcal{A}$ is ω-saturated and a_i, $i < \omega$, are such that for all $i < j < \omega$, $t^g(a_i/\mathcal{A}) = t^g(a_j/\mathcal{A})$ and for all $i < \omega$, $a_i \downarrow^s_\mathcal{A} \{a_j |\ j < i\}$, then for all $i_0 < ... < i_n < \omega$ and $j_0 < ... < j_n < \omega$, $t^g((a_{i_0}, ..., a_{i_n})/\mathcal{A}) = t^g((a_{j_0}, ..., a_{j_n})/\mathcal{A})$.

(vii) (Symmetry) If $A \downarrow^s_\mathcal{A} B$ and $\mathcal{A}$ is ω-saturated, then $B \downarrow^s_\mathcal{A} A$.

Proof. (i): We leave the furthermore part as an exercise and prove only the first claim: Now suppose not. We define finite tuples a_η and b_η and $f_\eta, g_\eta \in Aut(M)$ for all $\eta \in 2^{<\omega}$ by recursion on $dom(\eta)$ so that (B_η means the set $\{b_{\eta\restriction m} |\ m \le dom(\eta)\}$ and for $\eta \in 2^n$ $\eta^\frown(i)$ means the function $h \in 2^{n+1}$ such that $h(x) = \eta(x)$ if $x < n$ and $h(n) = i$):

(a) $t^g(a_{\eta^\frown(i)}/B_\eta) = t^g(a_\eta/B_\eta)$, $f_{\eta^\frown(i)} \in Aut(M/B_\eta)$, $g_{\eta^\frown(i)} = f_{\eta^\frown(i)} \circ g_\eta$, $f_\emptyset = g_\emptyset = id_M$, $a_\eta = g_\eta(a)$ and $g_\eta^{-1}(B_\eta) \subseteq A$,

(b) $b_{\eta^\frown(0)} = b_{\eta^\frown(1)}$ and $t^g(a_{\eta^\frown(0)}/b_{\eta^\frown(0)}) \ne t^g(a_{\eta^\frown(1)}/b_{\eta^\frown(0)})$.

We let $a_\emptyset = a$, $b_\emptyset = \emptyset$ and $f_\emptyset = g_\emptyset = id_M$. Suppose that we have defined the tuples for all $\eta \restriction n$, $n \le dom(\eta)$. Since $t^g(a/A)$ splits over $g_\eta^{-1}(B_\eta)$, there are $b, c \in g_\eta(A)$ such that $f(c) = b$ for some $f \in Aut(M/B_\eta)$ and $t^g(a_\eta b/\emptyset) \ne t^g(a_\eta c/\emptyset)$. We let $b_{\eta^\frown(0)} = b_{\eta^\frown(1)} = b$, $a_{\eta^\frown(0)} = a_\eta$, $a_{\eta^\frown(1)} = f(a_\eta)$, $f_{\eta^\frown(0)} = id_M$, $f_{\eta^\frown(1)} = f$, $g_{\eta^\frown(0)} = g_\eta$ and $g_{\eta^\frown(1)} = f \circ g_\eta$. Then for all $\eta \in 2^\omega$, by Exercise 3.14, we can find a_η such that for all $n < \omega$, $t^g(a_n/B_{\beta\restriction n}) = t^g(a_{\eta\restriction n}/B_{\eta\restriction n})$. But then for all $\eta, \eta' \in 2^\omega$, $t^g(a_\eta/B) \ne t^g(a_{\eta'}/B)$, where $B = \{b_\nu |\ \nu \in 2^{<\omega}\}$. This contradicts ω-stability.

(ii): W.o.l.g. $\mathcal{A} \subseteq A$. By (i), choose finite $C \subseteq \mathcal{A}$ such that $t^g(a/\mathcal{A})$ does not split over C. For all (finite tuples) $c \in A$, choose $d_c \in \mathcal{A}$ such that $t^g(d_c/C) = t^g(c/C)$ and denote $p_c = t^g(ad_c/\emptyset)$ (if $c \in \mathcal{A}$, then let $d_c = c$). Now d_c need not be unique but notice that by the choice of C, p_c does not depend on the choice of d_c. Then as in the proof of (i) above, there is b such that for all $c \in A$, $t^g(bc/\emptyset) = p_c$. Clearly b is as wanted.

(iii): Choose finite $C \subseteq \mathcal{A}$ such that neither $t^g(a/\mathcal{A}A)$ nor $t^g(b/\mathcal{A}A)$ split over C (e.g. choose C' for $t^g(a/\mathcal{A}A)$ and C'' for $t^g(b/\mathcal{A}A)$ and let $C = C' \cup C''$) and let $c \in \mathcal{A}A$. It is enough to show that $t^g(ac/\emptyset) = t^g(bc/\emptyset)$. Let $d \in \mathcal{A}$ be such that $t^g(d/C) = t^g(c/C)$. Then

$$t^g(ac/\emptyset) = t^g(ad/\emptyset) = t^g(bd/\emptyset) = t^g(bc/\emptyset).$$

(iv): By (ii) there is c such that $t^g(c/\mathcal{A}) = t^g(a/\mathcal{A})$ and $c \downarrow^s_{\mathcal{A}} A$. Then $t^g(a/\mathcal{A}A) \ne t^g(c/\mathcal{A}A)$. So there is $b' \in \mathcal{A}A$ such that $t^g(ab'/\emptyset) \ne t^g(cb'/\emptyset)$. But then by letting b be the sequence of elements of b' that do not belong to $\mathcal{A}$, $a \not\downarrow^s_{\mathcal{A}} b$ by (iii).

(v): Exercise.

(vi): Exercise.

