Part III: Weihrauch degrees

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Overview

- **Day 1** Computability on Uncountable Structures: Computability on Cantor space, represented spaces and synthetic topology
- **Day 2** Examples: The continuous functions on the unit interval; probability measures; the space of countable ordinals
- **Day 3** Non-computability: An introduction to Weihrauch degrees

A very short overview

- Weihrauch reducibility compares multivalued functions between represented spaces.
- The induced degrees have a rich algebraic structure.
- Many mathematical theorems can be interpreted as multivalued functions, with the associated Weihrauch degrees measuring the computational content of the theorem.
- The algebraic operations have logic-like meanings regarding such theorems.
- Many concrete theorems have been classified via Weihrauch reducibility; and this classification is reminiscent of reverse mathematics and Brouwerian counterexamples.
- Various techniques have been developed to prove separation results.

Outline

Introducing Weihrauch reducibility

Closed choice

Connections to other approaches

Some separation techniques

Some more classifications

Further reading

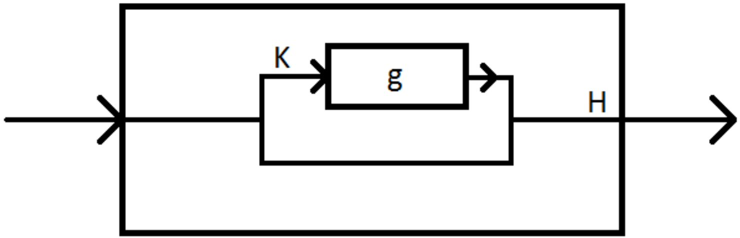
Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $H\langle \mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, GK\rangle$ is a realizer of f for every realizer G of g.



The idea behind Weihrauch reducibility

1. Identify a theorem

$$\forall x \in \mathbf{X} \,.\, \neg\neg D(x) \Rightarrow \exists y \in \mathbf{Y}\ T(x, y)$$

   with the multi-valued function $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $\mathrm{dom}(T) = D$ obtained by Skolemization.

2. Then compare theorems via Weihrauch-reducibility to learn about their *constructive content*.

Non-uniqueness

Observation (Brattka)

The Baire Category Theorem has 4 different incarnations as a multivalued function:

1. $\forall (A_i)_{i\in\mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{A}(\mathbf{X}))\ \exists x \in \mathbf{X}$
   $\left((\forall i \in \mathbb{N}\ A_i \text{ is nowhere dense}) \to x \notin \bigcup_{i\in\mathbb{N}} A_i\right)$
2. $\forall (A_i)_{i\in\mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{A}(\mathbf{X}))\ \exists n \in \mathbb{N}$
   $\left(\bigcup_{i\in\mathbb{N}} A_i = \mathbf{X} \to A_n \text{ is somewhere dense}\right)$
3. $\forall (A_i)_{i\in\mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{V}(\mathbf{X}))\ \exists x \in \mathbf{X}$
   $\left((\forall i \in \mathbb{N}\ A_i \text{ is nowhere dense}) \to x \notin \bigcup_{i\in\mathbb{N}} A_i\right)$
4. $\forall (A_i)_{i\in\mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{V}(\mathbf{X}))\ \exists n \in \mathbb{N}$
   $\left(\bigcup_{i\in\mathbb{N}} A_i = \mathbf{X} \to A_n \text{ is somewhere dense}\right)$

Algebraic structure

We have the following operations on Weihrauch degrees:

1. $f \sqcap g$, returning either an answer to f or an answer to g
2. $f \sqcup g$, letting us choose between f and g
3. $f \times g$, letting us both f and g in parallel
4. $f \star g$, letting us first use g, then f
5. $f \rightarrow g = \min\{h \mid g \leq_W f \star h\}$
6. $f^*, f^\diamond$ letting us use f finitely many times, in parallel or consecutively
7. $\widehat{f}$, letting us use f countably many times in parallel

Algebraic structure, continued

Theorem

- $(\mathfrak{W}, \sqcap, \sqcup)$ *is a distributive lattice.*
- *It lacks some infinite infima, and all non-trivial infinite suprema.*
- *It is not a Heyting algebra.*
- $(\mathfrak{W}, \sqcup, \times, ^*)$ *is a Kleene-algebra.*

Other reducibilities

Theorem

- *The Turing degrees embed into the Weihrauch degrees*
- *and they do so via the Medvedev degrees.*
- *In a non-canonic way, the many-one degrees also embed into the Weihrauch degrees.*

Closed choice

Definition

$\mathrm{C}_{\mathbf{X}} :\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ is defined via $x \in \mathrm{C}_{\mathbf{X}}(A)$ iff $x \in A$.

- If $\mathbf{X}$ is T_1 and $|\mathbf{X}| > 1$, then $\mathrm{C}_{\mathbf{X}}$ is not computable.
- $\mathrm{C}_{[0,1]^k} \equiv_{\mathrm{W}} \mathrm{C}_{\{0,1\}^{\mathbb{N}}}$.
- $\mathrm{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathrm{C}_{\{0,1\}^{\mathbb{N}}} \star \mathrm{C}_{\{0,1\}^{\mathbb{N}}}$ and $\mathrm{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \widehat{\mathrm{C}_{\{0,1\}}}$.
- $\mathrm{C}_{[0,1]} |_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.

