AXIOMATIC SET THEORY

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In this introduction to set theory we will concentrate, in addition to the basic theory of ordinals and cardinals, to the theory of constructible hierarchy L and to the theory of forcing. Both of these are techniques for showing consistency results i.e. that some claim is consistent with the theory ZFC of sets. In both cases we are more interested in how to apply the techniques than all the details in the development of the theories and thus we occasionally skip some proofs. Most of the skipped proofs can be found from K. Kunen's excellent book [Ku] which uses the approach to our topics mostly used also in these notes.

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1. Preliminaries

Set theory ZFC is a formal theory in the first-order language in the vocabulary $\{\in\}$, where $\in$ is a binary relation symbol (i.e. a predicate). However, for an obvious reason, we will not give formal proofs, instead our approach is semantic i.e. we pretend that we have a model V of ZFC and then work inside it using natural language and assume that the reader knows that these proofs can be translated to the proofs in the first-order logic. In fact, we will use natural language as much as possible (there are cases in which first-order formulas give the most convenient way of expressing claims). However, there are moments in which we can not avoid the use of the formal language and then we simply return to it. Also when we study forcing, we really need V and this is a potential problem, since even ZFC, and thus current mathematics, can not prove the existence of V (by Gödel's second incompleteness theorem). Similarly, for partially ordered sets P, we will assume the existence of so called P-generic filter G over V. And in general ZFC can not prove the existence of G either. We will address these problems when convenient, not necessarily as soon as they arise.

Elements of V are called sets and subsets of V, which are first-order definable with parameters, are called classes. If $\phi(v_0, ..., v_n)$ is a first-order $\{\in\}$-formula and $a_1, .., a_n$ are sets, then the expression $\phi(v_0, a_1, ..., a_n)$ is called a property and for a set x, we write $\phi(x, a_1, ..., a_n)$ (in the beginning of these notes, later we need to be more specific) if, using the notation from the course Mathematical Logic,

$$V \models_{s(x/0)(a_1/1),...,(a_n/n)} \phi$$

and say that x has the property $\phi(v_0, a_1, ..., a_n)$. Often we do not mention the parameters and just say that x has the property ϕ and write just $\phi(x)$. So the classes are families of all sets that have some fixed property ϕ.

1.1 Axioms

We start by giving the axioms of ZFC.

I Extensionality: If sets a and b have the same elements, then $a = b$.

Notice, that also the inverse of the implication in Extensionality holds (by identity axioms). And that from now on to determine a set, it is enough to describe its elements, e.g. $\{3, 8, i\}$, $\{n \in \mathbb{N} |\ n \text{ is even}\}$,..., and of course $\emptyset$. Also we extend the idea in Extensionality to classes i.e. two classes are considered the same if they have the same elements and a class and a set are considered the same if they have the same elements.

1.1.1 Exercise. *Show that every set is a class.*

II Foundation: Every non-empty set a has an $\in$-minimal element i.e. there is $x \in a$ such that for all $y \in a$, $y \notin x$.

III Pairing: For all sets a and b, (the class) $\{a, b\}$ is a set (i.e. there is a set x such that for all y, $y \in x$ iff $y = a$ or $y = b$).

Notice, that from Pairing it follows that for every set a, $\{a\}$ is a set.

1.1.2 Exercise. *Show that there is no set a such that $a \in a$ or sets a and b such that $a \in b \in a$.*

IV Separation (aka Comprehension): If a is a set and ϕ is a property, then $\{x \in a |\ \phi(x)\}$ is a set.

Notice that from Separation it follows that for all sets a and b, $a \cap b$ is a set, since $a \cap b = \{x \in a |\ x \in b\}$.

V Union: For every set a, the union $\cup a$ of the elements of a is a set ($x \in \cup a$ if $x \in b$ for some $b \in a$).

We will write $a \cup b$ for $\cup\{a, b\}$.

1.1.3 Exercise.
(i) Show that if a, b, c, d and e are sets, then $\{a, b, c, d, e\}$ is a set.
(ii) We write (a, b) for the set $\{a, \{a, b\}\}$. Show that
(a) (a, b) is indeed a set,
(b) if $(a, b) = (c, d)$, then $a = c$ and $b = d$.

VI Power Set: For every set a, the power set $P(a)$ of a is a set ($x \in P(a)$ if $x \subseteq a$ i.e. for every set y, if $y \in x$, then $y \in a$).

So far we have had no axiom that states that there exists even a single set. The next axiom says that there is an infinite set. However, it seems to assume that the empty set already exists. So should we not have an axiom that says this? There is no need for this: Even without any assumptions, in first-order logic one can always prove that there exists x such that $x = x$. So in the case of set theory, one can always prove the existence of at least one set.

1.1.4 Exercise.
(i) Show that the empty set $\emptyset$ exists.
(ii) For sets a and b, show that $a \times b = \{(x, y) |\ x \in a,\ y \in b\}$ is a set.

VII Infinity: There exists an inductive set i.e. a set a such that $\emptyset \in a$ and if $x \in a$, then also $x \cup \{x\} \in a$ (exercise: show that $x \cup \{x\}$ is a set).

When we talk about functions f from a set a to a set b, we always mean that $f = \{(x, f(x)) |\ x \in a\}$ is a set i.e. $f \in V$. We talk also about class functions:

1.1.5 Definition. *Let C be a class. We say that a function $F : C \to V$ is a class function if the graph of F is a class i.e. there is a property ϕ such that for all sets x, x has the property ϕ iff $x = (a, F(a))$ for some set $a \in C$.*

Notice the following: For all classes C and formulas $\phi(v_0, ..., v_n)$ there is a formula $\psi(v_1, ..., v_n)$ such that for all $a_1, ..., a_n$, $\phi(v_0, a_1, ..., a_n)$ defines a class function from C to V iff $\psi(a_1, ..., a_n)$.

VIII Replacement: If a is a set and $F : a \to V$ is a class function, then $\{F(x)|\ x \in a\}$ is a set.

1.1.6 Exercise.
(i) Show that if a is a set and $F : a \to V$ is a class function, then F is a function i.e. $\{(b, F(b))|\ b \in a\}$ is a set.
(ii) Show that if a and b are sets and $f : a \to b$ is a function, then it is a class function.
(iii) Suppose a is an inductive set. Show that there are no class functions $f : a \to V$ such that for all $x \in a$, $f(x \cup \{x\}) \in f(x)$.

IX Choice: If a is a set and every $x \in a$ is non-empty, then there is a function $f : a \to \cup a$ such that for all $x \in a$, $f(x) \in x$.

The theory that consists of all these axioms is called ZFC. If the Choice is left out, the resulting theory is called just ZF. Unless we state otherwise, we work in ZFC.

1.2 Recursive definitions

Notice that Choice is not used in this subsection.

1.2.1 Definition.
(i) If C is a class, then a class $<$ is called a partial ordering of C if the elements of $<$ are of the form (x, y), $x, y \in C$, and the following holds: if $(x, y) \in <$, then $(y, x) \notin <$ and if $(x, y), (y, z) \in <$, then $(x, z) \in <$. Instead of writing $(x, y) \in <$, we will simply write $x < y$.
(ii) A partial ordering $<$ is a linear ordering, if in addition, for all $x, y \in C$, $x < y$ or $x = y$ or $y < x$.
(iii) A partial ordering is well-founded if for all $x \in C$, $\{y \in C|\ y < x\}$ is a set and if a is a non-empty set such that every element of it belongs to C, then a has a $<$-minimal element. If in addition the partial ordering is a linear ordering, it is called a well-ordering.

If $<$ is a partial ordering of C, then by $\leq$ we mean the relation $a \leq b$ if $a < b$ or $a = b$.

1.2.2 Theorem. *Suppose C is a class and $<$ is a well-founded partial ordering of C. Let ϕ be a property and assume that for all $x \in C$, if every element of $\{y \in C |\ y < x\}$ has the property ϕ, then also x has it. Then every element of C has the property ϕ.*

Proof. Suppose not. Let $x \in C$ be such. We show first that we can choose x so that it is $<$-minimal element of C among those that do not have the property ϕ: If x is not such then the class a of all element of C which are smaller than x and do not have the property ϕ is non-empty and a set. Since $<$ is well founded, a has a $<$-minimal element. Clearly this is as wanted.

But if x is a minimal among those that do not have the property ϕ, then every element of $\{y \in C |\ y < x\}$ has the property, and so also x has it, a contradiction. □

1.2.3 Theorem. *Suppose C is a class, $<$ is a well-founded partial ordering of C and $G : V^2 \to V$ is a class function, where V^2 is the class of all pairs of elements of V. Then there is a unique class function $F : C \to V$ such that for all $x \in C$, $F(x) = G(F \restriction C_x, x)$, where $C_x = \{y \in C |\ y < x\}$. In particular, if $G : V \to V$ is a class function, then there is a unique class function $F : C \to V$ such that for all $x \in C$, $F(x) = G(F \restriction C_x)$.*

Proof. We say that $A \subseteq C$ is downward closed if $x < y \in A$ implies $x \in A$. We start with an exercise:

1.2.3.1 Exercise. *Suppose that a set $A \subseteq C$ is downward closed and $f, g : A \to V$ (recall Exercise 1.1.6) are such that for all $z \in A$, $f(z) = G(f \restriction C_z, z)$ and $g(z) = G(g \restriction C_z, z)$ (notice that $C_z \subseteq A$). Show that $f = g$. Conclude that if F exists, it is unique.*

Now let ϕ be the following property of sets a: a is of the form (x, y) where $x \in C$ and y is such that there is a function $f_x : C_x \to V$ such that $y = G(f_x, x)$ and for all $z \in C_x$, $f_x(z) = G(f_x \restriction C_z, z)$. We will show that for every $x \in C$, there is a set y such that (x, y) has the property ϕ. Then since by Exercise 1.2.3.1, such y is unique (since f_x is unique), ϕ defines a class function $C \to V$.

To see that y exists, it is enough to show that f_x exists. We prove this by induction i.e. by using Theorem 1.2.2. So suppose that the claim holds for every $z \in C_x$. We notice

(*) if $z, w \in C$ and f_z and f_w exist, then $f_z \restriction (C_z \cap C_w)$ and $f_w \restriction (C_z \cap C_w)$ satisfy the requirements of Exercise 1.2.3.1 for $A = (C_z \cap C_w)$ and thus $f_z \restriction (C_z \cap C_w) = f_w \restriction (C_z \cap C_w)$.

So by (*), if C_x does not have maximal elements ($z \in C_x$ is maximal if there are no $y \in C_x$ such that $z < y$) $f_x = \bigcup_{z \in C_x} f_z$ is as wanted. (Notice that we use replacement axiom here.) On the other hand, if C_x has maximal elements, we simply let $f_x = (\bigcup_{z \in C_x} f_z) \cup \{(z, G(f_z, z)) |\ z \in C_x \textit{ is maximal}\}$. Again by (*), f_x is as wanted.

So we are left to prove that for all $x \in C$, $F(x) = G(F \upharpoonright C_x, x)$. By Theorem 1.2.2 it is enough to prove this under the assumption that this holds for all $z \in C_x$. Let f_x be as in the definition of ϕ. Then by Exercise 1.2.3.1, $F \upharpoonright C_x = f_x$. Thus $F(x) = G(f_x, x) = G(F \upharpoonright C_x, x)$. □

1.2.4 Exercise.

(i) Suppose C is a set and $<$ is a partial ordering of C. Show that $<$ and C_x are sets for all $x \in C$.

(ii) Show that $\{(z, G(f_z, z))|\ z \in C_x \text{ is maximal}\}$ from the proof of Theorem 1.2.3, is a set.

1.3 Ordinals

1.3.1 Definition.

(i) We say that a set a is transitive if $x \in y \in a$ implies $x \in a$ (i.e. $\cup a \subseteq a$ and notice that if a and b are transitive, then so is $a \cap b$).

(ii) We say that a set α is an ordinal if it is transitive and linearly ordered by $\in$. For ordinals α and β, one usually writes $\alpha < \beta$ instead of $\alpha \in \beta$ and $\alpha \leq \beta$ for $\alpha < \beta$ or $\alpha = \beta$.

(iii) The class of all ordinals is denoted by On.

1.3.2 Exercise.

(i) Show that ordinals are well-ordered by $\in$.

(ii) Show that $0 = \emptyset$ is an ordinal.

(iii) Show that if α is an ordinal, then also $\alpha + 1 = \alpha \cup \{\alpha\}$ is an ordinal.

(iv) Show that if a is a set of ordinals and for all $\alpha, \beta \in a$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$, then $\cup a$ is an ordinal.

(v) Show that if α is an ordinal and $\beta \in \alpha$, then β is an ordinal.

(vi) Show that if α and β are ordinals, then so is $\alpha \cap \beta$.

1.3.3 Lemma. *Let α and β be ordinals.*

(i) If $\alpha \subseteq \beta$, then either $\alpha = \beta$ or $\alpha \in \beta$.

(ii) Either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proof. (i): Suppose $\alpha \neq \beta$. Then $\beta - \alpha$ is not empty and thus it has the least element γ. If $\delta \in \gamma$, then $\delta \in \beta$ and so by the choice of γ, $\delta \in \alpha$. On the other hand, if $\delta \in \alpha$, then $\gamma \not\leq \delta$, because otherwise $\gamma \in \alpha$ and this is against our choice of γ. Thus since $\in$ linearly orders β, $\delta \in \gamma$. It follows that $\alpha = \gamma$ and so $\alpha \in \beta$.

(ii): Now by Exercise 1.3.2 (vi), $\gamma = \alpha \cap \beta$ is an ordinal. Then $\gamma = \alpha$ or $\gamma = \beta$ because otherwise by (i), $\gamma \in \alpha \cap \beta = \gamma$. In the first case $\alpha \subseteq \beta$ and in the other case $\beta \subseteq \alpha$. □

1.3.4 Exercise.

(i) Show that On is well-ordered by $\in$.

(ii) Show that $\alpha + 1$ is the least ordinal strictly greater than the ordinal α.

(iii) For a set a of ordinals show that $\cup a$ is the supremum of a (in particular, $\cup a$ is an ordinal).

1.3.5 Definition.

(i) We say that an ordinal α is a successor ordinal if $\alpha = \beta + 1$ for some ordinal β and otherwise α is called a limit ordinal. However, usually 0 is not considered a limit ordinal. (Obviously, it is not a successor ordinal either.)

(ii) By ω we denote the least limit ordinal $\neq 0$ (if such ordinal exists).

1.3.6 Lemma. *For every ordinal β there is a limit ordinal $\alpha > \beta$.*

Proof. We show first that ω exists. By Infinity, there is an inductive set b. Let $a = b \cap On$ and $\alpha = \cup a$. By Exercise 1.3.4 (iii), α is an ordinal. Also it is easy to see that a is inductive and thus α can not be a successor ordinal. So in particular ω exists.

Now for given ordinal β, choose a function $f : \omega \to On$ so that $f(0) = \beta$ and for successor ordinals $\gamma + 1 \in \omega$, $f(\gamma + 1) = f(\gamma) + 1$ (exercise: show that f exists and $rng(f) \subseteq On$, keep in mind that every ordinal in ω excluding 0, is a successor ordinal). Let $\alpha = \cup rng(f)$. Clearly α is as wanted. □

1.3.7 Exercise.

(i) Show that ω is the $\subseteq$-least inductive set (i.e. that it is an inductive set and if also a is an inductive set, then $\omega \subseteq a$). Conclude that there is no class function $f : \omega \to V$ such that for all $n \in \omega$, $f(n+1) \in f(n)$.

(ii) Suppose C is a class and $<$ is a partial ordering of C such that for all $x \in C$, $C_x = \{y \in C \mid y < x\}$ is a set. Show that $<$ is not well-founded iff there is $f : \omega \to C$ such that for all $\alpha \in \omega$, $f(\alpha + 1) < f(\alpha)$.

1.3.8 Theorem. *For every set a there are an ordinal α and a one-to-one and onto function $f : \alpha \to a$.*

Proof. Let b be the set of all non-empty subsets of a and g be the choice function for b. We define a class function $G : V \to V$ so that for all ordinals β and functions $h : \beta \to a$ with $rng(h) \neq a$, $G(h) = g(a - rng(h))$ and for all other sets x, $G(x) = a$. Let $F : On \to V$ be such that for all ordinals γ, $F(\gamma) = G(F \restriction \gamma)$ (by Theorem 1.2.3) and suppose that for some ordinal γ, $F(\gamma) = a$. Then by letting α be the least such ordinal, α and $f = F \restriction \alpha$ are clearly as wanted.

So it is enough to show that for some γ, $F(\gamma) = a$. Suppose not. Then (by Separation) F^{-1} is a class function from a subset of a onto On. Thus by Replacement On is a set. Thus $\beta = \cup On$ is an ordinal. So $\beta \in \beta + 1 \in On$ and thus $\beta \in \beta$, a contradiction. □

1.3.9 Exercise. *(Zermelo's well-ordering theorem) Every set can be well-ordered.*

In fact, under e.g. ZF, Zermelo's well-ordering theorem is equivalent with Choice: To get Choice, simply choose a well-ordering $<$ for $\cup a$ and then for every $x \in a$, let $f(x)$ be the $<$-least element of x. There are a lot of claims that in ZF are equivalent with Choice, e.g. Zorn's lemma and the claim that every vector space has a basis.

The sets V_α in the next exercise form so called cumulative hierarchy.

1.3.10 Exercise. *We define* V_α *for all ordinals* α *as follows:* $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$, *and for limit ordinals* α, $V_\alpha = \cup_{\gamma<\alpha} V_\gamma$. *Show that*

(i) $\alpha \mapsto V_\alpha$ *is a class functions,*

(ii) for $\gamma < \alpha$, $V_\gamma \subseteq V_\alpha$,

(iii) for all sets a *there is an ordinal* α *such that* $a \in V_\alpha$,

(iv) for all α, V_α *is transitive,*

(v) for all α, $V_\alpha \cap On = \alpha$.

1.3.11 Exercise. *For all* x *let* $TC(x)$ *be the class of those* y *for which there are* $0 < n < \omega$ *and* x_i, $i \leq n$, *such that* $x_0 = y$, $x_n = x$ *and for all* $i < n$, $x_i \in x_{i+1}$ *(i.e.* $TC(x)$ *is the least transitive set* a *such that* $x \subseteq a$*).*

(i) Show that for all x, $TC(x)$ *is a set.*

(ii) Show that the relation $x \in TC(y)$ *is a well-founded partial ordering of* V.

1.3.12 Exercise. *Prove Exercise 1.3.10 (iii) by induction on the relation* $x \in TC(y)$.

1.3.13 Exercise. *Show that a set is an ordinal iff it is a transitive set of transitive sets. Hint: Show first that if* a *is a transitive set of transitive sets, then every* $b \in a$ *is a transitive set of transitive sets.*

1.3.14 Exercise. *Show that* $a \in V_\omega$ *iff* $TC(a)$ *is finite.*

1.3.15 Exercise. *Suppose* C *is a set and* $<$ *is a well-founded partial order of* C. *Show that there is a well-ordering* $<^*$ *of* C *such that for all* $x, y \in C$, $x < y$ *implies* $x <^* y$. *Hint: start by looking function* $ht : C \to On$ *such that* $ht(x) = 0$ *if* x *is* $<$*-minimal and otherwise* $ht(x) = \cup\{ht(y) + 1 |\ y < x\}$ *(ht=height).*

1.4 Cardinals

1.4.1 Definition. *We say that sets* a *and* b *have the same cardinality, if there is a one-to-one and onto function* $f : a \to b$.

1.4.2 Exercise.

(i) Show that the equicardinality relation from Definition 1.4.1 is an equivalence relation.

(ii) Show that if there is an onto function $f : a \to b$, *then there is a one-to-one function* $g : b \to a$ *and vice versa assuming that* $b \neq \emptyset$.