(vii): By (iii) and the definition of $A \downarrow^s_{\mathcal{A}} B$, it is enough to prove the claim under the assumption that A and B are finite i.e. that if $a \downarrow^s_{\mathcal{A}} b$ then $b \downarrow^s_{\mathcal{A}} a$. Suppose not. We may assume that $\mathcal{A}$ is countable i.e. if there is a counter example to our claim, there is one in which $\mathcal{A}$ is countable (exercise). By (i), $a \notin \mathcal{A}$ and $b \notin \mathcal{A}$. Choose a_i and b_i, $i < \omega$, so that for all $i < \omega$, $t^g(a_ib_i/\mathcal{A}) = t(ab/\mathcal{A})$ and $a_ib_i \downarrow^s_{\mathcal{A}} \{a_jb_j |\ j < i\}$. Notice that $i < j$ iff $b_j \downarrow^s_{\mathcal{A}} a_i$ and thus for all i, j, if $i \ne j$, then $t^g(a_ib_ia_jb_j/\mathcal{A}) \ne t^g(a_jb_ja_ib_i/\mathcal{A})$. Let $\mathbf{R}$ be the set of reals and $\mathbf{Q}$ the set of rational numbers. Now by (vi) and Exercise 3.14, we can find c_i and d_i, $i \in \mathbf{R}$, such that for all $i, j \in \mathbf{R}$, if $i < j$, then $t^g(c_id_ic_jd_j/\mathcal{A}) = t^g(a_0b_0a_1b_1/\mathcal{A})$. Let $B = \{c_i, d_i |\ i \in \mathbf{Q}\}$. Now for all $i, j \in \mathbf{R}$, if $i \ne j$, then $t^g(a_ib_i/\mathcal{A}B) \ne t^g(a_jb_j/\mathcal{A}B)$. This contradicts ω-stability. □

4.5 Exercise.

(i) (Monotonicity) Suppose $A \subseteq B \subseteq C \subseteq D$. Show that if $a \downarrow^s_A D$, then $a \downarrow^s_B C$.

(ii) Show that $ab \downarrow^s_A B$ iff $b \downarrow^s_A B$ and $a \downarrow^s_{Ab} B$.

(iii) (Transitivity) Suppose $\mathcal{A} \subseteq \mathcal{B} \subseteq C$ and $\mathcal{A}$ and $\mathcal{B}$ are ω-saturated. Show that $a \downarrow^s_{\mathcal{A}} C$ iff $a \downarrow^s_{\mathcal{A}} \mathcal{B}$ and $a \downarrow_{\mathcal{B}} C$. Hint: the shortest proof can be found using (i) above and Theorem 4.4 (ii) and (iii).

(iv) Show that K is κ-stable for all infinite κ. Hint: Notice that it is enough to count the number of types over ω-saturated models and then use Theorem 4.4 (i) and (iii).

4.6 Definition. *Let $\mathcal{A}$ be ω-saturated.*

(i) We say that a sequence $(a_i)_{i\in I}$ of possibly infinite tuples is independent over $\mathcal{A}$ if for all $i \in I$, $a_i \downarrow^s_{\mathcal{A}} \{a_j |\ j \in I - \{i\}\}$.

(ii) We say that $(a_i)_{i\in I}$ is a Morley sequence in $t^g(a/\mathcal{A})$ if $(a_i)_{i\in I}$ is independent over $\mathcal{A}$ and for all $i \in I$, $t^g(a_i/\mathcal{A}) = t^g(a/\mathcal{A})$.

Notice that if $(a_i)_{i<\alpha}$ is a Morley sequence in $t^g(a/\mathcal{A})$ and $\pi : \alpha \to \alpha$ is a permutation, then $(a_{\pi(i)})_{i<\alpha}$ is a Morley sequence in $t^g(a/\mathcal{A})$.

4.7 Exercise. *Suppose $\mathcal{A}$ is ω-saturated and a is a tuple.*

(i) Show that $(a_i)_{i<\alpha}$ is independent over $\mathcal{A}$ if for all $i < \alpha$, $a_i \downarrow^s_{\mathcal{A}} \{a_j |\ j < i\}$.

(ii) Show that $t^g((a_i)_{i<\alpha}/\mathcal{A}) = t^g((b_i)_{i<\alpha}/\mathcal{A})$ if both $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ are Morley sequences in $t^g(a/\mathcal{A})$.

(iii) Suppose $\mathcal{C}$ is countable and ω-saturated and $\mathcal{C} \subseteq \mathcal{A}$. Show that there is a maximal Morley sequence $(a_i)_{i<\alpha}$ in $t^g(a/\mathcal{C})$ from $\mathcal{A}$ i.e. $(a_i)_{i<\alpha}$ is a Morley sequence in $t^g(a/\mathcal{C})$, for all $i < \alpha$, $a_i \in \mathcal{A}$ and if $a' \in \mathcal{A} - \{a_i |\ i < \alpha\}$ is such that $t^g(a'/\mathcal{C}) = t(a/\mathcal{C})$, then $a' \not\downarrow^s_{\mathcal{C}} \{a_i |\ i < \alpha\}$. Show that if in addition $\mathcal{A}$ is ω_1-saturated, then α is uncountable.

5. Prime models

In this section we look prime models. Throughout this section we assume that K is a homogeneous ω-stable AEC. Again for the general homogeneous case, see [HS].

5.1 Definition. *We say that $t^g(a/A)$ is ω-isolated if there is finite $B \subseteq A$ such that for all b, if $t^g(b/B) = t^g(a/B)$, then $t^g(b/A) = t^g(a/A)$. In this case we also say that B witnesses that $t^g(a/A)$ is ω-isolated.*

5.2 Lemma. *Suppose $C \subseteq A$, C is finite and a is a finite tuple. Then there are b and finite $C \subseteq B \subseteq A$ such that $t^g(b/C) = t^g(a/C)$ and B witnesses that $t^g(b/A)$ is ω-isolated.*

Proof. Suppose not. For all $\eta \in 2^{<\omega}$, we define tuples b_η and finite sets $B_\eta \subseteq A$ by recursion on $dom(\eta)$ as follows:

We let $b_\emptyset = a$ and $B_\emptyset = C$. Now suppose b_η and B_η are defined. Then there is c such that $t^g(c/B_\eta) = t^g(b_\eta/B_\eta)$ but $t^g(c/A) \neq t^g(b_\eta/A)$. Let $b_{\eta\frown(0)} = b_\eta$, $b_{\eta\frown(1)} = c$ and $B_{\eta\frown(0)} = B_{\eta\frown(1)}$ be any finite D such that $B_\eta \subsetneq D \subseteq A$ and $t^g(c/D) \neq t^g(b_\eta/D)$. Let $B = \cup_{\eta\in 2^{<\omega}} B_\eta$ and again, by Exercise 3.14, we can find b_η for all $\eta \in 2^\omega$ such that for all $n < \omega$, $t^g(b_\eta/B_{\eta\restriction n}) = t^g(b_{\eta\restriction n}/B_{\eta\restriction n})$. But then for all $\eta, \eta' \in 2^\omega$, if $\eta \neq \eta'$, then $t^g(b_\eta/B) \neq t^g(b_{\eta'}/B)$. Since B is countable, this contradicts ω-stability. □

5.3 Definition. *Let $I = (I, <)$ be a well ordering.*

(i) We say that $(a_i, B_i)_{i\in I}$ is an ω-construction over A if writing $A_i = A \cup \{a_j \mid j < i\}$, for all $i \in I$, B_i witnesses that $t(a_i/A_i)$ is isolated.

(ii) Let $(a_i, B_i)_{i\in I}$ be an ω-construction over A. We say that $X \subseteq I$ is closed if for all $i \in X$, $B_i \subseteq A \cup \{a_j \mid j < i,\ j \in X\}$.

Notice that a union of closed sets is a closed set.

5.4 Exercise. *Let $((a_i, B_i))_{i\in I}$ be an ω-construction over A.*

(i) Show that for all finite $Y \subseteq I$ there is finite closed $X \subseteq I$ such that $Y \subseteq X$.

(ii) Suppose $X \subseteq I$ is closed. Show that $((a_i, B_i))_{i\in X}$ is an ω-construction over A.

(iii) Let $i \in I$ and $X = \{j \in I \mid j < i\}$. Show that $((a_i, B_i))_{i\in X}$ is an ω-construction over A and that $((a_i, B_i))_{i\in I-X}$ is an ω-construction over $A \cup \{a_j \mid j \in X\}$.

(iv) Suppose $B \subseteq A$ witnesses that $t^g(ab/A)$ is ω-isolated. Show that Bb witnesses that $t^g(a/Ab)$ is ω-isolated.

5.5 Lemma. *Suppose $((a_0, B_0), ..., (a_n, B_n))$ is an ω-construction over A and $B_n \subseteq A$. Then $((a_n, B_n), (a_0, B_0), ..., (a_{n-1}, B_{n-1}))$ is an ω-construction over A.*

Proof. Suppose not. Choose this counter example so that n is minimal. As in the definition, we write $A_i = A \cup \{a_0, ..., a_{i-1}\}$. Now

$$((a_0, B_0), ..., (a_{n-2}, B_{n-2}), (a_n, B_n))$$

is an ω-construction over A and so by the choice of n, also

$$((a_n, B_n), (a_0, B_0), ..., (a_{n-2}, B_{n-2}))$$

is an ω-construction over A. But then there is a'_{n-1} such that $t^g(a'_{n-1}/B_{n-1}) = t^g(a_{n-1}/B_{n-1})$ but $t^g(a'_{n-1}/A^*) \neq t^g(a_{n-1}/A^*)$, where $A^* = A \cup \{a_0, ..., a_{n-2}, a_n\}$. Since $t^g(a'_{n-1}/A_{n-1}) = t^g(a_{n-1}/A_{n-1})$, there is a function $f \in Aut(M/A_{n-1})$ such that $f(a'_{n-1}) = a_{n-1}$. Let $a'_n = f(a_n)$. Then $t^g(a'_n/B_n) = t^g(a_n/B_n)$ but $t^g(a'_n/A_n) \neq t^g(a_n/A_n)$, a contradiction. □

5.6 Exercise. *Let $((a_i, B_i))_{i\in I}$ be an ω-construction over A.*

(i) Suppose $<'$ is another well-ordering of I and we write $I' = (I, <')$. Show that if for all $i \in I$, $B_i \subseteq A \cup \{a_j \mid j <' i\}$, then $((a_i, B_i))_{i\in I'}$ is an ω-construction

over A. Hint: Suppose not. By Exercise 5.4 (i) we can assume that I in the counter assumption is finite and thus we can choose the counter assumption so that $|I|$ is minimal. Now apply Lemma 5.5 and Exercise 5.4 (iii).

(ii) Suppose $X \subseteq I$ is closed. Show that $((a_i, B_i))_{i \in I-X}$ is an ω-construction over $A \cup \{a_j |\ j \in X\}$.

(iii) Show that one can find an ordering $<'$ of I so that, letting $I' = (I, <')$, $((a_i, B_i))_{i \in I'}$ is an ω-construction over A and the order type of $(I, <')$ is $|I|$ i.e. that there is an order preserving onto function f from the cardinal $\kappa = |I|$ with the usual ordering to $(I, <')$. Hint: Enumerate $I = \{c_i |\ i < \kappa\}$ and then let $f(\alpha)$ be c_i if i is the least ordinal $\neq f(\beta)$ for any $\beta < \alpha$ and such that $B_{c_i} \subseteq A \cup \{a_{f(\beta)} |\ \beta < \alpha\}$.

5.7 Definition.

(i) We say that a model $\mathcal{A}$ is ω-primary over $A \subseteq \mathcal{A}$ if $\mathcal{A}$ is ω-saturated and there is an ω-construction $((a_i, B_i))_{i \in I}$ over A such that $\mathcal{A} = A \cup \{a_i |\ i \in I\}$.

(ii) We say that $f : A \to B$ is strong if for all $a \in A$, $t^g(a/\emptyset) = t^g(f(a)/\emptyset)$ i.e. there is $g \in Aut(M)$ such that $f \subseteq g$.

(iii) We say that a model $\mathcal{A}$ is ω-prime over $A \subseteq \mathcal{A}$ if for all ω-saturated $\mathcal{B}$ and strong $f : A \to \mathcal{B}$ there is a strong embedding $g : \mathcal{A} \to \mathcal{B}$ such that $f \subseteq g$.

For the next exercise we make the following definition: We say that a set A is ω-saturated if for all finite $B \subseteq A$ and b there is $a \in A$ such that $t^g(a/B) = t^g(b/B)$ (i.e. exactly as in the definition of ω-saturated model, only we do not require A to be a strong submodel of M, not even a model).

5.8 Exercise.

(i) Suppose A is an ω-saturated set.

(a) Show that A is a submodel of M (i.e. it generates a submodel whose universe is A).

(b) Show that if A is countable, then it is a K-model and $A \preceq M$. Hint: Find a strong onto map $f : A \to \mathcal{A}$ for some model $\mathcal{A}$, see Exercise 4.3 (ii) and the proof of Lemma 3.10.

(c) Show that A is a K-model and $A \preceq M$. Hint: (b) and Exercises 1.3 (iii) and 4.3 (i) ($\kappa = \omega$).

(ii) Suppose $((a_i, B_i))_{i<\alpha}$ is ω-construction over A and $\mathcal{B}$ is ω-saturated and $f : A \to \mathcal{B}$ is strong. Then there is strong $g : A \cup \{a_i |\ i < \alpha\} \to \mathcal{B}$ such that $f \subseteq g$. Conclude that for all A there is an ω-primary model over A and that ω-primary models over A are ω-prime models over A. Hint: For the first part of the conclusion, use Exercise 4.3 (i) to show that the construction can not go on forever (of course (i) above is also needed and there are other ways of seeing this first part of the conclusion but the one suggested in this hint is the most instructive).