1.4.3 Theorem. *(Cantor-Bernstein) For all sets* a *and* b, *if there are one-to-one functions* $f : a \to b$ *and* $g : b \to a$, *then* a *and* b *have the same cardinality.*

Proof. For all $n \in \omega$, we define sets A_n and B_n as follows: $A_0 = a$, $B_0 = b$, $A_{n+1} = g(f(A_n))$ and $B_{n+1} = f(g(B_n))$. Finally, let $A = \bigcap_{n<\omega} A_n$ and $B = \bigcap_{n<\omega} B_n$. Clearly, for $n < \omega$, $A_{n+1} \subseteq A_n$ and $B_{n+1} \subseteq B_n$. Also (e.g. draw a picture) $f \restriction (A_n - g(B_n))$ is one-to-one function from $A_n - g(B_n)$ onto $f(A_n) - B_{n+1}$, $g^{-1} \restriction (g(B_n) - A_{n+1})$ is one-to-one function from $g(B_n) - A_{n+1}$ onto $B_n - f(A_n)$ and $f \restriction A$ is one-to-one function from A onto B. By putting these together, the required one-to-one and onto function is found. □

1.4.4 Definition.

(i) We say that an ordinal α is a cardinal if there are no $\beta < \alpha$ and a one-to-one function from α to β.

(ii) We say that a set a is finite, if for all one-to-one functions $f : a \to a$, $rng(f) = a$.

1.4.5 Lemma. *ω and every $n \in \omega$ are cardinals. In fact, every $n \in \omega$ is finite.*

Proof. We start by proving the claim for the elements of ω. Clearly it is enough to show that they are finite. We prove this by induction (i.e. using Theorem 1.2.2, keeping in mind that all elements of ω, excluding 0, are successor ordinals and, in fact, the claim we prove is that every ordinal α is either finite or $\geq \omega$).

For $n = 0$, this is clear. So suppose that this holds for n and let $f : n+1 \to n+1$ be one-to-one. For a contradiction suppose that $rng(f) \neq n+1$. By applying a transposition, we may assume that $n \notin rng(f)$. But then $f \restriction n$ is a one-to-one function from n to a proper subset of n, a contradiction.

If ω is not a cardinal, then there are $n \in \omega$ and a one-to-one function $f : \omega \to n$. But then $f \restriction n+1$ contradicts what we just proved. □

1.4.6 Exercise.

(i) Show that an ordinal α is finite iff $\alpha \in \omega$.

(ii) Show that all infinite cardinals are limit ordinals.

(iii) Show that if a is a set of cardinals, then $\cup a$ is a cardinal.

1.4.7 Lemma. *For every set a, there is a unique cardinal κ for which there is a one-to-one function from κ onto a.*

Proof. Clearly there cannot be more than one such cardinal. So we prove just the existence: Let κ be the least ordinal such that there is a one-to-one function f from κ onto a (such κ exists by Theorem 1.3.8). It is enough to show that κ is a cardinal. If not, then there is $\alpha < \kappa$ and a one-to-one function $g : \kappa \to \alpha$. By Cantor-Bernstein, we can choose g so that it is also onto. But then α and $f \circ g^{-1}$ contradict the choice of κ. □

1.4.8 Definition. *Let a be a set. The unique cardinal κ for which there is a one-to-one function from κ onto a, is called the cardinality of a and is denoted by $|a|$. If the cardinality of a set is $\leq \omega$, we say that the set is countable.*

1.4.9 Exercise.

(i) Show that a set a is finite iff $|a| \in \omega$.

(ii) Show that $|a| \leq |b|$ iff there is a one-to-one function $f : a \to b$.

(iii) Show that if a and b are finite, then so are $a \cup b$ and $a \times b$.

The elements of ω are called natural numbers (i.e. this is how we interpret natural numbers in set theory) and thus ω is called also the set of natural numbers i.e. $\mathbb{N}$. We also write $0 = \emptyset$ as already mentioned and $1 = 0+1 = 0 \cup \{0\}$, $2 = 1+1$, $3 = 2+1$ etc. Recall that for all $n \in \omega$, $n = \{0, 1, ..., n-1\}$.

1.4.10 Theorem. *For all non-empty sets a and b, if one of them is infinite, then $|a \times b| = max\{|a|, |b|\}$.*

Proof. Clearly, it is enough to prove that for all infinite cardinals κ, $|\kappa \times \kappa| = \kappa$. For this it is enough to find a one-to-one function from $\kappa \times \kappa$ to κ. We order the elements of $On \times On$ so that $(\alpha, \beta) < (\gamma, \delta)$ if one of the following holds:
(i) $max\{\alpha, \beta\} < max\{\gamma, \delta\}$,
(ii) $\alpha < \gamma \leq max\{\alpha, \beta\} = max\{\gamma, \delta\}$,
(iii) $\alpha = max\{\alpha, \beta\} = max\{\gamma, \delta\} = \gamma$ and $\beta < \delta$.

1.4.10.1 Exercise. *Show that $<$ is a well-ordering of $On \times On$.*

Using Theorem 1.2.3, define $\Gamma : On \times On \to On$ so that for all $x \in On \times On$, $\Gamma(x)$ is the least ordinal (strictly) greater than every element in $rng(\Gamma \restriction (On \times On)_x)$ (for this notation, see Theorem 1.2.3).

1.4.10.2 Exercise. *Show that Γ is strictly increasing and that if $\Gamma(\alpha, \beta) = \gamma$ and $\gamma' < \gamma$, then there is $(\alpha', \beta') < (\alpha, \beta)$ such that $\Gamma(\alpha', \beta') = \gamma'$.*

By Exercise 1.4.10.2, it is enough to show that for infinite cardinals κ, $rng(\Gamma \restriction (\kappa \times \kappa)) \subseteq \kappa$. We do this by induction. The case when $\kappa = \omega$ is left as an exercise. So suppose $\kappa > \omega$. For a contradiction suppose that there are $\alpha, \beta < \kappa$ such that $\Gamma(\alpha, \beta) \geq \kappa$. Let $\lambda = max\{|\alpha|, |\beta|, \omega\} < \kappa$. Then by Exercise 1.4.10.2, $\Gamma^{-1} \restriction \kappa : \kappa \to (On \times On)_{(\alpha,\beta)}$ is one-to-one and by the induction assumption (from which it follows that if $|a|, |b| \leq \lambda$, then $|a \times b| \leq \lambda$), $|(On \times On)_{(\alpha,\beta)}| \leq |(max\{\alpha, \beta\} + 1) \times (max\{\alpha, \beta\} + 1)| \leq |\lambda \times \lambda| = \lambda < \kappa$, a contradiction. □

As a hint for the item (i) in next exercise we want to mention that the claim in the item can not be proved without Choice. If Choice is not assumed, it is possible that the set of reals is a countable union of countable sets and we will see later that the set of reals is not countable and this can be proved without Choice.

Also, instead of talking about functions $f : I \to X$ for some sets I and X, it is sometimes notationally convenient to talk about indexed sequences $(x_i)_{i \in I}$. So by an indexed sequence $(x_i)_{i \in I}$ we simply mean a function $f : I \to V$ such that for all $i \in I$, $f(i) = x_i$. Thus for $x : a \to V$, we sometimes also write x_i in place of $x(i)$.

1.4.11 Exercise.
(i) Suppose κ is an infinite cardinal and a is a set of cardinality $\leq \kappa$ such that also every element of it is of cardinality $\leq \kappa$. Show that $|\cup a| \leq \kappa$. In particular, for all sets a and b, if one of them is infinite, then $|a \cup b| = max\{|a|, |b|\}$.
(ii) For all infinite cardinals κ, show that there are sets $X_i \subseteq \kappa$, $i \in \kappa$, such that for all i, the cardinality of X_i is κ and for all $i \neq j$, $X_i \cap X_j = \emptyset$.
(iii) Show that the set of rational numbers is countable.

For sets a and b, by a^b we mean the set of all functions from b to a (e.g. $\mathbb{N}^n$). If $b = \beta$ is an ordinal we also write $a^{<\beta}$ for $\bigcup_{\alpha<\beta} a^\alpha$ and $a^{\leq\beta}$ for $\bigcup_{\alpha\leq\beta} a^\alpha$. On the level of notation, we also identify $f : 2 \to X$ with $(f(0), f(1))$ and thus think that $X \times X$ is the same as X^2, see the discussion on indexed sequences above.

1.4.12 Lemma. *For all cardinals* κ, $|P(\kappa)| = |2^\kappa|$ *and if* κ *is infinite, then* $|2^\kappa| = |(2^\kappa)^\kappa| = |2^{(\kappa\times\kappa)}| = |\kappa^\kappa|$.

Proof. For $|P(\kappa)| = |2^\kappa|$, just map every $a \subseteq \omega$ to its characteristic function. $|2^\kappa| = |2^{\kappa\times\kappa}|$ is clear by Lemma 1.4.10. To find a one-to-one function F from $2^{\kappa\times\kappa}$ onto $(2^\kappa)^\kappa$, simply for $\eta \in 2^{\kappa\times\kappa}$ let $\xi = F(\eta)$ be such that for all $n, m < \kappa$, $(\xi(n))(m) = \eta(n,m)$. Since $2^\kappa \subseteq \kappa^\kappa$, $|2^\kappa| \le |\kappa^\kappa|$. Finally since $\kappa \le |2^\kappa|$, it is easy to see that $|\kappa^\kappa| \le |(2^\kappa)^\kappa|$. □

One often denotes $|2^\kappa|$ by just 2^κ and similarly for κ^λ and $\kappa^{<\lambda}$. It is clear from the context which possibility we mean.

1.4.13 Theorem. *For all sets* a, $|P(a)| > |a|$.

Proof. Clearly it is enough to prove the claim in the cases when a is some cardinal κ, i.e. that $2^\kappa > \kappa$. For finite cardinals the claim is clear and so suppose κ is infinite. For a contradiction, suppose $2^\kappa \le \kappa$. Clearly, $2^\kappa \ge \kappa$ and thus, under the counter assumption, there is a one-to-one function f from κ onto 2^κ. Denote $f(\alpha)$ by ξ_α.

Let $g : \kappa \to 2$ be such that for all $\alpha < \kappa$, $g(\alpha) = 1 - \xi_\alpha(\alpha)$. Then $g \in 2^\kappa$ and so for some $\gamma < \kappa$, $g = \xi_\gamma$. Now $g(\gamma) = 1 - \xi_\gamma(\gamma) = 1 - g(\gamma)$, a contradiction. □

1.4.14 Definition. *Let* γ *be a limit ordinal.*

(i) The cofinality $cf(\gamma)$ *of* γ *is the least ordinal* α *such that there is a function* $f : \alpha \to \gamma$ *such that* $\cup rng(f) = \gamma$.

(ii) γ *is called regular if* $cf(\gamma) = \gamma$.

1.4.15 Exercise.

(i) Show that for all limit ordinals γ, $cf(\gamma)$ *is a regular cardinal. Conclude that regular ordinals are cardinals.*

(ii) Show that ω *is a regular cardinal.*

1.4.16 Definition. *If* κ *is a cardinal, then the least cardinal strictly greater than* κ *is denoted by* κ^+. *If* κ *is* λ^+ *for some cardinal* λ, *it is called a successor cardinal and otherwise it is a limit cardinal.*

1.4.17 Exercise.

(i) Show that for all ordinals α, *there is a cardinal* $\kappa > \alpha$.

(ii) Show that every infinite successor cardinal is regular.

We finish this section by defining a class function $\alpha \mapsto \omega_\alpha$ (sometimes ω_α is also denoted by $\aleph_\alpha$).

1.4.18 Definition. *We define* ω_α *for all ordinals* α *as follows:* $\omega_0 = \omega$, $\omega_{\alpha+1} = \omega_\alpha^+$ *and for limit ordinals* α, $\omega_\alpha = \cup_{\gamma<\alpha}\omega_\gamma$.

1.4.19 Exercise. *Show that for all infinite cardinals* κ, *there is* $\alpha \in On$ *such that* $\kappa = \omega_\alpha$.

1.5 Recursive definitions revisited

1.5.1 Definition. *Suppose X is a set.*

(i) Suppose α is an ordinal, $f : X^\alpha \to X$ is a function and $C \subseteq X$. We say that C is closed under f if for all $x \in C^\alpha$, $f(x) \in C$.

(ii) Suppose $Y \subseteq X$ and for all $i \in I$, α_i is an ordinal and $f_i : X^{\alpha_i} \to X$ is a function. Then by $C(Y, f_i)_{i\in I}$ we mean the $\subseteq$-least subset C of X such that it contains Y and is closed under every f_i, $i \in I$ (if such C exists).

1.5.2 Lemma. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). Then $C(Y, f_i)_{i\in I}$ exists.*

Proof. Just let $C(Y, f_i)_{i\in I}$ be the intersection of all sets $C \subseteq X$ which contain Y and are closed under every f_i (notice that X is such a set). □

1.5.3 Lemma. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). Suppose that ϕ is a property, every element of Y has it and for all $k \in I$ and $x \in C(Y, f_i)_{i\in I}^{\alpha_k}$ the following holds: If every x_j, $j < \alpha_k$, has the property, then also $f_k(x)$ has the property. Then every element of $C(Y, f_i)_{i\in I}$ has the property ϕ.*

Proof. Let C be the set of all elements of $C(Y, f_i)_{i\in I}$ that have the property ϕ. Then C contains Y and is closed under every f_i. Thus $C(Y, f_i)_{i\in I} \subseteq C$. □

1.5.4 Definition. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). For all ordinals α, we define $C_\alpha(Y, f_i)_{i\in I}$ as follows:*

(i) $C_0(Y, f_i)_{i\in I} = Y$,

(ii) $C_{\alpha+1}(Y, f_i)_{i\in I} = C_\alpha(Y, f_i)_{i\in I} \cup \{f_i(x) |\ i \in I,\ x \in (C_\alpha(Y, f_i)_{i\in I})^{\alpha_i}\}$,

(iii) if α is limit, then $C_\alpha(Y, f_i)_{i\in I} = \bigcup_{\beta<\alpha} C_\beta(Y, f_i)_{i\in I}$.

1.5.5 Exercise. *Show that $\alpha \mapsto C_\alpha(Y, f_i)_{i\in I}$ is a class function from On to $P(X)$ and that for all ordinals $\alpha < \beta$, $Y \subseteq C_\alpha(Y, f_i)_{i\in I} \subseteq C_\beta(Y, f_i)_{i\in I} \subseteq C(Y, f_i)_{i\in I}$.*

1.5.6 Lemma. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii) and κ be a regular cardinal. Suppose further that for all $i \in I$, $\alpha_i < \kappa$. Then $C(Y, f_i)_{i\in I} = C_\kappa(Y, f_i)_{i\in I}$.*

Proof. By Exercise 1.5.5, it is enough to show that $C_\kappa(Y, f_i)_{i\in I}$ is closed under every f_k, $k \in I$. For this let $x \in (C_\kappa(Y, f_i)_{i\in I})^{\alpha_k}$. Since κ is regular, there is $\gamma < \kappa$ such that $x \in (C_\gamma(Y, f_i)_{i\in I})^{\alpha_k}$ (Exercise, think of function $g : \alpha_k \to \kappa$ such that for all $\beta < \alpha_k$, $g(\beta)$ is the least ordinal δ for which $x_\beta \in C_\delta(Y, f_i)_{i\in I}$). But then $f_i(x) \in C_{\gamma+1}(Y, f_i)_{i\in I} \subseteq C_\kappa(Y, f_i)_{i\in I}$. □

2. Object theory

When one studies e.g. the theory of groups, one can use all the tools of ZFC in doing this i.e. one can use ZFC as a meta theory. However due to its foundational role, when one studies ZFC, it is the meta theory that is under study. So one has no tools to prove e.g. the existence of V unlike in the case of the theory of groups, using set theory one can construct all kinds of groups. However we still want, for technical reasons, to have both the meta theory, ZFC as introduced in Section 1, and the object theory as introduced in this section.

We start by introducing the first-order logic for the object theory. The formulas of this logic are like the Gödel numbers in the course 'Matemaattinen logiikka' but since we are working with set theory, we can choose the codes to be more formula-like than natural numbers, they will be functions from natural number to ω (in particular they are sets i.e. elements of V): First we choose codes for symbols of the first-order logic as follows: Code for (is 0, for) it is 1, for $\neg$ it is 2, for $\wedge$ it is 3, for $\exists$ it is 4, for $=$ it is 5, for $\in$ it is 6 and for v_i it is $7+i$. Then we define the set of (object) formulas as follows: ϕ is a formula if

(i) $dom(\phi)=3$ and $\phi(0),\phi(2)>6$ and $\phi(1)\in\{5,6\}$

or

(ii) there is formula ψ such that $\phi=\neg\psi$ (i.e. $dom(\phi)=dom(\psi)+1$, $\phi(0)=2$ and for all $i<dom(\psi)$, $\phi(i+1)=\psi(i)$),

or

(iii) there are formulas ψ and θ such that $\phi=(\psi\wedge\theta)$

or

(iv) there is a formula ψ and $i<\omega$ such that $\phi=\exists x_i\psi$.

By Lemma 1.5.2, this definition gives a set, in fact, a subset of V_ω. We will denote this set as $L_{\omega\omega}$.

The following remark is also a hint for Exercise 3.2.

2.1 Remark. *There are two technically convenient ways of defining $L_{\omega\omega}$ (exercise: Show that the two ways indeed define $L_{\omega\omega}$, $L_{\omega\omega}\subseteq V_\omega$ and $L_{\omega\omega}$ is first-order definable in the structure $(V_\omega,\in)$).*

The first one is the following: $\xi\in L_{\omega\omega}$ if there is $F: dom(\xi)+1\to P(\omega^{<\omega})$ such that $\xi\in F(dom(\xi))$ and

(a) $F(0)=F(1)=F(2)=\emptyset$,

(b) $\phi\in F(3)$ if (i) above holds,

(c) $\phi\in F(n+1)$, $3\le n\le dom(\xi)$, if $\phi\in F(n)$ or $dom(\phi)=n+1$ and one of (ii)-(iv) above holds with the additional requirement that $\psi,\theta\in F(n)$.

The second one is: $\xi\in L_{\omega\omega}$ if there is $F: dom(\xi)+1\to P(\omega^{<\omega})$ such that $\xi\in F(dom(\xi))$, for all $n\le dom(\xi)$, $F(n)$ is finite and

(a') $F(0)=F(1)=F(2)=\emptyset$,

(b') if $\phi\in F(3)$, then (i) above holds,

(c') if $\phi\in F(n+1)$, $3\le n<dom(\xi)$, then $\phi\in F(n)$ or $dom(\phi)=n+1$ and one of (ii)-(iv) above holds with the additional requirement that $\psi,\theta\in F(n)$.

The first of these is convenient e.g. when one defines the truth of the formulas of $L_{\omega\omega}$.

Notice also that for every formula ϕ on the meta level, there is a natural corresponding (i.e. 'Gödel number') $\phi^*\in L_{\omega\omega}$. E.g. if $\phi=\exists x_0 x_0=x_0$, then $dom(\phi^*)=5$, $\phi^*(0)=4$, $\phi^*(1)=7$, $\phi^*(2)=7$, $\phi^*(3)=5$ and $\phi^*(4)=7$. We will refer to ϕ^* as a code for ϕ or as the Gödel number of ϕ although ϕ^* is not a number. Notice also that the other direction may fail i.e. that there may be formulas $\phi \in L_{\omega\omega}$ such that ϕ is not a code of any formula from the meta level (under the assumption that is needed to make sense to this claim i.e. that V actually exists).

Now we can continue as in the course 'Matemaattinen logiikka': We can write the definition of ZFC as a formula on the meta level and it defines a subset of $L_{\omega\omega}$ which is our object ZFC, which we will denote ZFC^*. Notice that if ϕ is an axiom of ZFC, then it's Gödel number ϕ^* belongs to the object theory. We can also write the definition of being provable from ZFC as a formula on the meta level and this gives the notion of being provable on the object level. Notice that if $ZFC \vdash \phi$ on the meta level, the same is true on the object level i.e. $ZFC \vdash \phi$ implies $ZFC \vdash$ "$ZFC^* \vdash \phi^*$". (However, again assuming V exists, V may contain proofs that do not correspond any proofs on the meta level. E.g. V may think that ZFC^* is contradictory, while the existence of V guarantees that ZFC is not contradictory.)