(iii) Suppose $\mathcal{A}$ is ω-primary over A. Show that for all $a \in \mathcal{A}$, $t(a/A)$ is ω-isolated (i.e. ω-primary models over A are ω-atomic over A).

5.9 Theorem. *Suppose $f : A \to B$ is strong and onto, $\mathcal{A}$ is ω-primary model over A and $\mathcal{B}$ is ω-primary model over B. Then there is an isomorphism $g : \mathcal{A} \to \mathcal{B}$ such that $f \subseteq g$.*

Proof. Let $((a_i, B_i))_{i<\delta}$ and $((b_j, C_j))_{j<\nu}$ be the constructions. For all $\alpha \in On$, it is enough to find strong onto functions $f_\alpha : A_\alpha \to B_\alpha$ so that

(i) there are closed $X_\alpha \subseteq \delta$ and $Y_\alpha \subseteq \nu$ such that $A_\alpha = A \cup \{a_i|\ i \in X_\alpha\}$ $B_\alpha = B \cup \{b_i|\ i \in Y_\alpha\}$,

(ii) if $\alpha < \beta$, then $X_\alpha \subseteq X_\beta$, $Y_\alpha \subseteq Y_\beta$ and $f_\alpha \subseteq f_\beta$,

(iii) if $dom(f_\alpha) \neq \mathcal{A}$ or $rng(f_\alpha) \neq \mathcal{B}$, then $f_{\alpha+1} \neq f_\alpha$.

We let $f_0 = f$ and $X_0 = Y_0 = \emptyset$. If α is limit we let $f_\alpha = \cup_{\beta<\alpha} f_\beta$, $X_\alpha = \cup_{\beta<\alpha} X_\beta$ and $Y_\alpha = \cup_{\beta<\alpha} Y_\beta$. Finally for successor cases we notice that by Exercise 5.6 (ii), it is enough to show how f_1, X_1 and Y_1 are found. But this is an easy back and forth argument using ω-saturation of the models and Exercises 5.4 (i) and 5.8 (iii) (exercise). $\square$

5.10 Exercise. *Suppose $\mathcal{A}$ is ω-saturated.*

(i) Suppose $\mathcal{B}$ is ω-primary model over $\mathcal{A}A$ and $a \downarrow_{\mathcal{A}} A$. Show that $a \downarrow_{\mathcal{A}} \mathcal{B}$. Hint: Start by showing that it is enough to prove the claim under the assumption that A is finite and then show that for all $b \in \mathcal{B}$ there is finite $B \subseteq \mathcal{A}$ such that BA witnesses that $t^g(b/\mathcal{A}Aa)$ is ω-isolated. For this use Exercise 5.8 (iii) and non-splitting.

(ii) Suppose $(a_i)_{i<\alpha}$ is a Morley sequence in $t^g(a/\mathcal{A})$, $a \notin \mathcal{A}$, and $\mathcal{B}$ is ω-primary over $A \cup \{a_i|\ i < \alpha\}$. Show that $(a_i)_{i<\alpha}$ is a maximal Morley sequence in $t^g(a/\mathcal{A})$ from $\mathcal{B}$. Hint: Use (i).

6. Categoricity transfer

We say that K is κ-categorical if all K-model of power κ are isomorphic. In this section we assume that K is homogeneous, $LS(K) = \omega$ and that K is ω_1-categorical and we will show that K is κ-categorical for all uncountable κ. For a more general categoricity transfer theorem, see [Hy].

6.1 Lemma. *K is ω-stable.*

Proof. By Theorem 2.8 there is a K-model $\mathcal{A}$ of the form $EM(\omega_1, \Phi)$ for some Φ. Then $|\mathcal{A}| = \omega_1$.

6.1.1 Claim. *For all countable $A \subseteq \mathcal{A}$, the set $\{t^g(a/A)|\ a \in \mathcal{A}\}$ is countable.*

Proof. Exercise. Hint: Let $\{c_i|\ i < \omega_1\}$ be the skeleton. Now suppose the claim is not true and find countable sets $X \subseteq Y \subseteq \omega_1$ such that the set

$$\{t(a,/SH(\{c_i|\ i \in X\}))|\ a \in SH(\{c_i|\ i \in Y\})\}$$

is uncountable (which is impossible). $\square$ Claim 6.1.1.

If K is not ω-stable by Löwenheim-Skolem property, we can find a K-model $\mathcal{B}$ and countable $B \subseteq \mathcal{B}$ such that $|\mathcal{B}| = |\{t(b/B)|\ b \in \mathcal{B}\}| = \omega_1$. But then $\mathcal{B}$ can not be isomorphic with $\mathcal{A}$, a contradiction. $\square$

6.2 Corollary. *Every uncountable model is ω_1-saturated.*

Proof. By Exercise 4.3 (i) we can find ω_1-saturated $\mathcal{A}$ of power ω_1. By ω_1-categoricity, every model of size ω_1 is ω_1-saturated. Using Löwenheim-Skolem property, it is easy to see that if some uncountable model is not ω_1-saturated, then there is also a model of size ω_1 that is not ω_1-saturated. □

6.3 Definition.

(i) We say that $t^g(a/A)$ *is bounded if there is a set* $B\,(\subseteq M$ *of power* $< \lambda$, *see Section 3) such that if* $t^g(b/A) = t^g(a/A)$, *then* $b \in B$.