Suppose M is a non-empty class. We can think M as a model in the vocabulary $\{\in\}$ by interpreting $\in$ as the membership relation of V restricted to M. We will denote this model as $(M,\in)$ and call it an $\in$-model. If M is a proper class, then by Tarski's theorem, the truth in $(M,\in)$ need not be definable but if M is a set, it is: Just write the usual Tarski's truth definition as a formula $\Theta(x,y,z)$ on the meta level. (Using e.g. Lemma 1.5.6 or one can first define a function $F_M : \omega \to V$ so that $F_M(0) = F_M(1) = F_M(2) = \emptyset$ and for $n+1 > 2$, $(\phi,a) \in F_M(n+1)$ if $(\phi,a) \in F_M(n)$ or $dom(\phi) = n+1$ and e.g. if ϕ is of the form $\exists v_i\psi$, then $(\phi,a) \in F_M(n+1)$ iff there is $b \in M^k$, $k < \omega$, such that $a \restriction (dom(a) - \{i\}) \subseteq b$ and $(\psi,b) \in F_M(n)$. Then $\Theta(M,\phi,a)$ says that there is F_M as above such that $(\phi,a) \in F_M(dom(\phi))$. Notice that we can do the same as in Remark 2.1, i.e. also for truth, it is enough to have F_M only upto $dom(\phi)+1$.) For an $\in$-model M, $\phi(x) \in L_{\omega\omega}$, $x = (x_0,...,x_n)$, and $a = (a_0,...,a_n) \in M^n$ we write $(M,\in) \models \phi(a)$ for the formula $\Theta(M,\phi,a)$.

On the meta level, there is another way of talking about the truth in an $\in$-model and this time M may be a proper class: Let $\theta(x_0,b)$ define M. For each formula $\phi(x)$ on the meta level we define another formula $\phi^M(x) = \phi^M(x,b)$ as follows:

(i) If ϕ is atomic, then $\phi^M = \phi$,

(ii) if $\phi = \neg\psi$, then $\phi^M = \neg(\psi^M)$,

(iii) if $\phi = (\psi_0 \wedge \psi_1)$, then $\phi^M = (\psi_0^M \wedge \psi_1^M)$,

(iv) if $\phi = \exists x_i\psi$, then $\phi^M = \exists x_i(\theta(x_i,b) \wedge \psi^M)$.

If X is a class defined by ϕ, then by X^M we mean the class of all elements of M that satisfy ϕ^M (e.g. On^M). Notice that X^M is a class.

Notice also that $\phi \leftrightarrow \phi^V$.

2.2 Exercise. *Suppose that M is a non-empty set. Show that for all formulas $\phi(x)$, $x = (x_0,...,x_n)$, and $a \in M^{n+1}$, $\phi^M(a) \leftrightarrow M \models \phi^*(a)$, where $\phi^* \in L_{\omega\omega}$ is the Gödel number of ϕ.*

Foundational remark: Exercise 2.2 will not be needed in the form we stated it.

It is enough that we can prove the statement for each formula ϕ separately i.e. we do not need any induction priciples when we work with the finite sequences of symbols called first-order formulas.

2.3 Exercise. *Let M be a non-empty set and F a collection of formulas closed under subformulas. Suppose that for all $\psi(x) = \exists x_i \phi(x_i, x) \in F$ and $a \in M^n$, if $\psi(a)$ holds in V then there is $b \in M$ such that $\phi(b,a)$ holds in V. Show that for all $\phi(x) \in F$ and $a \in M^n$, $\phi(a) \leftrightarrow M \models \phi^*(a)$, where ϕ^* is the Gödel number of ϕ.*

2.4 Exercise. *Suppose F is a finite subset of ZFC. Show that there is $\alpha \in On$ such that $V_\alpha \models \phi^*$ for all $\phi \in F$.*

2.5 Exercise. *ZFC proves compactness theorem for $L_{\omega\omega}$, in particular, if for all finite $F \subseteq ZFC^*$, F has a model, then ZFC^* has a model. Also by Gödel's incompleteness theorem, ZFC does not prove the existence of a model of ZFC^*. Why these do not contradict Exercise 2.4?*

2.6 Exercise. *Suppose $M \in V$ is transitive (i.e. $x \in M$ implies $x \subseteq M$) and non-empty.*
(i) $On^M = M \cap On \in On$.
(ii) If $V_\omega \subseteq M$, then $V_n^M = V_n$ for all $n < \omega$.
(iii) If $\omega \in M$, then $\omega^M = \omega$.

From now on, formulas on the meta level are called just formulas and if the formula ϕ is on the object level we point this out by writing $\phi \in L_{\omega\omega}$ (or sometimes talk about Gödel numbers).

3. Constructible hierarchy

In this section we construct constructible hierarchy, originally due to K. Gödel, and prove the basic properties of it. In the text books L is usually constructed using Gödel functions but we will use more intuitive notion of being definable by a formula from $L_{\omega\omega}$ (this is common in the literature in general). It is easy to see that the two approached give the same L.

3.1 Definition.
(i) For all $\in$-models $M \in V$, we let $Def(M)$ be the set of all $X \subseteq M$ for which there are $\phi(x,y) \in L_{\omega\omega}$ and $a \in M^n$ such that $X = \{b \in M \mid M \models \phi(b,a)\}$.
(ii) We let $L_0 = \emptyset$, $L_1 = \{\emptyset\}$, $L_{\alpha+1} = L_\alpha \cup Def(L_\alpha)$ (for $\alpha \geq 1$) and for limit γ, $L_\gamma = \cup_{\alpha<\gamma} L_\alpha$.
(iii) We let $L = \cup_{\alpha \in On} L_\alpha$.

3.2 Exercise.
(i) Show that $F_L : On \to V$, $F_L(\alpha) = L_\alpha$ is a class function.
(ii) Show that L is a class.
(iii) Show that for all $\alpha \in On$, L_α is a transitive set (i.e. $x \in L_\alpha$ implies $x \subseteq L_\alpha$). Conclude that L is transitive and that $L_{\alpha+1} = Def(L_\alpha)$ for all $\alpha > 0$.
(iv) Show that for $\alpha < \beta$, $L_\alpha \in L_\beta$.

(v) *Show that* $L_\alpha \cap On = \alpha = On^{L_\alpha}$ *and for all* $\alpha \leq \omega$, $L_\alpha = V_\alpha$.

(vi) *Show that* $L_{\omega\omega} \in L_{\omega+1}$ *and* $L_{\omega\omega} = (L_{\omega\omega})^{L_\alpha}$ *for all* $\alpha > \omega$.

(vii) *Show that if* $\alpha > \omega$ *is a limit ordinal,* $M \in L_\alpha$ *is an* $\in$*-model,* $\phi(x) \in L_{\omega\omega}$ *and* $a \in M^n$, *then* $M \models \phi(a)$ *iff* $(M \models \phi(a))^L$ *iff* $(M \models \phi(a))^{L_\alpha}$.

(viii) *Show that there is a formula* $\phi(x,y)$ *such that the following holds:*

(a) *If* $\alpha > \omega$ *is a limit ordinal,* $\beta < \alpha$ *and* $F_L \restriction \beta \in L_\alpha$, *then for all* $a \in L_\alpha$, $L_\alpha \models \phi^*(a,\beta)$ *iff* $a = F_L \restriction \beta$ *(here* ϕ^* *is the Gödel number of* ϕ *and for* F_L, *see (i) above).*

(b) *If* β *is an ordinal and* $F_L \restriction \beta \in L$, *then for all* $a \in L$, $\phi^L(a,\beta)$ *iff* $a = F_L \restriction \beta$.

(ix) *Suppose* $\alpha > \omega$ *is a limit ordinal. Show that for all* $\beta < \alpha$, $F_L \restriction \beta \in L_\alpha$. *Conclude that for all limit ordinals* $\alpha > \omega$ *and* $\beta < \alpha$, $(L_\beta)^{L_\alpha} = L_\beta$ *(for* F_L, *see (i) above). Hint: Prove the claim by induction on* α *and for each* α, *by induction on* β. *For limit* β, *use induction assumption for* α *and for successor* $\beta = \gamma + 1$, *show that* $F_L \restriction \gamma \cup \{(\gamma, L_\gamma)\}$ *is as needed.*

(x) *Show that for all finite sets* F *of formulas and* $\alpha \in On$, *there is a limit ordinal* $\beta > \alpha$ *such that for all* $\phi(x) \in F$ *and* $a \in L_\beta^n$, $\phi^L(a)$ *holds iff* $L_\beta \models \phi^*(a)$ *(i.e.* ϕ^{L_β} *holds). Hint: See Exercises 2.3 and 2.4.*

The foundational remark from the previous section applies also to the following theorem.

3.3 Theorem. *For all axioms* ϕ *of* ZF, ϕ^L *holds.*

Proof. We prove this for the separation axiom, the rest are straight forward (exercise). Let $\psi(x,y)$ be a formula, $X \in L$ and $a \in L^n$. We need to show that the set $Y = \{b \in X \mid \psi^L(b,a)\}$ belongs to L. By Exercise 3.2 (x), there is β such that $X \in L_\beta$ and $Y = \{b \in X \mid L_\beta \models \psi^*(b,a)\}$ and thus $Y \in L_{\beta+1} \subseteq L$. □

To show that ϕ^L holds when ϕ is the choice, additional work is needed.

3.4 Definition.

(i) *Let* $<^*$ *be the lexicografical ordering of* $L_{\omega\omega}$ *i.e. for* $f : n \to \omega$ *and* $g : m \to \omega$, $f <^* g$ *if* $n < m$ *or* $n = m$ *and there is* $x < n$ *such that* $f(x) \neq g(x)$ *and for the least such* x, $f(x) < g(x)$. *(Exercise: This is a well-ordering of* $L_{\omega\omega}$.*)*

(ii) *For* $X \in L$, *let* $rk(X)$ *be the least ordinal* α *such that* $X \in L_\alpha$ *(notice that* $rk(X)$ *is a successor ordinal) and* $fm(X)$ *be the* $<^*$*-least formula* $\phi(x,y) \in L_{\omega\omega}$ *such that for some* $a \in L_{rk(x)-1}$, $X = \{b \in L_{rk(x)-1} \mid L_{rk(x)-1} \models \phi(b,a)\}$.

(iii) *We define a binary relation* $<_L$ *on* L *as follows: for* $X, Y \in L$, $X <_L Y$ *if one of the following holds:*

(a) $rk(X) < rk(Y)$,

(b) $rk(X) = rk(Y)$ *and* $fm(X) <^* fm(Y)$,

(c) $rk(X) = rk(Y)$ *and* $fm(X) = fm(Y) = \phi(x,y)$ *and there is* $a \in L^n_{rk(X)-1}$ $(y = (y_1, .., y_n))$ *such that* $X = \{b \in L_{rk(X)-1} \mid L_{rk(X)-1} \models \phi(b,a)\}$ *and for all* $a' \in L^n_{rk(X)-1}$, *if* $Y = \{b \in L_{rk(X)-1} \mid L_{rk(X)-1} \models \phi(b,a')\}$, *then* a *is smaller than* a' *in the lexicografical ordering of* $L^{<\omega}_{rk(X)-1}$ *that one gets from* $<_L$ *restricted to* $L_{rk(X)-1}$.

Notice that the property above is indeed a property i.e. expessable in the first-order logic and thus $<_L$ is a class.

We define $F_{lim} : On \to On$ so that for all $\alpha \in On$, $F_{lim}(\alpha)$ is the least (infinite) limit ordinal $> F_{lim}(\beta)$ for all $\beta < \alpha$.

3.5 Exercise.

(i) Show that $<^*$ *is a well-ordering of* $L_{\omega\omega}$.

(ii) Show that $<_L$ *is a well-ordering of* L *(in particular that it is a class).*

(iii) Show that if $F_{lim}(\alpha) = \alpha$ *and* $a, b \in L_\alpha$, $a <_L b$ *iff* $(a <_L b)^L$ *iff* $(a <_L b)^{L_\alpha}$. *Hint: See Exercises 3.2 (viii) and (ix) and prove that for all* $\alpha \in On$, $F_< \upharpoonright \alpha \in L_{F_{lim}(\alpha+1)}$, *where* $F_<(\beta) = <_L \upharpoonright L_\beta$ *for all* $\beta \in On$.

(iv) Show that for all cardinals $\kappa > \omega$, $F_{lim}(\kappa) = \kappa$ *and that for all* $\alpha < \kappa$ *there is* $\alpha < \beta < \kappa$ *such that* $F_{lim}(\beta) = \beta$.

3.6 Theorem. *Let* ϕ *be the axiom of choice. Then* ϕ^L *holds.*

Proof. Suppose that $X \in L$ is a set of non-empty sets. Let Y be the set of all $(x, y) \in X \times \cup X$ such that y is the $<_L$-least element of x. By Exercise 3.5, if κ is an infinite cardinal such that $X \in L_\kappa$, $Y \in L_{\kappa+1} \subseteq L$. Clearly Y is a choice function. ▫

By $V = L$ we mean the axiom $\forall x \exists y (y \in On \wedge x \in (F_L \upharpoonright (y+1))(y))$.

3.7 Corollary.

(i) $(V = L)^L$ *holds.*

(ii) If $\alpha > \omega$ *is a limit ordinal, then* $L_\alpha \models (V = L)^*$.

Proof. Immediate by Exercises 3.2 (viii) and (ix). ▫

For theories T and T' on the meta level, we write $Con(T)$ implies $Con(T')$ if the following holds: If there is a proof of contradiction from T', then there is a proof of contradiction also from T.

3.8 Theorem. $Con(ZFC)$ *implies* $Con(ZFC + V = L)$.

Proof. Suppose that there is a proof of contradiction from $ZFC + V = L$. Then there is a finite subset T of $ZFC + V = L$ from which the contradiction can be proved. Let $T^* = \{\phi^* | \ \phi \in T\}$. By Exercise 3.2 (x) and Corollary 3.7 (ii), there is a limit ordinal α such that $L_\alpha \models T^*$. Since ZFC proves soundness ('eheyslause' in the course Matemaattinen logiikka), ZFC proves that there is an $\in$-model $M \in V$ for ϕ^* where ϕ is the contradiction, e.g. $\forall x(x = x) \wedge \neg \forall x(x = x)$ (as mentioned above, if $T \vdash \phi$, then $ZFC \vdash$ "$T^* \vdash \phi^*$"). Clearly, ZFC proves also that there is no $\in$-model $M \in V$ for ϕ^*. Thus ZFC proves a contradiction. ▫

From now on in this section, excluding Exercises 3.14 and 3.15, we assume $ZFC + V = L$. By GCH we mean the following axiom: For all infinite cardinals κ, $2^\kappa = \kappa^+$. We finish this section by showing that $ZFC + V = L \vdash GCH$ (and thus $Con(ZFC)$ implies $Con(ZFC + GCH)$).

By $ZF - P$ we mean ZF without the power set axiom.

3.9 Theorem.

(i) For all regular cardinals $\kappa > \omega$ *and* $\phi \in ZF - P + V = L$, $L_\kappa \models \phi^*$.

(ii) There is finite $T \subseteq ZF - P + V = L$ *such that if* $M \in V$ *is a transitive* $\in$*-model and for all* $\phi \in T$, $M \models \phi^*$, *then for some limit ordinal* $\alpha > \omega$, $M = L_\alpha$, *in fact, one can choose* $\alpha = M \cap On$.

Proof. (i) Just go carefully through the proofs of ϕ^L holds for all $\phi \in ZFC + V = L$. For Replacement, one needs to know that for all $x \in L_\alpha$, $\alpha \geq \omega$, $|x| \leq |\alpha|$. For this, see Exercise 3.10 below.

(ii) Just go through carefully the proof that for all limit ordinals $\alpha > \omega$, $L_\alpha \models (V = L)^*$ (i.e. show that $F_L^M(\alpha) = F_L(\alpha)$ for all $\alpha \in On^M$) and choose T so that $V = L \in T$. Notice that Separation for the formula that defines the truth covers most of the use of Separation and that Union and Pairing alone quarantee that $V_\omega \subseteq M$. □

3.10 Exercise.

(i) Show that for all infinite ordinals α, $|L_\alpha| = |\alpha|$.

(ii) Suppose ϕ *is an axiom of* ZFC *got from the axiom skeema of replacement and* $\kappa > \omega$ *is regular. Show that* $L_\kappa \models \phi^*$.

3.11 Definition. *Let* $M \in V$. *We define Mostowski collapse* $C_M : M \cup \{M\} \to V$ *by letting* $C_M(x) = \{C_M(y) |\ y \in x \cap M\}$.

We say that an $\in$-model M is extensional if it satisfies the axiom of extensionality i.e. for all $x, y \in M$, if $\{z \in M |\ z \in x\} = \{z \in M |\ z \in y\}$, then $x = y$.

3.12 Exercise.

(i) Show that C_M *exists and is unique. Hint: Find a suitable well-founded partial order on* $M \cup \{M\}$.

(ii)Show that $C_M(M)$ *is a transitive set and if* M *is extensional, then* $C_M \restriction M$ *is an isomorphism between* $\in$*-models* $(M, \in)$ *and* $(C_M(M), \in)$.

3.13 Theorem. $ZFC + V = L$ *proves* GCH.

Proof. Let κ be an infinite cardinal. By exercise 3.10, it is enough to prove that if $X \subseteq \kappa$, then $X \in L_{\kappa^+}$. Notice that since $V = L$, V and L have the same cardinals, in particular $(k^+)^L = \kappa^+$.

Since $X \in L$, there is a regular cardinal $\lambda > \kappa$ such that $X \in L_\lambda$. By the downwars Löwenhein-Skolem theorem, we can find $M \in V$ such that $\kappa + 1 \cup \{X\} \subseteq M$, $|M| = \kappa$ and $(M, \in)$ is an elementary submodel of $(L_\lambda, \in)$ (i.e. for all $\psi(z) \in L_{\omega\omega}$ and $c \in M^m$, $M \models \psi(c)$ iff $L_\lambda \models \psi(c)$ and we write $M \preceq L_\lambda$ for this). Notice that M is extensional. Then by Theorem 3.9 (i) $C_M(M)$ is a transitive model of T^*, where T is as in Theorem 3.9 (ii) and thus by Theorem 3.9 (ii), there is a limit ordinal α such that $C_M(M) = L_\alpha$. Since $\kappa + 1 \subseteq M$, $C_M(X) = X$. Thus $X \in L_\alpha$. Since $|M| = \kappa$, $\alpha < \kappa^+$. □

We say that a cardinal κ is weakly inaccessible if it is a regular limit cardinal $> \omega$. A weakly inaccessible κ is inaccessible if for all $\lambda < \kappa$, $2^\lambda < \kappa$.

3.14 Exercise.

(i) Show that $ZFC^* \in L_{\omega+1}$. *(For* ZFC^*, *see page 15.)*

(ii) Suppose $V = L$. *Show that if* κ *is weakly inaccessible, then it is inaccessible and* $V_\kappa = L_\kappa$.

3.15 Exercise. *Suppose that* $ZFC \not\vdash$ *"*ZFC^* *is consistent" (cf. Gödel's second incompleteness theorem).*

(i) Show that ZFC *does not prove the existence of an inaccessible cardinal. Hint: Show that* $V_\kappa \models ZFC^*$ *for inaccessible* κ.

(ii) Show that ZFC *does not prove the existence of a weakly inaccessible cardinal. Hint: Apply (i) to* $ZFC^L = \{\phi^L | \ \phi \in ZFC\}$ *and use Exercise 3.14 or prove directly that* $L_\kappa \models ZFC^*$ *for weakly inaccessible* κ.

4. Diamonds

In this section we study diamonds that give a systematic method of making good guesses and they exists in L. Throughout this section we assume that $V = L$.

We start by defining cub and stationary set. We will take a closer look at these in Section 8.

The following definitions are usually made only for (regular) cardinals, but we will need the definition of cub also for limit ordinals (if e.g. $cf(\alpha) = \omega$, the definition of a stationary set does not make much sense).

4.1 Definition. *Let* $\alpha > \omega$ *be a limit ordinal.*

(i) $C \subseteq \alpha$ *is called cub (in* α*) if it is unbounded in* α *(i.e. for all* $\gamma < \alpha$ *there is* $\beta \in C$ *such that* $\beta > \gamma$*) and for all* $\gamma < \alpha$, *if* $\cup(C \cap \gamma) = \gamma$, *then* $\gamma \in C$.

(ii) $S \subseteq \alpha$ *is stationary if for all cub* $C \subseteq \alpha$, $S \cap C \neq \emptyset$.

4.2 Definition.

(i) We define a class function $F_\diamond : On \to V$. *For all* α, $F_\diamond(\alpha)$ *is a pair* (X_α, C_α) *where* $X_\alpha, C_\alpha \subseteq \alpha$ *and* C_α *is cub if* α *is a limit ordinal. And then the exact values can be define recursively as follows: We let* $F_\diamond(\alpha) = (X_\alpha, C_\alpha)$ *be the* $<_L$*-least pair such that for all* $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ *if* α *is a limit ordinal and such a pair exists and otherwise we let* $F_\diamond(\alpha) = (\emptyset, \alpha)$.

(ii) We let $C_\diamond \subseteq On$ *be the class of all limit ordinals* α *such that for all* $\beta < \alpha$, $F_\diamond \restriction \beta \in L_\alpha$ *and* $F_{lim}(\alpha) = \alpha$.

4.3 Exercise.

(i) Show that $F_\diamond$ *is a class function.*

(ii) Show that for all regular cardinals $\kappa > \omega$, $C_\diamond \cap \kappa$ *is cub in* κ. *Hint: By induction on* $\alpha < \kappa$, *show that* $F_\diamond \restriction \alpha \in L_\beta$ *for some* $\beta < \kappa$ *and for limit cases apply (iii) below.*

(iii) Show that there is a formula $\phi(x, y)$ *such that for all limit ordinals* $\alpha > \omega$, *if* $F_{\lim}(\alpha) = \alpha$, *then the following holds: for all* $\beta < \alpha$ *and* $a \in L_\alpha$, *if* $F_\diamond \restriction \beta \in L_\alpha$, *then* $L_\alpha \models \phi^*(a, \beta)$ *iff* $a = F_\diamond \restriction \beta$.

For all regular cardinals $\kappa > \omega$, we write $\diamond_\kappa$ for the sequence $(X_\alpha|\ \alpha < \kappa)$, where X_α is such that $F_\diamond(\alpha) = (X_\alpha, C_\alpha)$ for some C_α.

4.4 Theorem. *Suppose $\kappa > \omega$ is a regular cardinal and $\diamond_\kappa = (X_\alpha)_{\alpha<\kappa}$. For all $X \subseteq \kappa$, the set $\{\alpha < \kappa|\ X \cap \alpha = X_\alpha\}$ is stationary.*

Proof. Suppose not. Then there is a pair (X, C) such that $X, C \subseteq \kappa$, C is cub and for all $\alpha \in C$, $X \cap \alpha \neq X_\alpha$. Choose these so that, in addition, (X, C) is the $<_L$-least such pair. Notice that $(X, C) \in L_{\kappa^+}$ and in L_{κ^+} the defining property of the pair (X, C) can be expressed by a formula $\Phi(X, C, \kappa)$ as follows (i.e. for all $C', X' \in L_{\kappa^+}$, $L_{\kappa^+} \models \Phi^*(X', C', \kappa)$ iff $X' = X$ and $C' = C$): "(X, C) is the $<_L$-least pair such that C is a cub in κ and for all $\beta \in C$ and all a, if $\phi(a, \beta + 1)$ holds and $a(\beta) = (X', C')$, then $X' \neq X \cap \beta$", where ϕ is as in Exercise 4.3 (iii). Notice also that being $<_L$-least is expressible in L_{κ^+} by a formula by Exercise 3.5 (iii).

Now choose an $\in$-model $M \preceq L_{\kappa^+}$ such that $M \cap \kappa = \alpha \in \kappa$, $\alpha > \omega$, $|M| = |\alpha|$, $X, C, C_\diamond \cap \kappa, \kappa \in M$ (exercise: show that M exists) and let γ be such that $C_M(M) = L_\gamma$. Then γ is a limit ordinal $> \omega$ and $F_{lim}(\gamma) = \gamma$, $C_M \restriction (M \cap \kappa) = id$, $C_M(\kappa) = \alpha$, $C_M(Y) = Y \cap \alpha$ for all $Y \in \{X, C, C_\diamond \cap \kappa\}$ and $C_M(Y)$ is a cub in α for all $Y \in \{C, C_\diamond \cap \kappa\}$ (exercise). In particular, $\cup(C \cap \alpha) = \cup(C_\diamond \cap \alpha) = \alpha$ and thus $\alpha \in C \cap C_\diamond$ (and so $F_\diamond \restriction \beta \in L_\gamma$ for all $\beta < \alpha$). Finally, $L_\gamma \models \Phi^*(X \cap \alpha, C \cap \alpha, \alpha)$ and thus by Exercises 4.3 (iii) and 3.5 (iii) (Exercise: show that $F_{lim}(\gamma) = \gamma$), $(X \cap \alpha, C \cap \alpha)$ is the $<_L$-least pair such that $C \cap \alpha$ is cub in α and for all $\beta \in C \cap \alpha$, $X \cap \beta \neq X_\beta$. So $F_\diamond(\alpha) = (X \cap \alpha, C \cap \alpha)$ i.e. $X_\alpha = X \cap \alpha$. Since $\alpha \in C$, we have a contradiction. □

We will give two examples of the use of diamonds. The first contains a part of the combinatorial core behind a theorem from generalized descriptive set theory (this idea is used also elsewhere) and the other is a simplified version of a result due to S. Shelah from the theory of abstract elementary classes.

Let $\kappa > \omega$ be a regular cardinal. We make 2^κ a topological space by letting open sets be all the unions of basic open sets $N_\eta = \{\xi \in 2^\kappa|\ \eta \subseteq \xi\}$, $\eta \in 2^{<\kappa}$.

We let E_{ns} be the equivalence relation on 2^κ for which $\eta E_{ns} \xi$ if the set $\{\alpha \in \kappa|\ \eta(\alpha) \neq \xi(\alpha)\}$ is not stationary (exercise: show that E_{ns} is an equivalence relation, see Exercise 4.10).

We have also another equivalence relation $E \subseteq (2^\kappa)^2$ of which we assume the following: for all $\alpha < \kappa$, there is an equivalence relation $E_\alpha \subseteq (2^\alpha)^2$ such that for all $\eta, \xi \in 2^\kappa$ the following holds:

(*) if $\eta E \xi$, then the set $\{\alpha < \kappa|\ (\eta \restriction \alpha) E_\alpha (\xi \restriction \alpha)\}$ contains a cub and if $(\eta, \xi) \notin E$, then the set $\{\alpha < \kappa|\ (\eta \restriction \alpha, \xi \restriction \alpha) \notin E_\alpha\}$ contains a cub.

4.5 Theorem. *Suppose E and E_{ns} are as above. Then there is a continuous function $F : 2^\kappa \to 2^\kappa$ such that for all $\eta, \xi \in 2^\kappa$, $\eta E \xi$ iff $F(\eta) E_{ns} F(\xi)$.*

Proof. Let $(X_\alpha|\ \alpha < \kappa)$ be $\diamond_\kappa$ and for all $\alpha < \kappa$, denote the characteristic function of X_α by f_α (i.e. $f_\alpha : \alpha \to 2$, $f_\alpha(\gamma) = 1$ if $\gamma \in X_\alpha$). Now we can define F: $F(\eta)(\alpha) = 1$ if $(\eta \restriction \alpha) E_\alpha f_\alpha$ (and otherwise $F(\eta)(\alpha) = 0$). Exercise: Show that F is continuous. We are left to prove that $\eta E \xi$ iff $F(\eta) E_{ns} F(\xi)$.

$\Rightarrow$: Suppose $\eta E\xi$. Then there is cub $C \subseteq \kappa$ such that for all $\alpha \in C$, $(\eta \restriction \alpha)E_\alpha(\xi \restriction \alpha)$. Thus for all $\alpha \in C$, $F(\eta)(\alpha) = F(\xi)(\alpha)$ and thus $F(\eta)E_{ns}F(\xi)$.

$\Leftarrow$: Suppose that η and ξ are not E-equivalent and we prove that $F(\eta)$ and $F(\xi)$ are not E_{ns}-equivalent. Now there is cub $C \subseteq \kappa$ such that for all $\alpha \in C$, $(\eta \restriction \alpha, \xi \restriction \alpha) \notin E_\alpha$. Let S be the stationary set $\{\alpha < \kappa |\ \eta \restriction \alpha = f_\alpha\}$. Then $S^* = S \cap C$ is stationary (exercise) and for all $\alpha \in S^*$, $F(\eta)(\alpha) = 1$ and $F(\xi)(\alpha) = 0$. Thus $F(\eta)$ and $F(\xi)$ are not E_{ns}-equivalent. □

For two structure (in the same vocabulary) $\mathcal{A}$ and $\mathcal{B}$, we say that a function $f : \mathcal{A} \to \mathcal{B}$ is an embedding if for all atomic formulas $\phi(x)$ and $a = (a_1, ..., a_n) \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$, where $f(a) = (f(a_1), ..., f(a_n))$. If identity function from $\mathcal{A}$ to $\mathcal{B}$ is an embedding, we say that $\mathcal{A}$ is a submodel of $\mathcal{B}$ and write $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}_i$, $i < \alpha$, are such that for all $j < i < \alpha$, $\mathcal{A}_j \subseteq \mathcal{A}_i$, then by $\cup_{i<\alpha}\mathcal{A}_i$ we mean the structure $\mathcal{A}$ such that the universe $dom(\mathcal{A})$ of $\mathcal{A}$ is $\cup_{i<\alpha} dom(\mathcal{A}_i)$, for all relation symbols R (from the vocabulary), the interpretation $R^{\mathcal{A}}$ of R in $\mathcal{A}$ is $\cup_{i<\alpha} R^{\mathcal{A}_i}$, for all function symbols f, $f^{\mathcal{A}} = \cup_{i<\alpha} f^{\mathcal{A}_i}$ and for all constant symbols c, $c^{\mathcal{A}} = c^{\mathcal{A}_0}$ ($= c^{\mathcal{A}_i}$ for any $i < \alpha$).

Now for the second example, let us fix a class K of structures in a countable vocabulary L. We assume that K has the following four properties:

(1) If $\mathcal{A} \in K$ and $\mathcal{B} \cong \mathcal{A}$ (i.e. $\mathcal{A}$ and $\mathcal{B}$ are isomorphic), then $\mathcal{B} \in K$.

(2) If $\mathcal{A}_i$, $i < \alpha$, are models from K and for all $i < j < \alpha$, $\mathcal{A}_i \subseteq \mathcal{A}_j$, then $\cup_{i<\alpha}\mathcal{A}_i \in K$.

(3) K is ω-categorical i.e. if $\mathcal{A}, \mathcal{B} \in K$ are countably infinite, then $\mathcal{A} \cong \mathcal{B}$.

(4) The countable models of K do not have the amalgamation property (AP) i.e. there are countably infinite $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ such that $\mathcal{A} \subseteq \mathcal{B}, \mathcal{C}$ but there is no countable $\mathcal{D} \in K$ and an embedding $f : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{B} \subseteq \mathcal{D}$ and $f \restriction \mathcal{A} = id$.

We start by an exercise that tells that with diamonds one can guess much more than just sets.

4.6 Exercise.

(i) Show that there is $((X_\alpha, g_\alpha, f_\alpha) |\ \alpha < \omega_1)$ *such that for all* $X \subseteq \omega_1$, $g \in 2^{\omega_1}$ *and* $f \in \omega_1^{\omega_1}$ *the set* $\{\alpha < \omega_1 |\ X_\alpha = X \cap \alpha,\ g_\alpha = g \restriction \alpha,\ f_\alpha = f \restriction \alpha\}$ *is stationary. Hint: Triples* (X, g, f) *can be coded as subsets of* ω_1.

(ii) Show that there are $((g_\alpha, f_\alpha) |\ \alpha < \omega_1)$ *and stationary sets* $S_i \subseteq \omega_1$, $i < \omega_1$, *such that for all* $i < j < \omega_1$, $S_i \cap S_j = \emptyset$, *and for all* $i < \omega_1$ *and* $g \in 2^{\omega_1}$ *and* $f \in \omega_1^{\omega_1}$ *the set* $\{\alpha \in S_i |\ g_\alpha = g \restriction \alpha,\ f_\alpha = f \restriction \alpha\}$ *is stationary. Hint: Use (i) and guess triples* $(\{i\}, g, f)$.

We will also need the following observation about K:

4.7 Exercise.

(i) Show that in (4) above, one can choose $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *so that in addition,* $\mathcal{B} - \mathcal{A}$ *and* $\mathcal{C} - \mathcal{A}$ *are infinite. Hint: Use (2) and (3).*

(ii) Let $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *be as in (4) above. Show that there are no countable* $\mathcal{D} \in \mathcal{K}$ *and an embedding* $f : \mathcal{B} \to \mathcal{D}$ *such that* $\mathcal{C} \subseteq \mathcal{D}$ *and* $f \restriction \mathcal{A} = id$.

Notice that the contraposition of the following theorem looks probably more interesting: If $\mathcal{K}$ satisfies (1)-(3) from above and has upto isomorphism $< 2^{\omega_1}$ many models of size ω_1, then the countably infinite models of $\mathcal{K}$ have AP.

4.8 Theorem. *There are $\mathcal{A}_i \in K$, $i < 2^{\omega_1}$, of power ω_1 such that for all $i < j < 2^{\omega_1}$, $\mathcal{A}_i \not\cong \mathcal{A}_j$.*

Proof. Let $((g_\alpha, f_\alpha)|\ \alpha < \omega_1)$ and $S_i \subseteq \omega_1$, $i < \omega_1$, be as in Exercise 4.6 (ii) (w.o.l.g. we may assume that for all $\alpha < \omega_1$, $dom(f_\alpha) = \alpha$ and $g_\alpha \in 2^\alpha$ since only such pairs guess functions $g \in 2^{\omega_1}$ and $f \in \omega_1^{\omega_1}$ at α) and let $Y = \{\eta \in 2^{<\omega_1}|\ dom(\eta) \geq \omega\}$. For all $\eta \in Y$, we define models $\mathcal{A}_\eta$ as follows (we will construct these so that if $dom(\eta)$ is a limit ordinal, then the universe of $\mathcal{A}_\eta$ is $dom(\eta)$):

(i) If $dom(\eta) = \omega$, then $\mathcal{A}_\eta$ is any model from K whose universe is ω.

(ii) If $dom(\eta) = \alpha + 1$ and α is a successor ordinal, then $\mathcal{A}_\eta = \mathcal{A}_{\eta\restriction\alpha}$.

(iii) If $dom(\eta)$ is a limit ordinal, then $\mathcal{A}_\eta = \cup_{\omega<\alpha<dom(\eta)}\mathcal{A}_{\eta\restriction\alpha}$ (exercise: show that the universe of $\mathcal{A}_\eta$ is $dom(\eta)$, see (iv) below).

(iv) If $dom(\eta) = \alpha + 1$ and α is a limit ordinal, then we actually do something: Let $\xi_i \in 2^{\alpha+1}$, $i < 2$, be such that $g_\alpha \subseteq \xi_i$ and $\xi_i(\alpha) = i$. Since $\mathcal{A}_{g_\alpha}$ is isomorphic with $\mathcal{A}$ from (4) above, we can find $\mathcal{A}_{\xi_0}$ and $\mathcal{A}_{\xi_1}$ from K so that they can not be amalgamated over $\mathcal{A}_{g_\alpha}$ and their universe is $\alpha+\omega$ (by Exercise 4.7 and $\alpha+\omega$ is the least limit ordinal $> \alpha$). Now if $\eta = \xi_i$ for some $i < 2$, $\mathcal{A}_\eta$ is defined. Otherwise, there are two possibilities:

(a) f_α is an isomorphism from $\mathcal{A}_{g_\alpha}$ to $\mathcal{A}_{\eta\restriction\alpha}$: Choose $\mathcal{A}_\eta$ so that there is an isomorphism $f : \mathcal{A}_{\xi_i} \to \mathcal{A}_\eta$ such that $f_\alpha \subseteq f$, where $i = \eta(\alpha)$ and the universe of $\mathcal{A}_\eta$ is $\alpha + \omega$.

(b) f_α is not an isomorphism from $\mathcal{A}_{g_\alpha}$ to $\mathcal{A}_{\eta\restriction\alpha}$: We let $\mathcal{A}_\eta$ be any model from K such that the universe of $\mathcal{A}_\eta$ is $\alpha + \omega$ and $\mathcal{A}_{\eta\restriction\alpha} \subseteq \mathcal{A}_\eta$.

Let $(Z_i)_{i<2^{\omega_1}}$ list the subsets of ω_1. For all $i < 2^{\omega_1}$, let $\eta_i \in 2^{\omega_1}$ be such that for all $\alpha < \omega_1$, $\eta_i(\alpha) = 1$ iff $\alpha \in S_j$ for some $j \in Z_i$. Finally, for $i < 2^{\omega_1}$, let $\mathcal{A}_i = \cup_{\omega\leq\alpha<\omega_1}\mathcal{A}_{\eta_i\restriction\alpha}$. We show that these are as wanted.

Clearly, for all $i < 2^{\omega_1}$, $\mathcal{A}_i \in K$ and $|\mathcal{A}_i| = \omega_1$. So it is enough to show that if $i, j < 2^{\omega_1}$ and $i \neq j$, $\mathcal{A}_i \not\cong \mathcal{A}_j$. For a contradiction, suppose that $f : \mathcal{A}_i \cong \mathcal{A}_j$ and by symmetry, w.o.l.g. we may assume that there is $k \in Z_j - Z_i$. Now there is a cub $C \subseteq \omega_1$ such that for all $\alpha \in C$, α is a limit ordinal and $f \restriction \alpha$ is a bijection from α to α (and thus $f \restriction \alpha$ is an isomorphism from $\mathcal{A}_{\eta_i\restriction\alpha}$ to $\mathcal{A}_{\eta_j\restriction\alpha}$). So there is $\alpha \in C \cap S_k$ such that $g_\alpha = \eta_i \restriction \alpha$ and $f_\alpha = f \restriction \alpha$. In addition, we can choose α so that $\eta_j \restriction \alpha \neq g_\alpha$ since $g_\alpha = \eta_i \restriction \alpha$ and we can choose α to be as large as we want (this is to avoid spliting the proof into two cases).

Let $\xi_n = (\eta_i \restriction \alpha) \cup \{(\alpha, n)\}$ for $n < 2$ and $\xi_* = (\eta_j \restriction \alpha) \cup \{(\alpha, 1)\}$. Notice that $\xi_0 \subseteq \eta_i$ and $\xi_* \subseteq \eta_j$ (since $\alpha \in S_k$ and $k \in Z_j - Z_i$). Then by (iv)(a) above, there is an isomorphism $g : \mathcal{A}_{\xi_1} \to \mathcal{A}_{\xi_*}$ such that $f \restriction \mathcal{A}_{g_\alpha} = f_\alpha \subseteq g$. Since $\mathcal{A}_{\xi_*}$ is countable, there is $\alpha + 1 < \gamma < \omega_1$ such that $f^{-1}(\mathcal{A}_{\xi_*}) \subseteq \mathcal{A}_{\eta_i\restriction\gamma}$. But then $\mathcal{A}_{\xi_0} \subseteq \mathcal{A}_{\eta_i\restriction\gamma} \in K$, $\mathcal{A}_{\eta_i\restriction\gamma}$ is countable and $(f^{-1} \restriction \mathcal{A}_{\xi_*}) \circ g$ is an embedding of $\mathcal{A}_{\xi_1}$ to $\mathcal{A}_{\eta_i\restriction\gamma}$. Also since $f \restriction \mathcal{A}_{g_\alpha} = f_\alpha \subseteq g$, $[(f^{-1} \restriction \mathcal{A}_{\xi_*}) \circ g] \restriction \mathcal{A}_{g_\alpha} = id$. This contradicts the choice of $\mathcal{A}_{\xi_0}$

and $\mathcal{A}_{\xi_1}$ i.e. (4) above. □

4.9 Exercise. *In ZFC, show that if $\kappa > \omega$ is a regular cardinal and there are $X_\alpha \subseteq \alpha$, $\alpha < \kappa$, such that for all $X \subseteq \kappa$, the set $\{\alpha < \kappa | X_\alpha = X \cap \alpha\}$ is unbounded, then $\kappa^{<\kappa} = \kappa$ (which is equivalent with $|\{A \subseteq \kappa| |A| < \kappa\}| = \kappa$, exercise).*

4.10 Exercise. *Suppose $\kappa > \omega$ is regular.*

(i) Show that if $C, D \subseteq \kappa$ are cub, then also $C \cap D$ is cub.