(ii) $t^g(a/A)$ *is minimal if it is not bounded but for all* B, $a \not\downarrow_A B$ *implies that* $t^g(a/AB)$ *is bounded.*

6.4 Exercise.

(i) Suppose A *is countable. Show that* $t^g(a/A)$ *is bounded iff* $\{b|\ t^g(b/A) = t^g(a/A)\}$ *is countable. Hint: Use Theorem 4.4 (ii) and (v) and Exercise 4.3 (i).*

(ii) Prove the claim in (i) above without the assumption that A *is countable. Hint: First find countable* $B \subseteq A$ *such that* $t^g(a/A)$ *does not split over some finite* $C \subseteq B$ *and that for all finite* $C \subseteq B$ *and* $b \in A$, *there is* $c \in B$ *such that* $t^g(c/C) = t^g(b/C)$. *Then show that* $t^g(a/A)$ *is bounded iff* $t^g(a/B)$ *is bounded. For the non-trivial direction of this, pick a sequence* $(a_i)_{i<\alpha}$ *of distinct tuples with* $t^g(a_i/B) = t^g(a/B)$ *and show that there is a sequence* $(b_i)_{i<\alpha}$ *of distinct tuples with* $t^g(b_i/A) = t^g(a/A)$, *use the proof of Theorem 4.4 (ii) and (iii).*

(iii) Give an example of a homogeneous ω*-stable AEC such that for some* a, $t^g(a/\emptyset)$ *is bounded and* $\{b|\ t^g(b/\emptyset) = t^g(a/\emptyset)\}$ *is infinite (cooked up example is enough).*

6.5 Lemma. *For all* ω*-saturated* $\mathcal{A}$ *there is an element* a *such that* $t^g(a/\mathcal{A})$ *is minimal.*

Proof. It is enough to prove the claim for countable $\mathcal{A}$. Since all ω-saturated countable models are isomorphic, it is enough to find some countable $\mathcal{A}$ and a as in the claim. Now suppose that there are no such $\mathcal{A}$ and a.

Then by recursion on $i < \omega$ (using Theorem 4.4 (iv)) it is easy to construct countable ω-saturated $\mathcal{A}_i$ and elements a_i so that

(i) if $i < j < \omega$, then $\mathcal{A}_i \subseteq \mathcal{A}_j$ and $t^g(a_j/\mathcal{A}_i) = t^g(a_i/\mathcal{A}_i)$,

(ii) $t^g(a_i/\mathcal{A}_i)$ is not bounded,

(iii) $a_{i+1} \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$.

Let $\mathcal{A} = \cup_{i<\omega}\mathcal{A}_i$ and by Exercise 3.14, we can find a such that for all $i < \omega$, $t^g(a/\mathcal{A}_i) = t^g(a_i/\mathcal{A}_i)$. By Exercise 4.5 (i), $a \not\downarrow_{\mathcal{A}_i} \mathcal{A}$ for all $i < \omega$. This contradicts Theorem 4.4 (i). □

6.6 Theorem. K *is* κ*-categorical for all* $\kappa \geq \omega_1$.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are model of size $\kappa > \omega_1$. It is enough to show that they are isomorphic. Let $\mathcal{C}$ be a countable ω-saturated model. Since $\mathcal{A}$ and $\mathcal{B}$ are ω_1-saturated there are strong embeddings $f : \mathcal{C} \to \mathcal{A}$ and $g : \mathcal{C} \to \mathcal{B}$. By moving $\mathcal{A}$ and $\mathcal{B}$ by an automorphism of M, we may assume that $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{B}$.

Now let a be an element such that $t^g(a/\mathcal{C})$ is minimal and let $(a_i)_{i<\alpha}$ be a maximal Morley sequence in $t^g(a/\mathcal{C})$ from $\mathcal{A}$ and let $(b_i)_{i<\beta}$ be a maximal Morley sequence in $t^g(a/\mathcal{C})$ from $\mathcal{B}$. The heart of this argument is to prove the following claim.

6.6.1 Claim. *$\mathcal{A}$ is ω-primary over $\mathcal{C} \cup \{a_i|\ i < \alpha\}$ (and similarly for $\mathcal{B}$).*

Before proving this claim, let us see why proving it is enough. By Exercises 4.3 (i) and 5.8 (ii), $|\alpha| = |\beta| = \kappa$. So by re-indexing, we may assume that $\alpha = \beta = \kappa$. Let $f : \mathcal{C} \cup \{a_i|\ i < \kappa\} \to \mathcal{C} \cup \{b_i|\ i < \kappa\}$ be such that $f \restriction \mathcal{C} = id_{\mathcal{C}}$ and for all $i < \kappa$, $f(a_i) = b_i$. Then f is strong by Exercise 4.7 (ii). By Theorem 5.9, f extends to an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Proof of Claim 6.6.1. As above, we may assume that $\alpha = \kappa$. Now suppose that the claim is not true and let $\mathcal{D} \subseteq \mathcal{A}$ be an ω-primary model over $\mathcal{C} \cup \{a_i|\ i < \kappa\}$. Then $\mathcal{A} \neq \mathcal{D}$ and we can pick an element $d \in \mathcal{A} - \mathcal{D}$. Choose also a finite $D_0 \subseteq \mathcal{D}$ such that $d \downarrow^s_{D_0} \mathcal{A}$. Let $((c_i, C_i))_{i<\kappa}$ be an ω-construction of $\mathcal{D}$ over $\mathcal{C} \cup \{a_i|\ i < \kappa\}$ (we can choose the construction so that the length is κ by Exercise 5.6 (iii) although this is not important in this proof). Then there is finite closed $X \subseteq \kappa$ such that $D_0 \subseteq \mathcal{C} \cup \{a_i|\ i < \kappa\} \cup \{c_i|\ i \in X\}$. By re-indexing and re-ordering we may assume that

(*) $D_0 \subseteq \mathcal{C} \cup \{a_i|\ i < \omega\} \cup \{c_i|\ i \in X\}$, $X \subseteq \omega$ and for all $i \in X$, $C_i \subseteq \mathcal{C} \cup \{a_i|\ i < \omega\} \cup \{c_j|\ j < i\}$.

Thus we can find an ω-primary model $\mathcal{D}_1$ over $\mathcal{C} \cup \{a_i|\ i < \omega_1\}$ such that $D_0 \subseteq \mathcal{D}_1$.

6.6.2 Exercise. *Show that the ω-construction of $\mathcal{D}_1$ is also an ω-construction over $\mathcal{C} \cup \{a_i|\ i < \kappa\}$.*

By Exercise 6.6.2 we can continue the construction of $\mathcal{D}_1$ to a construction of an ω-primary model over $\mathcal{C} \cup \{a_i|\ i < \kappa\}$. By the uniqueness of such, we can assume that $\mathcal{D}$ is ω-primary over $\mathcal{D}_1 \cup \{a_i|\ i < \kappa\}$. Let $\mathcal{E} \subseteq \mathcal{A}$ be ω-primary over $\mathcal{D}_1 d$. By the choice of D_0, $d \downarrow_{\mathcal{D}_1} \mathcal{D}$ and so by applying Exercise 5.10, $\mathcal{E} \downarrow_{\mathcal{D}_1} \mathcal{D}$. Also if $e \in \mathcal{E} - \mathcal{D}$ and $t^g(e/\mathcal{C}) = t^g(a/\mathcal{C})$, then since $t^g(a/\mathcal{C})$ is minimal and $t^g(e/\mathcal{D}_1)$ is not bounded, $e \downarrow_{\mathcal{C}} \mathcal{D}_1$. It follows that

(**) if $e \in \mathcal{E}$ is such that $t^g(e/\mathcal{C}) = t^g(a/\mathcal{C})$, then $e \in \mathcal{D}_1$

since otherwise $e \downarrow_{\mathcal{C}} \{a_i|\ i < \kappa\}$ which would contradict the maximality of $(a_i)_{i<\kappa}$.