(ii) Show that if $C \subseteq \kappa$ is cub and $S \subseteq \kappa$ is stationary, then $C \cap S$ is stationary.

(iii) Show that E_{ns} is an equivalence relation.

(iv) Supose that for all $i < \kappa$, $C_i \subseteq \kappa$ is cub. We say that $C \subseteq \kappa$ is the diagonal intersection $\Delta_{i<\kappa} C_i$ of these sets if for all $\alpha < \kappa$, $\alpha \in C$ if $\alpha \in C_i$ for all $i < \alpha$. Show that C is cub. Conclude that for all $\alpha < \kappa$, $\cap_{i<\alpha} C_i$ is cub.

5. Squares

In this section we will look at a combinatorial principle called square (aka box). R. Jensen showed that $\square_\kappa$ holds in L for all regular cardinals κ. We will skip this proof but we will look at how to use the principle.

5.1 Exercise/Definition. *Suppose X is a set and $<$ is a well-ordering of X. Show that there are unique $\alpha \in On$ and a unique bijection $f : X \to \alpha$ such that for all $a, b \in X$, $a < b$ iff $f(a) < f(b)$. We write $ot(X, <)$ for this α. If $X \subseteq On$, then we write $ot(X)$ for $ot(X, \in)$.*

5.2 Definition. *Suppose κ is a regular cardinal. We write $\square_\kappa$ for the following principle: There are cub sets $C_\alpha \subseteq \alpha$ for all limit ordinals $\alpha < \kappa^+$ such that $ot(C_\alpha) \leq \kappa$ and for all limit ordinals $\beta < \alpha < \kappa^+$, if $\cup(C_\alpha \cap \beta) = \beta$, then $C_\beta = C_\alpha \cap \beta$.*

5.3 Fact. *(Jensen) $V = L$ implies that $\square_\kappa$ holds for all infinite regular cardinals κ.*

5.4 Exercise.

(i) Suppose κ is a regular cardinal (notice that then κ is a regular cardinal in L) and $(\kappa^+)^L = \kappa^+$. Show that $\square_\kappa$ holds.

(ii) Show that $\square_\omega$ holds.

5.5 Definition. *Suppose $\mu < \kappa$ are regular cardinals.*

(i) $S^\kappa_\mu = \{\alpha < \kappa | cf(\alpha) = \mu\}$.

(ii) $C \subseteq S^\kappa_\mu$ is μ-cub if it is unbounded and for all $\alpha < \kappa$ of cofinality μ the following holds: If $\cup(C \cap \alpha) = \alpha$, then $\alpha \in C$.

(iii) Suppose $X \subseteq \kappa$. A μ-cub-game $CG^\kappa_\mu(X)$ is the following game: There are two players I and II and the length of the game is μ. At each move $i < \mu$, first I chooses an ordinal $\alpha_i < \kappa$ and then II chooses $\beta_i < \kappa$ so that $\beta_i > \cup_{j \leq i} \alpha_j$. II wins if $\cup_{i<\mu} \beta_i \in X$.

(iv) A winning strategy of II in $G^\kappa_\mu(X)$ is a sequence $W = (f_i)_{i<\mu}$ such that for all $i < \mu$, $f_i : \kappa^{i+1} \to \kappa$ and if II plays according to W (i.e. always chooses $\beta_i = f_i((\alpha_j)_{j \leq i})$), then II wins the game.

5.6 Exercise.

(i) Show that if $\mu < \kappa$ *are regular, then* S^κ_μ *is stationary and that* $X \subseteq S^\kappa_\mu$ *is* μ*-cup iff there is a cub* $C \subseteq \kappa$ *such that* $C \cap S^\kappa_\mu = X$*.*

(ii) Suppose that κ *is a cardinal,* $\mu < \kappa^+$ *is regular and* $X \subseteq \kappa^+$*. Show that (a) implies (b) and that if* $\kappa^{<\mu} = \kappa$*, then (b) implies (a), where*

(a) X *contains a* μ*-cub.*

(b) II *has a winning strategy in* $CG^{\kappa^+}_\mu(X)$*.*

Hint: For (a)⇒(b): Let II play an increasing sequence of elements of the μ*-cub. For (b)⇒(a): Look at the set of ordinals closed under the winning strategy.*

Assuming the existence of a mahlo cardinal, it is possible to force a model in which there are a regular cardinal κ and $X \subseteq \kappa^+$ such that II has a winning strategy in $CG^{\kappa^+}_\kappa(X)$ but X does not contain a κ-cub set.

5.7 Theorem. *Suppose that* κ *is a reqular cardinal,* $\square_\kappa$ *holds and* $X \subseteq \kappa^+$*. Then the following are equivalent:*

(i) X *contains a* κ*-cub set.*

(ii) II *has a winning strategy in* $CG^{\kappa^+}_\kappa(X)$*.*

Proof. (i)⇒(ii): This follows from Exercise 5.6.

(ii)⇒(i): If $\kappa = \omega$, the claim follows from Exercise 5.6. So we assume that $\kappa > \omega$. Suppose (i) fails. Then $S = S^{\kappa^+}_\kappa - X$ is stationary by Exercise 5.6. Let $W = (f_i)_{i<\kappa}$ be a winning strategy of II and $C = (C_i)_{i\in J}$, $J = \{\alpha < \kappa^+ |\ \alpha \textit{ limit}\}$ witness that $\square_\kappa$ holds. For all $i \in J$, let $C^*_i = (\gamma^i_k |\ k < ot(C_i))$ be an increasing enumaration of C_i and $C^* = (C^*_i |\ i \in J)$. We can choose $\in$-models M_i, $i < \kappa^+$, so that

(a) $(\kappa+1) \cup \{\kappa^+, W, C, C^*, J\} \cup \{(\kappa^+)^{i+1} |\ i < \kappa\} \subseteq M_0$,

(b) for all $i < \kappa^+$, M_i is an elementary submodel of $V_{\kappa^{++}}$ (i.e. for all $\phi(x) \in L_{\omega\omega}$ and $a \in (M_i)^n$, $M_i \models \phi(a)$ iff $V_{\kappa^{++}} \models \phi(a)$ and $\kappa^{++} = (\kappa^+)^+$),

(c) for all $i < \kappa^+$, $|M_i| = \kappa$ and $M_i \cap \kappa^+ = \alpha_i \in On$,

(d) for all $i < j < \kappa^+$, $M_i \subseteq M_j$ (and thus M_i is an elementary submodel of M_j) and $\alpha_i < \alpha_j$,

(e) for $i < \kappa^+$ limit, $M_i = \cup_{j<i} M_j$ (and thus $\alpha_i = \cup_{j<i}\alpha_j$).

Now $D = \{\alpha_i |\ i < \kappa^+\}$ is cub (by (e) above) and thus there is $i < \kappa^+$ such that $\alpha_i \in S$. Denote $M = M_i$ and $\alpha = \alpha_i$ and notice that $cf(\alpha) = \kappa$ and $\alpha > \kappa$. Thus $ot(C_\alpha) = \kappa$ and then $C_\alpha = \{\gamma^\alpha_i |\ i < \kappa\}$ and recall that $\gamma^\alpha_i < \gamma^\alpha_j$ if $i < j$. We write γ_i for γ^α_i.

Now we play the game $CG^{\kappa^+}_\kappa(X)$ the following way: II follows her winning strategy W and I chooses at round i, the ordinal γ_i. Since II wins this game and $\cup_{i<\kappa}\gamma_i = \alpha \notin X$, there must be $i < \kappa$ such that $f_i((\gamma_j)_{j\le i}) > \alpha$.

We make the following observation: If $h : Y \to V_{\kappa^{++}}$ and $h, Y \in M$, then for all $x \in Y \cap M$, $h(x) \in M$. This is because M is an elemenry submodel of $V_{\kappa^{++}}$ and $h(x) \in V_{\kappa^{++}}$ for all $x \in Y$.

Now since $W, i \in M$, $f_i = W(i) \in M$ and thus $(\gamma_j)_{j\le i} \notin M$ (because otherwise $f_i((\gamma_j)_{j\le i}) \in M$ and so $f_i((\gamma_j)_{j\le i}) < \alpha = \kappa^+ \cap M$). On the other hand, there is a limit ordinal $\beta \in C_\alpha$ such that $\cup(C_\alpha \cap \beta) = \beta$ and $\beta > \gamma_i$, and thus $C_\alpha \cap \beta = C_\beta$ and $(\gamma_j)_{j \le i} = C^*_\beta \restriction (\gamma_i + 1)$. Since $C^* \in M$ and $\beta \in M$, $C^*_\beta \in M$ and since also $\gamma_i + 1 \in M$, $(\gamma_j)_{j \le i} = C^*_\beta \restriction (\gamma_i + 1) \in M$ since M is an elementary substructure of $V_{\kappa^{++}}$, a contradiction. □

5.8 Exercise. *Suppose $\kappa > \omega$ is a regular cardinal.*

(i) Suppose $\mu < \kappa$ and for $i < \mu$, $S_i \subseteq \kappa$ are such that $\cup_{i<\mu} S_i$ is stationary. Show that there is $i < \mu$ such that S_i is stationary. Hint: Exercise 4.10 (iv).

(ii) Suppose $\mu \le \kappa$ is regular and sets C_α, $\alpha < \kappa^+$ limit, witness that $\Box_\kappa$ holds. For all $\gamma \le \kappa$, let $S_\gamma = \{\beta \in S^{\kappa^+}_\mu \mid ot(C_\beta) = \gamma\}$. Show that there is $\gamma \le \kappa$ such that $S = S_\gamma$ is stationary and for all limit $\alpha < \kappa^+$ of cofinality $> \omega$, $S \cap \alpha$ is not stationary in α. (If $cf(\alpha) = \omega$, stationarity in α does not make much sense.) Hint: The limit points of C_α almost witness that $S \cap \alpha$ is not stationary.

5.9 Exercise. *Show that for all $\alpha < \omega_1$ the following holds: For all stationary $S \subseteq \omega_1$ there is $A \subseteq S$ such that $ot(A) = \alpha + 1$ and A is closed i.e. for all $\gamma < \cup A$, if $\cup(A \cap \gamma) = \gamma$, then $\gamma \in A$. Hint: By induction on α, for the limit cases one can use Exercise 5.6 (ii) for $X = \omega_1 - S$.*

6. Generic extension

In forcing the strategy to show that some first-order sentence ϕ is not provable from ZFC is to first find a suitable partial-order $P = (P, <)$ (i.e. a set P together with a partial-ordering $<$ of it) and a P-generic filter G over V and then construct a generic extension $V[G]$ and finally show that $V[G]$ is a model of ZFC together with $\neg\phi$. However, this construction can not be done inside V. It follows that we will work on the object side. The first approximation for this would be to pick an $\in$-model of ZFC^* but in fact it turns out that it is enough to assume, and this will be important, that it is enough to assume that $V \models \phi^*$ (i.e. ϕ^V) for all $\phi \in ZFC$ We call this model V. So from now on by V we mean this and not the model of ZFC in which we pretend to be working as in Section 2. We know that ZFC does not prove the existence of a model of ZFC^* (assuming ZFC is consistent) and we do not have even tools to ask if ZFC proves the existence of our V. We will return to this problem in Section 9.

To be able to talk about V in the meta theory, we think it as a constant. Since we assume only ϕ^V for $\phi \in ZFC$, by definability in V we mean definability in the sense of our meta language (i.e. by formulas of the form ϕ^V or ϕ^*) and not by formulas of $L_{\omega\omega}$ (we have e.g. Separation only for formulas ϕ^*, not for all $L_{\omega\omega}$-formulas). Notice also that if $ZFC \vdash \phi$, then ϕ^V holds.

Also now our meta theory knows that V is an $\in$-model and because of this, the meta theory thinks that V is well-founded also in the following sense (below when we talk about well-founded partial orderings of classes of V we mean in the sense of Definition 1.3.1 applied in V): there are no $a_i \in V$, $i < \omega$, such that for all $i < \omega$, $a_{i+1} \in a_i$ in V. We also assume that the meta theory thinks that V is transitive, in particular, every element x of V is the set of all elements $y \in V$ that V thinks are elements of x (i.e. $x = x \cap V$).

6.1 Exercise. *Show that the transitivity assumption can be made without loss of generality.*

Finally, we are going to assume that V is countable. Recall that in Section 9, we will look at the questions: why all our additional assumptions are harmless and why finding $V[G]$ such that $V[G] \models \neg\phi$ shows that ZFC does not prove ϕ (assuming ZFC is consistent).

Notice that since we have added V as a constant, the set of elements of V is definable without parameters and so for all formulas ϕ, ϕ^V does not contain parameters. If ϕ^V contains just one element, we use ϕ^V also to denote that element e.g. ω^V is the element of V that satisfies the definition of ω in V.

Since we have assumed that V is transitive, for many sentences ϕ (with parameters from V), $\phi^V \leftrightarrow \phi$ is true (i.e. provable from in our meta theory) and when this is the case, we say that ϕ is absolute for V.

6.2 Exercise. *Let $a, b \in V$. Show that*
(i) $(a \in b)^V \leftrightarrow (a \in b)$.
(ii) $(\text{"}a \text{ is a partial order"})^V \leftrightarrow (\text{"}a \text{ is a partial order"})$.
(iii) $(a \in On)^V \leftrightarrow (a \in On)$.
(iv) $\omega^V = \omega$.
(v) $V_\omega^V = V_\omega$. *Hint: Show that if $x \subseteq y \in V$ is finite, then $x \in V$.*
(vi) $\forall x(\text{"}x \text{ is a formula"} \leftrightarrow (x \in V \wedge (\text{"}x \text{ is a formula"})^V))$.

For this and the next section, we fix a partial order $P = (P, <) \in V$ with a largest element 1 (these are often called po-sets). For $a, b \in P$, we write $a||b$ if there is $c \in P$ such that $c \leq a, b$. If there is no such c, we write $a \perp b$.

6.3 Definition.
(i) We say that $D \in V$ is dense in P if $D \subseteq P$ and for all $a \in P$, there is $b \in D$ such that $b \leq a$.
(ii) We say that $G \subseteq P$ is a filter if the following holds:
(a) $1 \in G$,
(b) if $a \in G$ and $b \geq a$, then $b \in G$,
(c) if $a, b \in G$, then there is $c \in G$ such that $c \leq a, b$.
(iii) We say that G is P-generic over V if it is a filter and for all $D \in V$, if D is dense in P, then $G \cap D \neq \emptyset$.

6.4 Exercise.
(i) For all $D \in V$, $(\text{"}D \text{ is dense in } P\text{"})^V \leftrightarrow (\text{"}D \text{ is dense in } P\text{"})$.
(ii) Show that if G is P-generic over V and $p \in P$ is such that for all $q \in G$, $p||q$, then $p \in G$.
(iii) Suppose that for all $a \in P$, there are $b, c \in P$ such that $b, c < a$ and $b \perp c$. Show that if G is P-generic over V, then $G \notin V$.

(iv) Show that for all $p \in P$, there exists a P-generic G over V such that $p \in G$.

(v) Suppose G is P-generic over V, $p \in G$ and $C \subseteq P$, $C \in V$, is dense below p i.e. for all $q \le p$ there is $r \le q$ such that $r \in C$. Show that $C \cap G \neq \emptyset$.

(vi) Suppose C is dense below p and $q \le p$. Show that C is dense below q.

When I talk about dense sets D or sets D dense below some $p \in P$, I mean that in addition $D \in V$ (I may forget to mension this).

6.5 Definition. *The set V^P of P-names is defined as follows:*

(i) $\emptyset$ is a P-name.

(ii) For all $\alpha \in On^V$, and $p_i \in P$, $i < \alpha$, if for all $i < \alpha$, τ_i is a P-name and $\tau = \{(\tau_i, p_i) |\ i < \alpha\} \in V$, then τ is a P-name.

6.6 Exercise.

(i) Show that V^P is a class in V i.e. that there is a formula ϕ and $a \in V^n$ such that $V^P = \{x \in V |\ \phi^V(x, a)$ holds$\}$.

(ii) Show that the following binary relation $<^*$ on elements of V^P is a well-founded partial order: $\tau <^* \sigma$ if there are $n < \omega$, $\tau_i \in V^P$, $i \le n$, and $p_i \in P$, $i < n$, such that $\tau_0 = \tau$, $\tau_n = \sigma$ and for all $i < n$, $(\tau_i, p_i) \in \tau_{i+1}$.

(iii) Show that $<^*$ is a class in V.

6.7 Definition. *Let G be P-generic over V.*

(i) For all $\tau \in V^P$, τ_G is defined as follows: ($\emptyset_G = \emptyset$ and) $\tau_G = \{\sigma_G |\ \exists p \in G$ such that $(\sigma, p) \in \tau\}$.

(ii) $V[G] = \{\tau_G |\ \tau \in V^P\}$.

As usually, we can see that $\tau \mapsto \tau_G$ is a function and thus $V[G]$ exists. We think $V[G]$ as an $\in$-model. Notice that $V[G]$ is transitive (exercise). We use the same notations with $V[G]$ as with V. So e.g. $\phi^{V[G]}$ denotes the formula that says that ϕ is true in $V[G]$.

If $a \in V[G]$, then it has a name τ i.e. a P-name such that $a = \tau_G$. This name is often denoted by $\dot{a}$ (or $\hat{a}$, see below), but we are not very strict with this. Notice that $\dot{a}$ is not unique.

6.8 Definition. *For each $a \in V$, we define the standard name $\hat{a}$ for a as follows: ($\hat{\emptyset} = \emptyset$ and) $\hat{a} = \{(\hat{b}, 1) |\ b \in a\}$.*

6.9 Exercise.

(i) Show that $V[G]$ is transitive.

(ii) Show that for all P-generic G over V, $\hat{a}_G = a$ and conclude that $V \subseteq V[G]$.

We finish this section by defining the forcing notion $\Vdash$. In the next section we give another definition for $\Vdash$ and we prove that the two definitions are equivalent as well as the very basic properties of this notion.

We start by defining the forcing language, This is a language that works on the object side only, it does not really have a counterpart on the meta side. So in the definition below we describe a sentence of the meta language that expresses what it means to be a formula in the forcing language. By a forcing language we mean the first-order logic in the vocabulary $\{\in\} \cup \{\tau |\ \tau \in V^P\}$, where the P-names τ are considered as constants. So every $L_{\omega\omega}$-formula belongs to the forcing language. When we write a formula of this forcing language, we usually point out what are the constants. So $\phi(\tau_1, ..., \tau_n)$ means a formula in which no other constants than $\tau_1, ..., \tau_n$ appear. Notice that we can think $\phi(\tau_1, ..., \tau_n)$ as a formula we have got from an $L_{\omega\omega}$-formula by replacing variables by constants and then for a P-generic G over V, $\phi((\tau_1)_G, ..., (\tau_n)_G)$ is the $L_{\omega\omega}$-formula with parameters from $V[G]$. Also if $\phi(x_1, ..., x_n)$ is a first-order formula in the meta language, then by $\phi(\tau_1, ..., \tau_n)$ we mean the formula in the forcing language that we get from ϕ^* by replacing variables x_i by constants τ_i (in their free appearances). When we write $\phi(x_1, ..., x_n)$, we mean that $x_1, ..., x_n$ list all free variables of the formula ϕ (in the meta language) and then $\phi(\tau_1, ..., \tau_n)$ is a sentence of the forcing language. It is these sentences that we are mainly interested in (although they do not form a nice object).

6.10 Definition. *Let $\phi(\tau_1, ..., \tau_n)$ be a sentence in the forcing language and $p \in P$. We say that p forces $\phi(\tau_1, ..., \tau_n)$ and write $p \Vdash \phi(\tau_1, ..., \tau_n)$ (or, if needed, $p \Vdash_P \phi(\tau_1, ..., \tau_n)$), if for all P-generic G over V, the following holds: If $p \in G$, then $V[G] \models \phi((\tau_1)_G, ..., (\tau_n)_G)$.*

6.11 Exercise.
(i) Suppose $p \Vdash \phi$ and $q \leq p$. Show that $q \Vdash \phi$.
(ii) Show that $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$.
(iii) Suppose $p \Vdash \phi$ and $\vdash \phi \rightarrow \psi$. Show that $p \Vdash \psi$.

By $\dot{G}$ we denote the P-name $\{(\hat{p}, p) |\ p \in P\}$.

6.12 Exercise. *Show that $\dot{G}_G = G$ and conclude that $G \in V[G]$.*

6.13 Exercise. *Let $P \in V$ be the set of all functions $f : n \rightarrow \omega$, $n < \omega$, ordered by $f \leq g$ if $g \subseteq f$ and let G be P-generic over V. Suppose $\pi : \omega^2 \rightarrow \omega$ belongs to V and is one-to-one and onto (in V, but this is absolute). Let $F = \cup G$. Show that F is a function from ω to ω and that for all $n, m < \omega$, there is $k < \omega$ such that $F(\pi(n, k)) = m$.*

7. Forcing

We define an ordering $<^*$ to $(V^P)^2$ so that $(\tau, \sigma) <^* (\tau', \sigma')$ if $\tau <^* \tau'$ and $\sigma <^* \sigma'$ (see Exercise 6.6 (ii)).

7.1 Exercise. *Show that $\leq^*$ is well-founded.*

7.2 Definition. *For $p \in P$ and P-names τ and σ, the relation $p \Vdash^* \tau = \sigma$ is defined as follows: $p \Vdash^* \tau = \sigma$ if both (a) and (b) below hold:*

(a) for all $q \leq p$ and $(\tau', s) \in \tau$, if $q \leq s$, then there are $r \leq q$ and $(\sigma', t) \in \sigma$ such that $r \leq t$ and $r \Vdash^* \tau' = \sigma'$.

(b) for all $q \leq p$ and $(\sigma', t) \in \sigma$, if $q \leq t$, then there are $r \leq q$ and $(\tau', s) \in \tau$ such that $r \leq s$ and $r \Vdash^* \tau' = \sigma'$.

Notice that (a) above is equivalent with the following (and similarly for (b)): For all $(\tau', s) \in \tau$ if $q \le p, s$, then the set $\{r \in P | \ \exists(\sigma', t) \in \sigma \ s.t. \ r \le t \ and \ r \Vdash^* \tau' = \sigma'\}$ is dense below q.

7.3 Exercise.

(i) Show that the set $\{(p, \tau, \sigma) | \ p \Vdash^* \tau = \sigma\}$ *is a class in* V*. Hint: Think of the function* F *such that* $F(\tau, \sigma)$ *is the set of all* $p \in P$ *s.t.* $p \Vdash \tau = \sigma$*.*

(ii) Show that if for all $q \le p$ *there is* $r \le q$ *such that* $r \Vdash^* \tau = \sigma$*, then* $p \Vdash^* \tau = \sigma$*.*

(iii) Show that if $p \Vdash^* \tau = \sigma$ *and* $q \le p$*, then* $q \Vdash^* \tau = \sigma$*.*

7.4 Lemma. *Suppose* G *is* P*-generic over* V*.*

(i) If $p \in G$ *and* $p \Vdash^* \tau = \sigma$*, then* $\tau_G = \sigma_G$*.*

(ii) If $\tau_G = \sigma_G$*, then there is* $p \in G$ *such that* $p \Vdash^* \tau = \sigma$*.*

Proof. (i): We prove this by induction on $<^*$. By symmetry it is enough to show that $\tau_G \subseteq \sigma_G$. For this it is enough to show the following: If $(\tau', s) \in \tau$ and $s \in G$, then $\tau'_G \in \sigma_G$. Let $p' \in G$ be such that $p' \le p, s$. By the definition of $\Vdash^*$ and the definition of a P-generic filter over V, we can find $(\sigma', t) \in \sigma$ and $r \in G$ such that $r \le p', t$ and $r \Vdash^* \tau' = \sigma'$ (exercise, hint: use the remark after Definition 7.2 and Exercise 6.4 (v)). By the induction assumption, $\tau'_G = \sigma'_G$ and thus $\tau'_G \in \sigma_G$.

(ii): We prove this by induction on $<^*$. We show that there is $p \in G$ such that (a) from Definition 7.2 holds. Similarly we see that there is $p \in G$ such that (b) from Definition 7.2 holds. This suffices (exercise). For a contradiction suppose that there is no such $p \in G$.

For all $p \in P$, let $q(p)$ and $(\tau'(p), s(p))$ witness the failure of (a) in the case that these elements exists (keep in mind that $q(p) \le p$). If they do not exists, we say that p is good. Now $\{q \in P | \ \exists p \in P, \ p \text{ is not good and } q \le q(p)\} \cup \{q \in P | \ q \text{ is good}\}$ is dense in P (exercise). Since we assumed that there is no good $p \in G$, there must be $p \in G$ (since G is a generic filter) such that $q(p) \in G$ (if $q \in G$ and $q \le q(p) \le p$, then $q(p), p \in G$).

But then for all $(\sigma', t) \in \sigma$ such that $t \in G$, for no $r \in G$, $r \Vdash^* \tau'(p) = \sigma'$ (by Exercise 7.3 (iii)). Thus by the induction assumption, for all $(\sigma', t) \in \sigma$ such that $t \in G$, $\tau(p)_G \neq \sigma'_G$. Since $\tau'(p)_G \in \tau_G$, we have a contradiction. □

7.5 Definition. *Suppose* $p \in P$ *and* τ *and* σ *are* P*-names. We define* $p \Vdash^* \tau \in \sigma$ *as follows:* $p \Vdash^* \tau \in \sigma$ *if for all* $q \le p$*, there are* $r \le q$ *and* $(\sigma', t) \in \sigma$ *such that* $r \le t$ *and* $r \Vdash^* \tau = \sigma'$*.*

7.6 Exercise.

(i) Show that the set $\{(p, \tau, \sigma) | \ p \Vdash^* \tau \in \sigma\}$ *is a class in* V*.*

(ii) Show that if for all $q \le p$ *there is* $r \le q$ *such that* $r \Vdash^* \tau \in \sigma$*, then* $p \Vdash^* \tau \in \sigma$*.*

(iii) Show that if $p \Vdash^* \tau \in \sigma$ *and* $q \le p$*, then* $q \Vdash^* \tau \in \sigma$*.*

7.7 Lemma. *Suppose G is P-generic over V.*
(i) If $p \in G$ and $p \Vdash^* \tau \in \sigma$, then $\tau_G \in \sigma_G$.
(ii) If $\tau_G \in \sigma_G$, then there is $p \in G$ such that $p \Vdash^* \tau \in \sigma$.

Proof. As the proof of Lemma 7.4 (using Lemma 7.4). □

7.8 Definition. *Let $p \in P$ and $\phi = \phi(\tau_1, ..., \tau_n)$ be a sentence in the forcing language. We define $p \Vdash^* \phi$ as follows:*
(i) If ϕ is an atomic formula, we have already defined $p \Vdash^* \phi$.
(ii) If $\phi = \neg\psi$, then $p \Vdash^* \phi$ if there is no $q \le p$ such that $q \Vdash^* \psi$.
(iii) If $\phi = \psi \wedge \theta$, then $p \Vdash^* \phi$ if $p \Vdash^* \psi$ and $p \Vdash^* \theta$.
(iv) If $\phi = \exists v_k \psi(v_k, \tau_1, ..., \tau_n)$, then $p \Vdash^* \phi$ if for all $q \le p$ there are a P-name τ and $r \le q$ such that $r \Vdash^* \psi(\tau, \tau_1, ..., \tau_n)$.

Recall from Section 6, that if $\phi(x_1..., x_n)$ is a formula (in the meta language), then $\phi(\tau_1, ..., \tau_n)$ is the formula one gets from ϕ^* by replacing variables x_i by P-names τ_i.

7.9 Exercise.
(i) Show that for all formulas $\phi(x_i, ..., x_n)$, the set

$$\{(p, \tau_1, ..., \tau_n) |\ p \Vdash^* \phi(\tau_1, ..., \tau_n)\}$$

is a class in V.
(ii) Show that if $p \Vdash^* \phi$ and $q \le p$, then $q \Vdash^* \phi$.
(iii) Show that if for all $q \le p$ there is $r \le q$ such that $r \Vdash^* \phi$, then $p \Vdash^* \phi$.

7.10 Theorem. *Suppose G is P-generic over V.*
(i) If $p \in G$ and $p \Vdash^* \phi(\tau_1, ..., \tau_n)$, then $V[G] \models \phi((\tau_1)_G, ..., (\tau_n)_G)$.
(ii) If $V[G] \models \phi((\tau_1)_G, ..., (\tau_n)_G)$, then for some $p \in G$, $p \Vdash^* \phi(\tau_1, ..., \tau_n)$.

Proof. We prove the claims simultaneously by induction on ϕ. If ϕ is an atomic formula, then we have already proved this. We prove the claims in the case $\phi = \neg\psi$, the two other cases are left as an exercise.

(i): For a contradiction suppose $V[G] \models \psi$. Then by the induction assumption, there is $q \in G$ such that $q \Vdash^* \psi$. By Exercise 7.9, we may assume that $q \le p$, a contradiction.

(ii): For a contradiction, suppose that there is no such $p \in G$ i.e. for all $p \in G$ there is $q_p \in P$ such that $q_p \le p$ and $q_p \Vdash^* \psi$. But then as in the proof of Lemma 7.4, we can find $p \in G$ such that $q_p \in G$. By the induction assumption $V[G] \models \psi$, a contradiction. □

7.11 Corollary. *$p \Vdash \phi$ iff $p \Vdash^* \phi$.*

Proof. From right to left the claim follows immediately from Theorem 7.10 (i). For the other direction, by Exercise 7.9 (ii), it is enough to show that for all $q \le p$, there is $r \le q$ such that $r \Vdash^* \phi$. But this is clear by Theorem 7.10 (ii). □

We finish this section by showing that $V[G]$ satisfies all the axioms of ZFC.

7.12 Theorem. *Let ϕ be an axiom of ZFC. Then $\phi^{V[G]}$ holds.*

Proof. For extensionality, foundation and infinity, the claim is immediate by our construction of $V[G]$. We prove separation, the rest are similar.

Let τ and $\tau_1, ..., \tau_n$ be P-names and $\phi(v_0, ..., v_n)$ be a formula. Let G be P-generic over V. We need to show that the set

$$a = \{x \in \tau_G | \ \phi(x, (\tau_1)_G, ..., (\tau_n)_G)^{V[G]}\}$$

is in $V[G]$. For this we need to find a P-name for a.

We let σ be the set of all pairs (δ, p) such that

(i) $p \in P$ and for some $q \geq p$, $(\delta, q) \in \tau$,

(ii) $p \Vdash \phi(\delta, \tau_1, ..., \tau_n)$.

Notice that by Exercise 7.9 (i), σ is a P-name (i.e. is in V). We are left to show that $\sigma_G = a$.

$\sigma_G \subseteq a$: Suppose $\delta'_G \in \sigma_G$. Then there are $p \in G$ and δ such that $(\delta, p) \in \sigma$ and $p \Vdash \delta' = \delta$. But then $\delta'_G = \delta_G \in \tau_G$ and $\phi(\delta_G, (\tau_1)_G, ..., (\tau_n)_G)^{V[G]}$ holds i.e. $\delta'_G \in a$.

$a \subseteq \sigma_G$: Suppose $\delta'_G \in a$. Then $\delta'_G \in \tau_G$ and so there are $p \in G$ and δ such that $p \Vdash \delta' = \delta$ and $(\delta, q) \in \tau$ for some $q \geq p$. Also for some $p' \in G$, $p' \Vdash \phi(\delta', \tau_1, ..., \tau_n)$. Clearly we may assume that $p' = p$. But then by Exercise 6.11, $p \Vdash \phi(\delta, \tau_1, ..., \tau_n)$ and thus $\delta_G \in \sigma_G$ and so also $\delta'_G \in \sigma_G$. □

7.13 Exercise.

(i) Show that the pairing axiom is true in $V[G]$.

(ii) Show that the union axiom is true in $V[G]$.

(iii) Show that the power set axiom is true in $V[G]$.

7.14 Exercise.

(i) Suppose that $\exists v_k \psi$ is a formula (of the meta language). Show that if $p \Vdash \exists v_k \psi(v_k, \tau_1, ..., \tau_n)$, then there is a P-name τ such that $p \Vdash \psi(\tau, \tau_1, ..., \tau_n)$. Hint: For all $q \in P$ pick σ_q such that $q \Vdash \psi(\sigma_q, \tau_1, ..., \tau_n)$ if there is such a name. Enumerate these σ_q as σ_i, $i < \alpha$. We say that q is good if there is $i < \alpha$ such that $q \Vdash \psi(\sigma_i, \tau_1, ..., \tau_n)$ and for all $r \leq q$, if $r \Vdash \psi(\sigma_j, \tau_1, ..., \tau_n)$, then $j \geq i$. Start by showing that good conditions are dense below p. By looking the beginning of Section 8, another method can be found.

(ii) Suppose $C \in V$ is a set of P-names. Show that in $V[G]$ there is a function $f : C \to V[G]$ such that for all $\tau \in C$, $f(\tau) = \tau_G$.

7.15 Lemma. *Let G be P-generic over V. Then $On^{V[G]} = On^V$.*

Proof. Clearly $On^V \subseteq On^{V[G]}$. So for a contradiction, suppose that there is $\alpha \in On^{V[G]}$ such that $\alpha \notin On^V$. Then $On^V \subseteq \alpha$. Let τ be a P-name such that $\tau_G = \alpha$ and let A be the set of all P-names σ such that $(\sigma, q) \in \tau$ for some $q \in P$. Let $\kappa \in V$ be a cardinal for which there is a bijection $f : A \times P \to \kappa$ (in V). Let κ^+ be the successor of κ in V.

Then there is some $p \in P$ such that $p \Vdash \hat{\kappa^+} \subseteq \tau$. And so for all $\gamma \in \kappa^+$ there is $(\delta_\gamma, p_\gamma) \in A \times P$ such that $p_\gamma \Vdash \delta_\gamma = \hat{\gamma}$. By Corollary 7.11 and Exercise 7.9 (i), we can choose δ_γ and p_γ so that the function $g : \kappa^+ \to A \times P$, $g(\gamma) = (\delta_\gamma, p_\gamma)$ is in V and clearly it is an injection. Thus $f \circ g$ is an injection from κ^+ to κ, a contradiction. □

8. Negation of continuum hypothesis

In this section we prove the consistency of the negation of the continuum hypothesis.

8.1 Definition. *Let $P = (P, <)$ be a partial order.*

(i) We say that $A \subseteq P$ is an antichain if for all $p, q \in A$, if $p \neq q$, then $p \perp q$. For $B \subseteq P$, we say that $A \subseteq B$ is a maximal antichain in B if it is an antichain and no antichain $A' \subseteq B$ is a proper extension of A (equivalently there is no $p \in B$ such that $A \cup \{p\}$ is an antichain, which is absolute). A is a maximal antichain below $p \in P$ if it is a maximal antichain in $\{q \in P | \; q \leq p\}$. A is a maximal antichain if it is a maximal antichain in P (equivalently a maximal antichain below 1).

(ii) For a cardinal κ, we say that P has κ-cc (chain condition) if $|A| < \kappa$ for all antichains $A \subseteq P$.

Again when we talk about antichains A we mean that in addition $A \in V$.

8.2 Exercise.

(i) Suppose $A \subseteq \{q \in P | \; q \leq p\}$ is an antichain (in V). Show that A is a maximal antichain below p iff $D = \{r \in P | \; \exists q \in A(r \leq q)\}$ is dense below p.

(ii) Show that if $A \subseteq P$ is a maximal antichain below p (and in V), G is P-generic over V and $p \in G$, then $G \cap A$ is a singleton.

(iii) Show that for all $B \subseteq P$ and antichain $A \subseteq B$ (both in V), there is a maximal antichain C in B such that $A \subseteq C$.

(iv) Suppose that $B \subseteq \{q \in P | \; q \leq p\}$ is dense below p and $A \subseteq B$ is a maximal antichain in B. Show that A is a maximal antichain below p.

Recall that by ω_1 we denote the least cardinal $> \omega$ i.e. ω^+. ω_1-cc is usually called ccc (countable chain condition, which, of course, should be called countable antichain condition but that is not the case).

8.3 Theorem. *Suppose that in V the following holds: P has κ-cc and $cf(\lambda) = \gamma \geq \kappa$. Let G be P-generic over V. Then in $V[G]$, $cf(\lambda) = \gamma$.*

Proof. For a contradiction, suppose that in $V[G]$ there are $\theta < \gamma$ and $f : \theta \to \lambda$ such that $\cup(rng(f)) = \lambda$. Let $\dot{f}$ be a P-name such that $\dot{f}_G = f$. When this happens, we say that $\dot{f}$ is a P-name for f.

8.3.1 Exercise. *Show that there is a P-name τ and $p \in G$ such that $p \Vdash \tau = \dot{f}$ and 1 forces that τ is a function from $\hat{\theta}$ to $\hat{\lambda}$.*

So we may assume that 1 forces that $\dot{f}$ is a function from $\hat{\theta}$ to $\hat{\lambda}$.

8.3.2 Exercise. *Show that for all $\alpha<\theta$, there is a maximal antichain $A_\alpha\subseteq P$ such that for all $p\in A_\alpha$, there is β_p for which $p\Vdash\dot{f}(\hat{\alpha})=\hat{\beta}_p$.*

For all $\alpha<\theta$, let $\delta_\alpha=\cup\{\beta_p+1|\ p\in A_\alpha\}$. By κ-cc and the assumption that $cf(\lambda)\geq\kappa$, $\delta_\alpha<\lambda$. Let $\delta=\cup\{\delta_\alpha|\ \alpha<\theta\}$. Since $\theta<cf(\lambda)$, $\delta<\lambda$. But clearly, $rng(f)\subseteq\delta$, a contradiction. □

8.4 Corollary. *If in V, P has κ-cc, κ is regular and $\lambda\geq\kappa$ is a cardinal, then λ is a cardinal also in $V[G]$.*

Proof. Clearly it is enough to prove this under the additional assumption that λ is regular (exercise). But then the claim follows immediately from Theorem 8.3. □

Theorem 8.3 gives an alternative way of proving Lemma 7.15.

8.5 Corollary. *Suppose in V, P is a partial order and G is P-generic over V. Then for all $\alpha\in V[G]$, $(\alpha\in On)^{V[G]}$ iff $\alpha\in V$ and $(\alpha\in On)^V$.*

Proof. By Exercise 6.2 (ii), it is enough to show that $(\alpha\in On)^{V[G]}$ implies that $\alpha\in V$. For this it is enough to find a cardinal $\lambda\in V$ such that in $V[G]$, $\alpha<\lambda$ (as above). Let $\dot{\alpha}$ be such that $\dot{\alpha}_G=\alpha$. Then there are (in V) a cardinal κ and a function f such that $dom(f)=\kappa$ and

$$rng(f)=\{\tau\in TC(\dot{\alpha})|\ \exists p\in P((\tau,p)\in\dot{\alpha})\}.$$

Now in V, choose a regular cardinal λ so that $\lambda>\kappa$ and $\lambda>|P|$. Then By Corollary 8.4, λ is a cardinal also in $V[G]$. Also by Exercise 7.14, in $V[G]$, there is a function g such that $dom(g)=\kappa$ and for all $\gamma<\kappa$, $g(\gamma)=f(\gamma)_G$. Then clearly $\alpha\subseteq rng(g)$ and thus $|a|<\lambda$. But then $\alpha<\lambda$. □

Let us recall the following definition from Section 4:

8.6 Definition. *Let $\kappa>\omega$ be a regular cardinal.*

(i) $C\subseteq\kappa$ is called cub (closed and unbounded) if it is unbounded in κ (i.e. for all $\alpha<\kappa$ there is $\beta\in C$ such that $\beta>\alpha$) and for all $\alpha<\kappa$, if $\cup(C\cap\alpha)=\alpha$, then $\alpha\in C$.