As above, by massaging the construction of $\mathcal{D}_1$ and re-indexing $\{a_i|\ i < \omega_1\}$ we can assume that (*) above holds for the construction of $\mathcal{D}_1$ and some finite closed $X \subseteq \omega_1$ and so we can find ω-primary $\mathcal{D}_0$ over $\mathcal{C} \cup \{a_i|\ i < \omega\}$ so that $D_0 \subseteq \mathcal{D}_0$, $\mathcal{D}_1$ is ω-primary over $\mathcal{D}_0 \cup \{a_i|\ i < \omega_1\}$ and the ω-construction of $\mathcal{D}_0$ over $\mathcal{C} \cup \{a_i|\ i < \omega\}$ is also an ω-construction over $\mathcal{C} \cup \{a_i|\ i < \omega_1\}$.

By recursion on $i \leq \omega_1$ we define models $\mathcal{E}_i$ of power ω_1 and elements d_i (only for $i < \omega_1$) so that $\mathcal{E}_0 = \mathcal{D}_1$, $\mathcal{E}_{i+1}$ is ω-primary over $\mathcal{E}_i d_i$, $d_i \downarrow_{D_0} \mathcal{E}_i$ and $t^g(d_i/\mathcal{D}_0) = t^g(d/\mathcal{D}_0)$: We let $d_0 = d$ and $\mathcal{E}_1 = \mathcal{E}$. If i is limit, $\mathcal{E}_i = \cup_{j<i}\mathcal{E}_j$. If $\mathcal{E}_i$ is defined, by ω_1-categoricity and Exercise 4.3 (iii), there is $f \in Aut(M/\mathcal{D}_0)$ such that $f(\mathcal{E}_i) = \mathcal{D}_1$. We let $\mathcal{E}_{i+1} = f^{-1}(\mathcal{E})$ and $d_i = f^{-1}(d)$. Now for all $i \leq \omega_1$ we define ω-saturated models $\mathcal{E}^*_i$ as follows: $\mathcal{E}^*_0 = \mathcal{D}_0$, $\mathcal{E}^*_{i+1} \subseteq \mathcal{E}_{i+1}$ is ω-primary over $\mathcal{E}^*_i d_i$ and for limit i, $\mathcal{E}^*_i = \cup_{j<i}\mathcal{E}^*_j$.

6.6.3 Exercise. *Show that if* $e \in \mathcal{E}^*_{i+1}$ *and* $t^g(e/\mathcal{C}) = t^g(a/\mathcal{C})$, *then* $e \in \mathcal{E}^*_i$. *Hint: Start by showing that* $\mathcal{E}^*_{i+1} \downarrow_{\mathcal{E}^*_i} \mathcal{E}_i$ *and use (**) above.*

But now by Exercise 6.6.3, if $e \in \mathcal{E}^*_{\omega_1}$ is such that $t^g(e/\mathcal{C}) = t^g(a/\mathcal{C})$, then $e \in \mathcal{D}_0$. This contradicts ω_1-saturation of $\mathcal{E}^*_{\omega_1}$, e.g. it contradicts Exercises 4.7 (iii) and 5.10 (ii). $\square$ Claim 6.6.1.

$\square$

In the proof of Claim 6.6.1 above we proved a bit more than what we claimed:

6.7 Corollary. *If* $\mathcal{A}$ *is uncountable* $\mathcal{C} \subseteq \mathcal{A}$ *is countable and* ω*-saturated,* $t^g(a/\mathcal{C})$ *is minimal,* $(a_i)_{i<\alpha}$ *is a maximal Morley sequence in* $t^g(a/\mathcal{C})$ *from* $\mathcal{A}$ *and* $\mathcal{B} \subseteq \mathcal{A}$ *is* ω*-primary over* $\mathcal{C} \cup \{a_i \mid i < \alpha\}$, *then* $\mathcal{B} = \mathcal{A}$. $\square$

6.8 Exercise.

(i) If $\mathcal{A}$ *is uncountable* $\mathcal{C} \subseteq \mathcal{A}$ *is countable and* ω*-saturated,* $t^g(a/\mathcal{C})$ *is minimal,* $(a_i)_{i<\alpha}$ *is a maximal Morley sequence in* $t^g(a/\mathcal{C})$ *from* $\mathcal{A}$ *and* $\mathcal{B} \subseteq \mathcal{A}$ *contains* $\mathcal{C} \cup \{a_i \mid i < \alpha\}$, *then* $\mathcal{B} = \mathcal{A}$.

(ii) Show that every uncountable K*-model* $\mathcal{A}$ *is saturated. Hint: Start by showing that it is enough to prove this in the case* $|\mathcal{A}|$ *is a successor cardinal.*

7. Geometries with application

Throughout this section we assume that K has AP, JEP, arbitrarily large models, $LS(K) = \omega$ and K is ω_1-categorical and universal:

7.1 Definition. *We say that* K *is universal if for all* $A \subseteq \mathcal{A}$, *(*$SH(A) \in K$ *and)* $SH(A) \preceq \mathcal{A}$.

Notice that universality implies that $\preceq$ is the submodel relation.

7.2 Exercise. *Show that* K *is homogeneous.*

We will show that that K is essentially a class of vector spaces (or trivial structures). For a more general result, see [HK].

The argument is based on a study of geometries:

7.3 Definition. *Let* X *be any set and* $cl : P(X) \to P(X)$, *where* $P(X)$ *is the set of all subsets of* X. *We say that* (X, cl) *is a pregeometry (a.k.a. matroid) if the following holds for all* $A \subseteq B \subseteq X$ *and elements* $a, b \in X$:

(i) $A \subseteq cl(A) \subseteq cl(B) = cl(cl(B))$,

(ii) if $a \in cl(A)$, *then there is finite* $C \subseteq A$ *such that* $a \in cl(C)$,

(iii) (Steinitz exchange principle) if $a \in cl(Ab) - cl(A)$, *then* $b \in cl(Aa)$.