(ii) $S\subseteq\kappa$ is stationary if for all cub $C\subseteq\kappa$, $S\cap C\neq\emptyset$.

We recall from Exercise 4.10 that if $\kappa>\omega$ is a regular cardinal and for all $\alpha<\kappa$, $C_\alpha\subseteq\kappa$ is cub, then $\Delta_{\alpha<\kappa}C_\alpha$ is cub.

8.7 Exercise. *Show that there is stationary $S\subseteq\omega_1$ such that ω_1-S is stationary. Hint: Let X be the set of all limit ordinals $\alpha<\omega_1$ and for each $\alpha\in X$, pick an increasing sequence $(\alpha_i)_{i<\omega}$ of ordinals $<\alpha$ such that $\cup_{i<\omega}\alpha_i=\alpha$. For all $i<\omega$, define $f_i:X\rightarrow\omega_1$ so that $f_i(\alpha)=\alpha_i$. Then apply Lemma 8.8 below and show that at least one of the sets is as wanted.*

8.8 Lemma. *(Fodor's lemma aka pressing down) Suppose that $\kappa>\omega$ is a regular cardinal, $S\subseteq\kappa$ is stationary and $f:S\rightarrow\kappa$ is such that for all $\alpha\in S$, $f(\alpha)<\alpha$. Then there is stationary $S'\subseteq S$ and $\alpha<\kappa$ such that $f(\gamma)=\alpha$ for all $\gamma\in S'$.*

Proof. Suppose that there are no such S' and α. Then for all $\alpha < \kappa$, there is cub $C_\alpha \subseteq \kappa$ such that for all $\gamma \in C_\alpha \cap S$, $f(\gamma) \neq \alpha$. Let $\gamma \in (\Delta_{\alpha<\kappa} C_\alpha) \cap S$. Then for all $\alpha < \gamma$, $f(\gamma) \neq \alpha$, a contradiction. □

Recall that by $|\alpha|^{<\kappa}$ we mean the cardinality of the set $\{f : \beta \to |\alpha| \mid \beta < \kappa\}$ which is the same as the cardinality of the set $\{f : \beta \to \alpha| \ \beta < \kappa\}$.

8.9 Lemma. *(Δ-lemma) Suppose $\lambda > \kappa$ are regular cardinals, for all $\alpha < \lambda$, $|\alpha|^{<\kappa} < \lambda$ and A be a set. For all $i < \lambda$, let $A_i \subseteq A$ be a set of size $< \kappa$. Then there is an unbounded $X \subseteq \lambda$ and $Y \subseteq A$ such that for all $i, j \in X$, if $i \neq j$, then $A_i \cap A_j = Y$.*

Proof. Without loss of generality we may assume that $A = \lambda$. Let $S = \{\gamma < \lambda| \ cf(\gamma) = \kappa\}$. By Exercise 5.6 (i), S is stationary.

Define $f : S \to \lambda$ so that $f(\gamma) = \cup(A_\gamma \cap \gamma)$. Notice that for all $\gamma \in S$, $f(\gamma) < \gamma$. By Fodor's lemma, there is stationary $S' \subseteq S$ and $\alpha < \lambda$ such that $f(\gamma) = \alpha$ for all $\gamma \in S'$. By the pigeon hole principle and the assumption that $|\alpha + 1|^{<\kappa} < \lambda$, there is $Y \subseteq (\alpha + 1)$ and unbounded $X' \subseteq S'$ such that for all $\gamma \in X'$, $A_\gamma \cap \gamma = Y$.

By induction on $i < \lambda$, we choose ordinals $\gamma_i \in X'$ as follows:

(i) $\gamma_0 = min(X' - (\alpha + 1))$,

(ii) for $i > 0$, $\gamma_i = min(X' - \cup\{(\gamma_j \cup \bigcup A_{\gamma_j}) + 1| \ j < i\})$.

Then Y and $X = \{\gamma_i| \ i < \lambda\}$ are as wanted. □

We recall:

8.10 Definition. *By CH (continuum hypothesis) we mean the claim $2^\omega = \omega_1$.*

Now we are ready to prove the consistency of the negation of continuum hypothesis. We present the proof the way forcing constructions are usually presented and in the next section we study the reason why the proof shows the claim (and we recall what it is that we claim).

8.11 Theorem. *(Cohen) Con(ZFC) implies Con(ZFC+$\neg$CH)*

Proof. In V, let κ be a cardinal $> \omega_1$ and P be the partial order of all functions $p : X_p \to 2$, $X_p \subseteq \kappa \times \omega$ finite, ordered by inverse inclusion i.e. $p \leq q$ if $q \subseteq p$.

8.11.1 Exercise. *Show that (in V) P has ccc. Hint: Suppose that $\{p_i| \ i < \omega_1\}$ is an antichain and start by applying Δ-lemma to the set $\{X_{p_i}| \ i < \omega_1\}$.*

Let G be P-generic over V and then from $V[G]$ we find the function $F = \cup G : \kappa \times \omega \to 2$ and sets $X_\alpha = \{n < \omega| \ F(\alpha, n) = 1\}$, $\alpha < \kappa$.

8.11.2 Exercise.

(i) Show that, indeed, $dom(\cup G) = \kappa \times \omega$.

(ii) Show that for all $\alpha < \beta < \kappa$, $X_\alpha \neq X_\beta$.

By Corollaries 8.4 and 8.5, V and $V[G]$ have the same cardinals and thus in $V[G]$, $2^\omega \geq \kappa > \omega_1$. □

8.12 Exercise. *Suppose κ, P and G are as in the proof of Theorem 8.11.*

(i) Suppose that (in V) $cf(\kappa)=\omega$. Show that in $V[G]$, $2^{\omega}>\kappa$. Hint: Show by diagonalizing that if $cf(\kappa)=\omega$, then $\kappa^{\omega}>\kappa$.

(ii) Suppose that (in V) κ is regular and that GCH holds. Show that in $V[G]$, for all cardinals $\lambda<\kappa$, $2^{\lambda}=\kappa$. Hint: Look at the set X of P-names τ of the following form: For all $\alpha<\lambda$ there is a maximal antichain A_{α} and for all $p\in A_{\alpha}$, $\beta^{\alpha}_{p}<2$ such that

$$\tau=\{((\alpha,\hat{\beta^{\alpha}_{p}}),p)\mid \alpha<\lambda,\ p\in A_{\alpha}\}.$$

(Such names are called nice in [Ku].) Start by showing that if in $V[G]$, f is a function from λ to 2, then $f=\tau_G$ for some $\tau\in X$.

9. Why forcing works

The proof of Theorem 8.11 shows that if, on the meta level, there is a proof of CH from ZFC, then on the meta level there is a proof of contradiction from ZFC (and in fact there is a mechanical method of forming a proof of contradiction from any proof of CH, making the forcing a constructive method). The reason for this is the following:

So suppose that we are given a proof $\mathcal{D}$ of CH from ZFC. Let T be the finite set of axioms of ZFC used in the proof $\mathcal{D}$. Then, by looking at the proofs of Theorems 7.12 and 8.11, one can can find a finite set T^* of axioms of ZFC such that ZFC$\cup\{\phi^V\mid \phi\in T^*\}\cup\{$"$V$ is countable and transitive"$\}$ proves the exisitence of (P and G and) $V[G]$ and $\psi^{V[G]}$ for every $\psi\in T\cup\{\neg\mathrm{CH}\}$. Now using Lemma on constants from the course Mathematical logic, we get that ZFC proves

$$\forall V((\text{"}V\text{ is countable and transitive"}\wedge\bigwedge_{\phi\in T^*}\phi^V)\rightarrow\exists V^*(\bigwedge_{\psi\in T\cup\{\neg\mathrm{CH}\}}\psi^{V^*})).$$

Using Exercises 2.3, 2.4 and 3.12 (ii) one gets:

9.1 Exercise. *For all finite $T'\subseteq ZFC$, ZFC proves that there exists a countable and transitive $\in$-model V such that for all $\phi\in T'$, ϕ^V holds.*

Thus ZFC proves that there exists V^* such that for all $\phi\in T\cup\{\neg CH\}$, ϕ^{V^*} holds.

On the other hand, since T proves CH, ZFC proves that "$T\vdash$ CH" (as in the proof of Gödel's second incompleteness theorem in the course Mathematical logic). Since ZFC also proves soundness (see the course Mathematical Logic), ZFC proves CH^{V^*}. Thus ZFC proves that there is V^* in which a contradiction holds. As we saw in the course Mathematical logic, ZFC also proves that there is no model, in particular, no V^* which satisfies a contradiction and thus we have a proof of a contradiction from ZFC.

9.2 Exercise. *Does the proof of Theorem 8.11 show that ZFC+"$ZFC^*\vdash CH^*$" is inconsistent? If it does, explain why and if it does not, again explain why.*

10. Continuum hypothesis

In Section 3 we proved the consistence of CH by showing that it is true in L. In this section we show how to prove this by using forcing.

10.1 Definition. *We say that partial order P is κ-closed if for all $\alpha < \kappa$ and all $p_i \in P$, $i < \alpha$, the following holds: If for all $i < j < \alpha$, $p_j \leq p_i$, then there is $p \in P$ such that $p \leq p_i$ for all $i < \alpha$.*

10.2 Theorem. *Suppose P is κ-closed, G is P-generic over V, $X \in V$, $Y \in V[G]$, $Y \subseteq X$ and in $V[G]$, $|Y| < \kappa$. Then $Y \in V$.*

Notice that above we do not yet know that κ is a cardinal in $V[G]$.

Proof. So in $V[G]$, there is $\lambda < \kappa$ and $f : \lambda \to X$ such that $Y = rng(f)$ and let $\dot{f}$, $\dot{Y}$ and $\hat{A}$ be P-names for f, Y and $A = P(X)^V \in V$. Then there is $p \in P$ which forces that $\dot{Y} = rng(\dot{f})$ and $dom(\dot{f}) = \hat{\lambda}$ and $\dot{Y} \notin \hat{A}$ and $\dot{Y} \subseteq \hat{X}$.

For all $\gamma \leq \lambda$, we construct $p_\gamma \in P$ and $x_{\gamma+1} \in X$ as follows:

(i) $p_0 = p$,

(ii) $p_{\gamma+1}$ is such that $p_{\gamma+1} \leq p_\gamma$ and for some $x_{\gamma+1} \in X$, $p_{\gamma+1}$ forces that $\dot{f}(\hat{\gamma}) = \hat{x_{\gamma+1}}$,

(iii) if γ is a limit ordinal, then p_γ is any element of P such that $p_\gamma \leq p_i$ for all $i < \gamma$.

Let $Z = \{x_{\gamma+1} | \ \gamma < \lambda\} \in V$. Then p_λ forces that $rng(\dot{f}) = \hat{Z} \in \hat{A}$, a contradiction. $\square$

10.3 Exercise. *Suppose P is κ-closed, $\lambda \leq \kappa$ is a cardinal (in V) and G is P-generic over V. Show that λ is a cardinal in $V[G]$.*

10.4 Theorem. *Con(ZFC) implies Con(ZFC+CH).*

Proof. Let $\kappa = 2^\omega$ and let P be the set of all functions $f : \alpha \to \kappa$, $\alpha < \omega_1$, ordered by the inverse inclusion. Clearly P is ω_1-closed. Let G be P-generic over V and $f = \cup G \in V[G]$.

10.4.1 Exercise. *Show that f is a surjection from $(\omega_1)^V$ onto κ.*

By Theorem 10.2, in $V[G]$ there are no new subsets of ω and thus $P(\omega)^V = P(\omega)^{V[G]}$. Also by Exercise 10.3, $(\omega_1)^V = (\omega_1)^{V[G]}$ and so, in $V[G]$, $|P(\omega)| \leq \omega_1$ and thus CH holds. $\square$

In the literature it is common to say that some claim ϕ is consistent when one actually means that Con(ZFC) implies Con(ZFC+ϕ).

10.5 Exercise. *Prove by forcing the consistency of the following claims:*

(i) $2^\omega = \omega_1$ *and* $2^{\omega_1} = \omega_2$.

(ii) $2^\omega = 2^{\omega_1} = \omega_2$. *Try to find a simple proof that does not use Exercise 8.12 (ii).*

(iii) $2^\omega = \omega_1$ *and* $2^{\omega_1} > \omega_2$.

11. Iterated forcing - the starting point

In forcing, finding a suitable partial order is the main difficulty (keeping in mind that one also has to show that the partial order works). From the literature one can find several methods that are developed to help one to find these partial orders. The most used method is iterated forcing. We start by going back to Exercise 10.5 and do the constructions in a very complicated way. This helps in the next section.

In iterations the requirement that the partial order $P = (P, \le, 1)$ must satisfy that $p \le q$ and $q \le p$ implies that $p = q$, causes technical inconveniences. Thus we lift this requirement i.e. we require only that $\le$ is transitive and reflexive. Then pEq if $p \le q$ and $q \le p$ is an equivalence relation and P/E is a partial order in the old sense when one defines $p/E \le q/E$ if $p \le q$ and P and P/E work in forcing exactly the same way (exercise). And if one wants, one can replace all partial orders P with P/E everywhere below.

Throughout this section $P = (P, \le, 1)$ is a partial order (in V and in our new sense).

11.1 Definition.

(i) We say that $Q = (\dot{Q}, \dot{\le}, \dot{1}) = (Q, \le, 1)$ *is a* P*-name of a partial order if* $\dot{Q}$, $\dot{\le}$ *and* $\dot{1}$ *are* P*-names and* 1 *forces that* $\dot{\le}$ *is a partial order of* $\dot{Q}$ *with the largest element* $\dot{1}$ *and* $(\dot{1}, 1) \in Q$. *We will write* Q *for* $\dot{Q}$ *etc. It should be clear from the context what we mean.*

(ii) $P \star Q$ *is the set*

$$\{(p,\tau) |\ p \in P,\ \exists q \in P((\tau, q) \in Q),\ p \Vdash \tau \in Q\}$$

ordered by the following partial order: $(p,\tau) \le (q,\sigma)$ *if* $p \le q$ *and* $p \Vdash \tau \le \sigma$. *(The largest element is* $(1,1)$*.) The set of those* P*-names* τ *for which there is* $p \in P$ *such that* $(\tau, p) \in Q$ *is denoted by* $Dom(Q)$.

(iii) $i : P \to P \star Q$ *is the function* $i(p) = (p, 1)$.

From now on we let Q be a P-name for a partial order and i as in Definition 11.1 (iii).

11.2 Exercise. i *is a complete embedding (see [Ku]), in particular,*

(i) if $p, q \in P$ *and* $p \le q$, *then* $i(p) \le i(q)$,

(ii) if $p, q \in P$, *then* $p \perp q$ *iff* $i(p) \perp i(q)$,

(iii) if $(p,\tau) \in P \star Q$ *and* $q \le p$, *then* $(p,\tau) || i(q)$.

11.3 Exercise. *Suppose* K *is* $P \star Q$*-generic over* V. *Show that* $K_P = i^{-1}(K)$ *is* P*-generic filter over* V.

11.4 Definition.

(i) If G *is a* P*-generic over* V *and* $H \subseteq Q_G$, *then* $G \star H$ *is the set of those* $(p,\tau) \in P \star Q$ *such that* $p \in G$ *and* $\tau_G \in H$.

(ii) If K *is* $P \star Q$*-generic over* V *and* $G = K_P$, *then* K_Q *is the set of those* τ_G *such that for some* $q \in P$, $(q,\tau) \in K$. *Notice that* $(q,\tau) \in K$ *implies that* $q \in G$.

11.5 Lemma. *Suppose K is $P\star Q$-generic over V, $G=K_P$ and $H=K_Q$. Then H is a Q_G-generic filter over $V[G]$, $K=G\star H$ and $V[K]=V[G][H]$.*

Proof. We start by showing that if $x\in H$, $y\in Q_G$ and $y\geq_G x$, then $y\in H$, the rest of the claim that H is a filter is left as an exercise. Since $x\in H$, there are $p\in G$ and τ such that $(p,\tau)\in K$ and $x=\tau_G$. Since $y\in Q_G$, there is $\sigma\in dom(Q)$ such that $y=\sigma_G$. But then there is $r\in G$ such that $r\Vdash\sigma\geq\tau\wedge\sigma\in Q$. Then $(r,1)\in K$ and we can find $(q,\delta)\in K$ such that $(q,\delta)\leq(r,1),(p,\tau)$. Then $q\Vdash\delta\leq\tau\leq\sigma$ and $q\leq r$. It follows that $(q,\delta)\leq(r,\sigma)\in P\star Q$ and so $(r,\sigma)\in K$ i.e. $y=\sigma_G\in H$.

H is Q_G-generic over $V[G]$: For this let δ be a P-name for a dense subset of Q_G i.e. some $p\in G$ forces that δ is a dense subset of Q. But then $D=\{(q,\tau)\in P\star Q|\ q\leq p,\ q\Vdash\tau\in\delta\}\cup\{(q,\tau)\in P\star Q|\ q\perp p\}$ is dense in $P\star Q$ (exercise). Thus there is $(q,\tau)\in K\cap D$. Since K is a filter, $q||p$ and so $\tau_G\in\delta_G\cap H$.

$K=G\star H$: The direction $\subseteq$ is immediate by the definitions and so we prove only that $G\star H\subseteq K$: So suppose $(p,\tau)\in G\star H$. Then $p\in G$ i.e. $(p,1)\in K$ and $\tau_G\in H$ i.e. for some $q\in P$, $(q,\tau)\in K$ (and p forces that $\tau\in Q$ since $G\star H\subseteq P\star Q$). Since K is a filter there is some $(r,\rho)\in K$ such that $(r,\rho)\leq(p,1),(q,\tau)$. But then $(r,\rho)\leq(p,\tau)$ and so $(p,\tau)\in K$.

$V[K]=V[G][H]$ is left as an exercise. (Hint: Suppose that $V\subseteq V'$ are transitive countable $\in$-models that satisfy ϕ^* for all $\phi\in ZFC$, $P\in V$ is a partial order and $G\in V'$ is P-generic over V. Start by showing that $V[G]\subseteq V'$.)

11.6 Exercise. *Suppose G is P-generic over V and H is Q_G-generic over $V[G]$. Then $G\star H$ is $P\star Q$-generic filter over V. Hint: Suppose $D\in V$ is dense in $P\star Q$ and let $\tau=\{(\delta,p)|\ (p,\delta)\in D\}$. Start by showing that τ_G is dense in Q_G.*

11.7 Exercise. *Suppose P has ccc, $X\in V$ and τ is a P-name of which 1 forces that $\tau\subseteq\hat{X}$ and that τ is countable. Show that there exists a countable $Y\subseteq X$ in V such that $1\Vdash\tau\subseteq\hat{Y}$. Hint: Choose a P-name $\dot{f}$ such that 1 forces that $\dot{f}$ is a function from $\hat{\omega}$ onto τ and repeat the argument from the proof of Theorem 8.3.*

11.8 Lemma. *If P has ccc and 1 forces that Q has ccc, then $P\star Q$ has ccc.*

Proof. For a contradiction, suppose $\{(p_i,\tau_i)\in P\star Q|\ i<\omega_1\}$ is an antichain. Let $\delta=\{(\hat{p_i},p_i)|\ i<\omega_1\}$.

11.8.1 Exercise. *Show that if G is P-generic over V, then δ_G is a countable subset of P in $V[G]$. Hint: Any two elements of δ_G are compatible.*

Thus by Exercise 11.7, there is countable $Y\subseteq P$ such that $1\Vdash\delta\subseteq\hat{Y}$. But since for all $i<\omega_1$, $p_i\Vdash\hat{p_i}\in\delta$, the set $\{p_i|\ i<\omega_1\}$ is countable. Thus there is an uncountable set $X\subseteq\omega_1$ such that for all $i,j\in X$, $p_i=p_j=p$. Since 1 forces that Q has ccc, there are $q\leq p$ and $i,j\in X$, $i\neq j$, such that $q\Vdash\tau_i||\tau_j$. But then $(p_i,\tau_i)||(p_j,\tau_j)$, a contradiction. □

12. Finite support iteration

Now we are ready to define finite support iterations:

12.1 Definition. *We say that* $(P_\gamma, Q_\gamma)_{\gamma\le\alpha}$ *is a finite support iteration if the following holds:*

(i) P_0 *is the one element partial order* $\{\emptyset\}$.

(ii) $Q_\gamma = (Q_\gamma, \le, 1)$ *is a* P_γ*-name for a partial order.*

(iii) $P_{\gamma+1}$ *is the set of all functions* p *with domain* $\gamma+1$ *such that* $p\restriction\gamma \in P_\gamma$ *and* $\tau = p(\gamma)$ *is such that for some* $q \in P_\gamma$, $(\tau, q) \in Q_\gamma$ *and* $p\restriction\gamma \Vdash \tau \in Q_\gamma$. $P_{\gamma+1}$ *is ordered so that* $p \le q$ *if* $p\restriction\gamma \le q\restriction\gamma$ *and* $p\restriction\gamma \Vdash p(\gamma) \le q(\gamma)$. *(Notice that then* $P_{\gamma+1}$ *is isomorphic with* $P_\gamma \star Q_\gamma$.*)*

(iv) For limit γ, P_γ *is the set of all functions* p *with domain* γ *such that for all* $\beta<\gamma$, $p\restriction\beta \in P_\beta$ *and the support*

$$supp(p) = \{\beta < dom(p) |\ p(\beta) \ne 1\}$$

is finite. P_γ *is ordered so that* $p \le q$ *if for all* $\beta < \gamma$, $p\restriction\beta \le q\restriction\beta$.

Notice that for all $0 < \beta \le \alpha$, the maximal element of P_β is the element $p \in P_\beta$ such that $p(\gamma) = 1$ for all $\gamma < \beta$. Also by an easy induction one can see that for all $\beta \le \alpha$ and $p \in P_\beta$, $supp(p)$ is finite.

To show that $p \in P_\alpha$ it is enough to show that p is a function with domain α, for all $\beta < \alpha$, $p(\beta) \in dom(Q_\beta)$, $supp(p)$ is finite and that for all $\beta < \alpha$, $p\restriction\beta \Vdash p(\beta) \in Q_\beta$. And $p \le q$ iff for all $\beta<\alpha$, $p\restriction\beta \Vdash p(\beta) \le q(\beta)$ (exercise).

From now on, $(P_\gamma, Q_\gamma)_{\gamma\le\alpha}$ is a finite support iteration. Notice that Q_α does not play a role in the definition of P_α (it is there for notational reasons).

12.2 Definition. *For* $\gamma \le \beta \le \alpha$, *by* $i_{\gamma\beta}$ *we mean the function from* P_γ *to* P_β *such that for all* $p \in P_\gamma$, $i_{\gamma\beta}(p)$ *is the element* $q \in P_\beta$ *for which* $q\restriction\gamma = p$ *and for all* $\delta \in \beta - \gamma$, $q(\delta) = 1$.

12.3 Exercise. *Suppose* $\gamma \le \beta \le \alpha$, $p, p' \in P_\gamma$ *and* $q, q' \in P_\beta$.

(i) If $q \le q'$, *then* $q\restriction\gamma, q'\restriction\gamma \in P_\gamma$ *and* $q\restriction\gamma \le q'\restriction\gamma$.

(ii) If $p \le p'$, *then* $i_{\gamma\beta}(p) \le i_{\gamma\beta}(p')$.

(iii) If $q\restriction\gamma \perp q'\restriction\gamma$, *then* $q \perp q'$.

(iv) If $supp(q) \cap supp(q') \subseteq \gamma$, *then* $q \perp q'$ *iff* $q\restriction\gamma \perp q'\restriction\gamma$.

(v) $p \perp p'$ *iff* $i_{\gamma\beta}(p) \perp i_{\gamma\beta}(p')$.

(vi) Suppose $p = q\restriction\gamma$ *and* $p' \le p$. *Show that* $r = (q - p) \cup p' \in P_\beta$ *and* $r \le q$.

12.4 Corollary. *Suppose* $\gamma < \beta \le \alpha$, G *is* P_β*-generic over* V *and* $G' = i^{-1}_{\gamma\beta}(G)$. *Then* G' *is* P_γ*-generic over* V.

Proof. As in the previous section. □.

12.5 Exercise. *Suppose that for all* $\gamma < \alpha$, 1 *forces that* Q_γ *has ccc. Show that* P_α *has ccc. Hint: Prove by induction on* $\beta \le \alpha$ *that* P_β *has ccc. The successor steps follow immediately from Lemma 11.8 and for limit cases, make a counter assumption and use Exercise 12.3 (iv) and (in the case* $cf(\beta) = \omega_1$*)* Δ*-lemma for the supports of the elements in the antichain.*

12.6 Definition. *Let $\gamma < \alpha$ and G be P_γ-generic over V. By P^γ_G we mean the set of all functions p with domain $\alpha - \gamma$ such that for some $q \in G$, $q \cup p \in P_\alpha$. We partially order P^γ_G so that $p \le p'$ if there is some $q \in G$, $q \cup p, q \cup p' \in P_\alpha$ and $q \cup p \le q \cup p'$. (Exercise: Show that this is a partial order.)*

By P^γ we mean a P_γ-name $(\dot{P}^\gamma, \dot{\le}, \dot{1})$ for a partial order so that for all P_γ-generic G over V, $(P^\gamma)_G = P^\gamma_G$ (exists by Exercise 7.14 (i)). We may always choose $\dot{P}^\gamma$ to be the set $\{(\hat{q}, p) |\ p \in P_\gamma,\ dom(q) = \alpha - \gamma, p \cup q \in P_\alpha\}$. As usually, by P^γ we denote also $\dot{P}^\gamma$ etc.

12.7 Exercise. *Suppose $p, q \in P_\alpha$, $\gamma < \alpha$, $p_0 = p \restriction \gamma \le q \restriction \gamma = q_0$ and denote $p_1 = p \restriction (\alpha - \gamma)$ and $q_1 = q \restriction (\alpha - \gamma)$. Show that if p_0 forces that $\hat{p_1} \le \hat{q_1}$ (in the ordering of P^γ), then $p \le q$. Hint: Suppose not and pick the least $\beta \le \alpha$ such that $p \restriction \beta \not\le q \restriction \beta$. Then $\beta = \delta + 1$ for some $\gamma \le \delta < \alpha$ and $p \restriction \delta$ does not force that $p(\delta) \le q(\delta)$ i.e. there is $r \in P_\delta$ such that $r \le p \restriction \delta$ and r forces that $p(\delta) \not\le q(\delta)$. Now pick P_γ-generic G such that $r \restriction \gamma \in G$.*

The following lemma is not the most useful form of splitting iterated forcing into pieces but still gives some idea of what is going on and suffices for our purposes (in fact we will not even need the full claim).

12.8 Lemma. *Suppose $\gamma < \alpha$, G is P_α-generic over V, $G_\gamma = i^{-1}_{\gamma\alpha}(G)$ and $G^\gamma = \{p \in P^\gamma_{G_\gamma} |\ \exists q \in P_\gamma (q \cup p \in G)\}$. Then G^γ is $P^\gamma_{G_\gamma}$-generic over $V[G_\gamma]$ and $V[G] = V[G_\gamma][G^\gamma]$.*

Proof. This is basically the same than what was done in Section 11 and we prove only that G^γ is $P^\gamma_{G_\gamma}$-generic over $V[G_\gamma]$: For this, let τ be a P_γ-name such that τ_{G_γ} is a dense subset of $P^\gamma_{G_\gamma}$. Then there is $p' \in G_\gamma$ such that it forces that τ is a dense subset of P^γ (keep in mind that $P^\gamma_{G_\gamma} = (P^\gamma)_{G_\gamma}$). For any $q \in P_\alpha$, we denote $q_0 = q \restriction \gamma$ and $q_1 = q \restriction (\alpha - \gamma)$. Let $p \in P_\alpha$ be such that $p_0 \le p'$.

12.8.1 Exercise. *Show that it is enough to find $q \le p$ such that q_0 forces that $\hat{q_1} \in \tau$. Hint: As in the proof of Lemma 11.5.*

Since p_0 forces that τ is dense in P^γ, there is δ such that p_0 forces that $\delta \le \hat{p_1}$ and $\delta \in \tau$ (notice that p_0 forces that $\hat{p_1} \in P^\gamma$, exercise). Let H be P_γ-generic over V such that $p_0 \in H$. Then in $V[H]$ there are $r \in P^\gamma_H$ such that $\delta_H = r$. Let $s \in H$ be such that $s \cup r \in P_\alpha$ and $s' \in H$ such that it forces that $\delta = \hat{r}$. By Exercise 12.3 (vi), we may assume that $s = s' \le p_0$ and then, by Exercise 12.7, $q = s \cup r$ is as wanted. ▫

12.9 Lemma. *Let G be P_α-generic over V and $G(\gamma) = \{p(\gamma)_{G_\gamma} |\ p \in G\}$. Then $G(\gamma)$ is $(Q_\gamma)_{G_\gamma}$-generic over $V[G_\gamma]$ and $V[G_{\gamma+1}] = V[G_\gamma][G(\gamma)]$.*

Proof. By Corollary 12.4, $G_{\gamma+1}$ is $P_{\gamma+1}$-generic over V and by definitions, $P_{\gamma+1}$ is isomorphic with $P_\gamma \star Q_\gamma$. By checking the isomorphism and using Lemma 11.5, $G'(\gamma) = \{p(\gamma)_{G_\gamma} |\ p \in G_{\gamma+1}\}$ is $(Q_\gamma)_{G_\gamma}$-generic over $V[G_\gamma]$. But clearly $G(\gamma) = G'(\gamma)$. Then also the second claim follows from Lemma 11.5. ▫

12.10 Exercise. *Suppose that* $cf(\alpha) \geq \omega_1$, *for all* $\gamma < \alpha$, 1 *forces that* Q_γ *has ccc,* $p \in P_\alpha$ *forces that* τ *is a function from* $\hat{\omega}$ *to* $\hat{\omega}$ *and* G *is* P_α*-generic over* V *such that* $p \in G$. *Show that there is* $\gamma < \alpha$ *such that* $\tau_G \in V[G_\gamma]$. *Hint: The hint for Exercise 8.12 (ii).*

13. Dominating number

As an application of iterated forcing, we look at dominating number.

13.1 Definition.

(i) For $f, g \in \omega^\omega$, *we write* $f <^* g$ *and say that* g *eventually dominates* f, *if there is* $n \in \omega$ *such that for all* $n < m < \omega$, $f(m) < g(m)$.

(ii) We let D *to be the set of all dominating families i.e. the set of all those* $A \subseteq \omega^\omega$ *such that for all* $f \in \omega^\omega$ *there is* $g \in A$ *which eventually dominates* f. *By* $\mathbf{d}$ *(dominating number) we mean* $min\{|A| \mid A \in D\}$.

13.2 Exercise.

(i) Show that $\mathbf{d} \geq \omega_1$.

(ii) Suppose that CH holds in V *and let* P *be the set of all* $p : X_p \to \omega$ *such that* $X_p \subseteq \omega_2 \times \omega$ *is finite. We partially order* P *by inverse inclusion (as before) i.e.* $p \leq q$ *if* $q \subseteq p$. *Let* G *be* P*-generic over* V. *Show that* $\mathbf{d} \geq \omega_2$ *in* $V[G]$. *Hint: Suppose not. Then there is* $f : \omega_1 \times \omega \to \omega$ *such that* $\{f_\alpha | \ \alpha < \omega_1\}$ *is a dominating family, where* $f_\alpha(n) = f(\alpha, n)$. *Then find a nice name (see Exercise 8.12) for this function and notice that there is* $\gamma < \omega_2$ *such that if* p *appears in any of the maximal antichains of the nice name, then* $X_p \subseteq \gamma \times \omega$. *Now look at the function* $g(n) = \cup G(\gamma, n)$. *The proof of Theorem 13.6 below may also give ideas.*

Dominating number is one example of so called cardinal invariants. Another example of such invariants is $Cov(M)$ i.e. the least cardinal κ for which there are meager (aka meagre aka of first category) subsets A_i, $i < \kappa$, of reals $\mathbf{R}$ such that $\bigcup_{i<\kappa} A_i = \mathbf{R}$. It is known that $Cov(M) \leq \mathbf{d}$.

13.3 Definition. *By* $P_{\mathbf{d}}$ *we mean the partial order* $(P_{\mathbf{d}}, \leq, (\emptyset, \emptyset))$, *where* $P_{\mathbf{d}}$ *is the set of pairs* $p = (f_p, F_p)$ *such that* $f_p : n_p \to \omega$ *for some* $n_p < \omega$ *and* F_p *is a finite set of functions from* ω *to* ω. $P_{\mathbf{d}}$ *is ordered so that* $p \leq q$ *if* $f_q \subseteq f_p$, $F_q \subseteq F_p$ *and for all* $n_q \leq i < n_p$ *and* $h \in F_q$, $f_p(i) > h(i)$.

13.4 Exercise.

(i) Show that $P_{\mathbf{d}}$ *has ccc.*

(ii) Let G *be* $P_{\mathbf{d}}$*-generic over* V *and* $g = \bigcup_{p \in G} f_p$. *Show that* g *is a function from* ω *to* ω.

13.5 Lemma. *Let* G *be* $P_{\mathbf{d}}$*-generic over* V *and* $g = \bigcup_{p \in G} f_p$. *Then for all* $h : \omega \to \omega$ *from* V, $h <^* g$.

Proof. Suppose not and let $\dot{g}$ be a $P_{\mathbf{d}}$-name for g (i.e. for all $P_{\mathbf{d}}$-generic H over V, $\dot{g}_H = \bigcup_{p \in H} f_p$). Then there are $h : \omega \to \omega$ and $p \in G$ such that p forces the negation of $\hat{h} <^* \dot{g}$. Let $q \in P_{\mathbf{d}}$ be such that $f_q = f_p$ and $F_q = F_p \cup \{h\}$ and let H be $P_{\mathbf{d}}$-generic over V such that $q \in H$. Then for all $i \geq n_p$, $h(i) < (\dot{g}_H)(i)$. Since $q \leq p$, we have a contradiction. ▫

13.6 Theorem. *Con(ZFC) implies Con(ZFC+$\mathbf{d} = \omega_1 < 2^\omega$).*

Proof. By Theorem 8.11, we may assume that $2^\omega > \omega_1$ in V. Let $(P_g, Q_\gamma)_{\gamma \leq \omega_1}$ be a finite support iteration such that for all $\gamma \leq \omega_1$, Q_γ is a P_γ-name for $(P_{\mathbf{d}})^{V[G_\gamma]}$ (i.e. for all P_γ-generic G over V, $(Q_\gamma)_{G_\gamma}$ satisfies in $V[G_\gamma]$ the definition of $P_{\mathbf{d}}$).

Let G be P_{ω_1}-generic over V. By Exercise 13.4 (i) and Lemma 11.8, P_{ω_1} has ccc and thus $(\omega_1)^V = (\omega_1)^{V[G]}$ and in $V[G]$ $2^\omega > \omega_1$ by Corollary 8.4. Thus it is enough to show that in $V[G]$, $A = \{f_\gamma |\ \gamma < \omega_1\}$, where $f_\gamma = \bigcup_{p \in G(\gamma)} f_p$, see Lemma 12.9, has the property that for all $g : \omega \to \omega$, there is $f \in A$ such that $g <^* f$. But this is clear: By Exercise 12.10, there is $\gamma < \omega_1$ such that $g \in V[G_\gamma]$. By Lemma 12.9, $G(\gamma)$ is $(Q_\gamma)_{G_\gamma}$-generic over $V[G_\gamma]$ and thus by Lemma 13.5, $g <^* f_\gamma$. ▫

14. Further exercises

In this section, in the form of exercises we look at how to kill stationary subsets of ω_1 by forcing (killing stationary subsets of $\kappa > \omega_1$ is much harder). Recall that by $X^{<\omega}$ we mean the set of all functions $f : n \to X$, $n < \omega$.

14.1 Definition. *Let $P = (P, \leq, 1)$ be a partial order.*

(i) $\Gamma(P)$ is a game of two players, I and II and it lasts ω rounds. At each round $n < \omega$ first I chooses some $p_n \in P$ and then II chooses $q_n \in P$. II must choose so that $q_n \leq p_n$ and in rounds $n > 0$, I must choose so that $p_n \leq q_{n-1}$. II wins if there is $q \in P$ such that for all $n < \omega$, $q \leq q_n$.

(ii) Winning strategy for I in $\Gamma(P)$ is a function $\sigma : P^{<\omega} \to P$ such that no matter how II plays I wins if at each round $n < \omega$, I chooses $\sigma(q_0, ..., q_{n-1})$.

(iii) We say that P is hopeless for II, if I has a winning strategy in $\Gamma(P)$.

14.2 Exercise. *Suppose that P is not hopeless for II and G is P-generic over V.*

(i) Suppose $X \in V$, $Y \in V[G]$, $Y \subseteq X$ and Y is countable. Show that $Y \in V$.

(ii) Show that $\omega_1^V = \omega_1^{V[G]}$.

Fix $S \subseteq \omega_1$ so that $\omega_1 - S$ is stationary (S may also be stationary). By $P(S)$ we mean the set of all strictly increasing $f : \alpha + 1 \to \omega_1$, $\alpha < \omega_1$, such that $rng(f) \cap S = \emptyset$ and for all limit $\gamma \leq \alpha$, $f(\gamma) = \cup_{\beta < \gamma} f(\beta)$ and we order $P(S)$ by inverse inclusion.

14.3 Exercise.

(i) Show that for all $f : \omega_1^{<\omega} \to \omega_1$, the set $C_f = \{\alpha < \omega_1 |\ f(\alpha^{<\omega}) \subseteq \alpha\}$ is cub.

(ii) Show that $P(S)$ is not hopeless for II. Hint: For a contradiction, suppose that σ is a winning strategy for I. Then think the case when at each round $n < \omega$, II plays so that she first chooses some $\gamma_n < \omega_1$ so that $\gamma_n > \cup rng(p_n)$ and then answers by $q_n = p_n \cup \{(dom(p_n), \gamma_n)\}$. Then apply (i) to the function $f(\gamma_0, ..., \gamma_{m-1}) =$

$\cup rng(p_m)$, *where for all* $i \leq m$, $p_i = \sigma(q_0, ..., q_{i-1})$ *and for all* $i < m$ $q_i = p_i \cup \{(dom(p_i), \gamma_i)\}$ *(if for some* $i < m$, $q_i \notin P(S)$, *let* $f(\gamma_0, ..., \gamma_{m-1}) = 0$*).*

14.4 Exercise. *Let* G *be* $P(S)$*-generic over* V *and* $C = rng(\cup G)$. *Show that* C *is a cub subset of* ω_1 *and* $C \cap S = \emptyset$ *(i.e.* S *is not stationary in* $V[G]$*).*

In Exercise 14.4 the assumption that $\omega_1 - S$ is stationary is necessary:

14.5 Exercise. *Suppose* P *is a partial order,* G *is* P*-generic over* V, $C \subseteq \omega_1$ *is in* V *and* $(\omega_1)^V = (\omega_1)^{V[G]}$.

(i) Show that C *is a cub subset of* ω_1 *in* V *iff* C *is a cub subset of* ω_1 *in* $V[G]$.

(ii) Since it is possible that $\omega_1 - S$ *is not cub, why the direction from right to left in (i) does not contradict Exercise 14.4?*

References

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