7.4 Exercise. *Suppose* (X, cl) *is a pregeometry and* $Y \subseteq X$. *For all* $A \subseteq X$, *let* $cl_Y(A) = cl(YA)$. *Show that* (X, cl_Y) *is a pregeometry.*

7.5 Definition. *Suppose* (X, cl) *is a pregeometry and* $A = \{a_i \mid i < \alpha\} \subseteq X$. *We say that* A *is* cl*-independent if for all* $i < \alpha$, $a_i \notin cl(\{a_j \mid j < \alpha,\ j \neq i\}$. *We say that* A *is maximal* cl*-independent set from* B *if it is* cl*-independent,* $A \subseteq B$ *and for all* $a \in B - A$, $a \in cl(A)$.

7.6 Theorem. *Suppose (X, cl) is a pregeometry. If A and A' are maximal cl-independent sets from $B \subseteq X$, then $|A| = |A'|$.*

Proof. We skip the proof in this course. The proof can be found from the lecture notes of the course Model theory. □

7.7 Definition. *Suppose (X, cl) is a pregeometry and $A, B, C \subseteq X$.*

(i) We write $dim(A/B) = |D|$ if D is a maximal cl_B-independent set from A. We write $dim(A)$ for $dim(A/\emptyset)$.

(ii) We write $A \downarrow^{cl}_C B$ if for all finite $A' \subseteq A$ $dim(A'/CB) = dim(A'/C)$.

(iii) We say (X, cl) is modular, if for all $A, B \subseteq X$, $A \downarrow^{cl}_{cl(A) \cap cl(B)} B$.

(iv) We say that (X, cl) is locally modular if $(X, cl_{\{a\}})$ is modular for some $a \in X$.

(v) We say that (X, cl) is trivial (a.k.a. disintegrated) if for all (non-empty) $A \subseteq X$, $cl(A) = \cup_{a \in A} cl(\{a\})$ (so a ranges over the elements of A).

7.8 Exercise. *Suppose that (X, cl) is a pregeometry and $A, B, C \subseteq X$.*

(i) Show that $dim(AB/C) = dim(B/C) + dim(A/BC)$.

(ii) Show that $dim(A/B) = dim(A/cl(B)) = dim(cl_B(A)/B)$.

(iii) Show that if (X, cl) is modular then so is (X, cl_Y) for any $Y \subseteq X$.

(iv) Show that trivial pregeometries are modular.

From now on we fix a countable ω-saturated $\mathcal{A}$ and an element a such that $t^g(a/\mathcal{A})$ is minimal. We need to a bit careful on how we choose these: We pick first any countable ω-saturated $\mathcal{A}^*$ and an element $a \notin \mathcal{A}^*$ so that $t^g(a/\mathcal{A}^*)$ is minimal. Then we choose a Morley sequence $(a_i)_{i<\omega}$ in $t^g(a/\mathcal{A}^*)$ so that $a \downarrow^s_{\mathcal{A}^*} \{a_i |\ i < \omega\}$. Then we let $\mathcal{A}$ be ω-primary over $\mathcal{A}^* \cup \{a_i |\ i < \omega\}$. Notice that $t^g(a/\mathcal{A})$ is still minimal and that $\mathcal{A}$ and $\mathcal{A}^*$ are isomorphic.

The goal is to show that every uncountable K-model $\mathcal{B} \supseteq \mathcal{A}$ is essentially a vector space. From now on, when we talk about K-models $\mathcal{B}$, unless we state otherwise, we mean that is uncountable and $\mathcal{A} \subseteq \mathcal{B}$ (since every uncountable $\mathcal{B}$ contains a copy of $\mathcal{A}$ this is without loss of generality). We write $D_{\mathcal{B}}$ for the set $\{b \in \mathcal{B} |\ t^g(b/\mathcal{A}) = t^g(a/\mathcal{A})\}$.

7.9 Definition. *For $X \subseteq D_{\mathcal{B}}$, by $cl_{\mathcal{B}}(X)$ we mean the set of all $b \in D_{\mathcal{B}}$ such that $t^g(b/\mathcal{A}X)$ is bounded.*

7.10 Exercise. *Show that for an element $a \in D_{\mathcal{B}}$ and $A \subseteq D_{\mathcal{B}}$, $a \in cl_{\mathcal{B}}(A)$ iff $a \not\downarrow^s_{\mathcal{A}} A$.*

7.11 Lemma. *$(D_{\mathcal{B}}, cl_{\mathcal{B}})$ is a pregeometry.*

Proof. We prove Steinitz exchange principle, the rest is left as an exercise. So suppose $a \in cl_{\mathcal{B}}(Ab) - cl_{\mathcal{B}}(A)$ and for a contradiction suppose that $b \notin cl_{\mathcal{B}}(Aa)$. W.o.l.g. A is finite (exercise). Let c list the elements of A. Then by Exercise 7.10, $b \downarrow^s_{\mathcal{A}} ca$ and $a \downarrow^s_{\mathcal{A}} c$. From the first of these it follows by monotonicity that $b \downarrow^s_{\mathcal{A}c} a$ and from the second it follows by symmetry that $c \downarrow^s_{\mathcal{A}} a$. By Exercise 4.5 (ii), $cb \downarrow^s_{\mathcal{A}} a$ and so by symmetry $a \downarrow^s_{\mathcal{A}} cb$. By Exercise 7.10, $a \notin cl_{\mathcal{B}}(Ab)$, a contradiction. □

7.12 Exercise.

(i) Suppose $\mathcal{A}_0$ *is countable and* ω*-saturated,* $(a_i)_{i<\alpha}$ *is a Morley sequence in* $t^g(a_0/\mathcal{A}_0)$ *and* $\alpha \geq \omega$*. Show that* $SH(\mathcal{A}_0 \cup \{a_i|\ i<\alpha\})$ *is* ω*-saturated. Conclude that for all* $A \subseteq D_{\mathcal{B}}$*,* $SH(\mathcal{A}A)$ *is* ω*-saturated and thus also* ω*-primary over* $\mathcal{A}A$*. Conclude also that if* $(a_i)_{i<\alpha}$ *is a maximal Morley sequence in* $t^g(a/\mathcal{A})$ *from* $\mathcal{B}$*, then* $\mathcal{B} = SH(\mathcal{A} \cup \{a_i|\ i<\alpha\})$*. Hint: Notice that for uncountable* α *the* ω*-saturation is clear and keep in mind our choice of* $\mathcal{A}$ *and* a*.*

(ii) Show that for all $A \subseteq D_{\mathcal{B}}$*,* $cl_{\mathcal{B}}(A) = SH(\mathcal{A} \cup A) \cap D_{\mathcal{B}}$*.*

(iii) Show that if $(D_{\mathcal{B}}, cl_{\mathcal{B}})$ *is locally modular, then it is modular. Hint: Use our choice of* $\mathcal{A}$ *and* a*.*

7.13 Definition. *Let* G *be a permutation group of a set* X *(i.e. a subgroup of* $Sym(X)$*). By* G_Y*,* $Y \subseteq X$ *we mean the set of all* $g \in G$ *such that* $g(x) = x$ *for all* $x \in Y$*. We write* $[Y]$ *for the set of all* $x \in X$ *such that* $g(x) = x$ *for all* $g \in G_Y$*. We say that* (G, X) *is quasi-Urbanic if for all finite* $Y \subseteq X$ *and elements* $x, y \in X - [Y]$*, there is* $g \in G_Y$ *such that* $g(x) = y$*.*

7.14 Exercise. *Let* $G = \{g \restriction D_{\mathcal{B}}|\ g \in Aut(\mathcal{B}/\mathcal{A})\}$*. Show that* $(G, D_{\mathcal{B}})$ *is quasi-Urbanic and that* $[A] = cl_{\mathcal{B}}(A)$ *for all* $A \subseteq D_{\mathcal{B}}$*.*

7.15 Theorem. *One of the following holds:*

(i) $(D_{\mathcal{B}}, cl_{\mathcal{B}})$ *is trivial.*

(ii) There are a vector space V*, a subspace* $W \subseteq V$ *and a bijection* $f : D_{\mathcal{B}} \to V - W$ *such that for all* $A \subseteq D_{\mathcal{B}}$*,* $f(cl_{\mathcal{B}}(A)) = span(W \cup f(A)) - W$*.*

Proof. This follows immediately from [Zi] Theorem B and Exercises 7.14 and 7.12 (iii) (and the fact that $cl_{\mathcal{B}}(\emptyset) = \emptyset$). In this course we skip the proof of Theorem B from [Zi]. □

So in particular, $(D_{\mathcal{B}}, cl_{\mathcal{B}})$ is modular. We will use this to coordinatize all elements of $\mathcal{B}$ using elements of $D_{\mathcal{B}}$. Notice also that W in Theorem 7.15 (ii) and the field over which V is a vector space must be of size $\leq \omega$ (exercise).

7.16 Lemma. *For all* $a \in \mathcal{B}$*, there is finite dimensional* $X_a \subseteq D_{\mathcal{B}}$ *such that* $X_a = cl_{\mathcal{B}}(X_a)$ *and the following holds:*

(i) If $f \in Aut(\mathcal{B}/\mathcal{A}a)$*, then* $f(X_a) = X_a$*.*

(ii) $a \in SH(\mathcal{A}X_a)$*.*

Proof. Let W be the set of finite dimensional $X \subseteq D_{\mathcal{B}}$ such that $cl_{\mathcal{B}}(X) = X$ and $a \in SH(\mathcal{A}X)$. Since $\mathcal{B} = SH(\mathcal{A}D_{\mathcal{B}})$, it is easy to see that $W \neq \emptyset$. Thus it is enough to show that W is closed under intersections (since then X_a is the $\subseteq$-least element of W). So suppose $X, Y \in W$. Clearly $cl_{\mathcal{B}}(X \cap Y) = X \cap Y$. Denote $Z = X \cap Y$. We want to show that $a \in SH(\mathcal{A}Z)$.

Let $\mathcal{C} = SH(\mathcal{A}Z)$. Then $\mathcal{C}$ is countable and ω-saturated by Exercise 7.12 (i). Also pick $d = (d_0, ..., d_n) \in X - Z$ such that $cl_{\mathcal{B}}(Zd) = X$ $e = (e_0, ..., e_m) \in Y - Z$ such that $cl_{\mathcal{B}}(Ze) = Y$. We can choose these so that both sequences are $(cl_{\mathcal{B}})_Z$-independent. Then $\mathcal{D} = SH(\mathcal{A}X) = SH(\mathcal{C}d)$ is ω-primary over $\mathcal{C}d$ and similarly

$\mathcal{E} = SH(\mathcal{A}Y)$ is ω-primary over $\mathcal{C}e$, again by Exercise 7.12 (i). By modularity, $d \downarrow^{cl}_{Z} e$ and so $d \downarrow^{s}_{\mathcal{C}} e$ (exercise). So $\mathcal{D} \cap \mathcal{E} = \mathcal{C}$ by Exercise 5.10 (i) and thus $a \in \mathcal{C}$. $\square$

7.17 Definition. *Let $a = (a_0, ..., a_n) \in \mathcal{B}$ and $B = (b_0, ..., b_m) \in \mathcal{B}$. We say that a is quantifier free definable from b over $\mathcal{A}$ if for all $i \leq n$, there is a quantifier free formula ϕ_i and $c_i \in \mathcal{A}$ such that for all $d \in \mathcal{B}$, $\mathcal{B} \models \phi_i(d, b, c_i)$ iff $d = a_i$. We say that a and b are quantifier free inter-definable over $\mathcal{A}$ if a is quantifier free definable from b over $\mathcal{A}$ and b is quantifier free definable from a over $\mathcal{A}$.*

7.18 Theorem. *For all elements $a \in \mathcal{B}$ there is $b = (b_0, ..., b_n) \in D_{\mathcal{B}}$ such that a and b are quantifier free inter-definable over $\mathcal{A}$.*

Proof. X_a be as in Lemma 7.16 and $b = (b_0, ..., b_n) \in X_a$ be such that $cl_{\mathcal{B}}(b) = X_a$. Then a is quantifier free definable from b over $\mathcal{A}$ by Lemma 7.16 (ii). For the other direction it is enough to show that $X_a \subseteq SH(\mathcal{A}a)$. If not then one can easily contradict Lemma 7.16 (i) (exercise, keep in mind that $SH(\mathcal{A}a)$ is ω-saturated). $\square$

7.19 Exercise. *Find K that satisfies the assumptions of this section, countable ω-saturated K-model $\mathcal{A}$ and an element a such that $t^g(a/\mathcal{A})$ is not minimal (and thus the coordinatization is not trivial).*

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