Non-deterministic computability

Definition

A non-deterministic Type-2 machine with advice space $\mathbf{Z}$ computes a multivalued function $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ as follows:

1. On input $x \in \mathbf{X}$, guess some $z \in \mathbf{Z}$.
2. Either: Halt and reject the guess.
3. Or: Run indefinitely, and output some $y \in f(x)$.

Such that for any $x \in \mathbf{X}$ there is some $z \in \mathbf{Z}$ leading to case 3.

Observation

f is non-deterministically computable with advice space $\mathbf{Z}$ iff $f \leq_W C_{\mathbf{Z}}$.

Restrictions of closed choice for calibration

1. Connected closed choice is equivalent to Brouwer's Fixed Point theorem.
2. All-or-unique choice characterizes finding Nash equilibria in bimatrix games and Gaussian elimination.
3. Convex choice characterizes the Browder-Goehde-Kirk Fixed Point theorem.

lim and its iterations

Consider $\lim :\subseteq \mathcal{C}(\mathbb{N}, \mathbb{N}^{\mathbb{N}}) \to \mathbb{N}^{\mathbb{N}}$, $\mathrm{EC} : \mathcal{O}(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ and $\mathrm{Diff} :\subseteq \mathcal{C}(\mathbb{R}, \mathbb{R}) \to \mathcal{C}(\mathbb{R}, \mathbb{R})$.

Proposition (v. Stein 1989, Brattka & Gherardi 2012)

$\lim \equiv_W EC \equiv_W \mathit{Diff} \equiv_W \widehat{\mathrm{C}_{\mathbb{N}}} \equiv_W (\mathrm{id} : \mathbb{R}_< \to \mathbb{R})$

Proposition (v. Stein 1989)

$\lim <_W \lim \star \lim <_W \lim \star \lim \star \lim =: \lim^{(3)} <_W \dots$

Proposition (Brattka 2005)

$f \leq_W \lim^{(n)}$ iff f is effectively Σ^0_{n+1}-measurable.

A connection to reverse mathematics

Observation

Typically, $f \leq_W \mathrm{C}_{\{0,1\}^{\mathbb{N}}}$ corresponds to provability in WKL_0. If f proves WKL_0, then usually $\mathrm{C}_{\{0,1\}} \leq_W f$ (equivalent, $\mathrm{C}_{\{0,1\}^{\mathbb{N}}} \leq_W \widehat{f}$.)

However:

Example

Both Δ^0_1-determinancy and Σ^0_1-determinancy are equivalent to WKL_0 over *RCA*$_0$. However, Δ^0_1-determinancy is computable, whereas Σ^0_1 is Weihrauch-equivalent to $\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$.

Absorption theorems

Theorem (BRATTKA, DE BRECHT & P. 2012)

Let $f : \mathbf{X} \to \mathbf{Y}$ *be a function and* $\mathbf{Y}$ *be admissible. Then* $f \leq_W \mathrm{C}_{\{0,1\}^\mathbb{N}} \star g$ *implies* $f \leq_W g$.

Small digression: This was used to prove the computable Jayne Rogers theorem by DE BRECHT and P..

Absorption theorems II

Definition

Call $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ *fractal*, if there is some $g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ such that $f \equiv_W g$ and for any clopen $A \subseteq \mathbb{N}^\mathbb{N}$, $g|_A \equiv_W f$ or $g|_A \equiv_W 0$. Call f *closed fractal*, if g can be chosen total.

Proposition (LE ROUX & P. 2013)

For fractal f, we have that $f \leq_W g \star \mathrm{C}_{\{0,\ldots,n\}}$ implies $f \leq_W g$.

Proposition (LE ROUX & P. 2013)

For closed fractal f, we have that $f \leq_W g \star \mathrm{C}_\mathbb{N}$ implies $f \leq_W g$.

The displacement principle

Given some collection $\mathfrak{A}$ of closed sets, let $\overline{\mathfrak{A}}$ be all limits of elements of $\mathfrak{A}$. The limit notion is $\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{i\geq n} A_i}$. Let $2\mathfrak{A}$ be all sets containing two disjoint elements of $\mathfrak{A}$.

Theorem (Brattka, Le Roux, Miller & P.)

If $\mathrm{C}_{\{0,1\}} \times f \leq_W \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A}}$*, then* $f \leq_W \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A}\cap 2\overline{\mathfrak{A}}}$*.*

Corollary

Let $\mathfrak{U}(\mathbf{X})$ *be the collection of all closed singletons of* $\mathbf{X}$*, and* $\mathbf{X}$ *compact. Then* $\mathrm{C}_{\{0,1\}} \times f \leq_W \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A}}$ *implies* $f \leq_W \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A}\setminus\mathfrak{U}(\mathbf{X})}$*.*

Corollary

$\mathrm{C}_{\{0,1\}} \times CC_{[0,1]} \nleq_W CC_{[0,1]}$.

Bolzano Weierstrass

Theorem

Every sequence $(x_n)_{n \in \mathbb{N}}$ *in* $[0,1]$ *has at least one accumulation point.*

Theorem (Brattka, Gherardi & Marcone 2012)

Finding an accumulation point of a sequence in $[0,1]$ *is Weihrauch-equivalent to* $\mathrm{C}_{[0,1]} \star \lim$*.*

Idea: $\lim$ gets us the set of accumulation points as a closed set, $\mathrm{C}_{[0,1]}$ then picks. This is all these operations can do.

Analytic functions

- Translating a continuous function that happens to be analytic to an analytic function is equivalent to $\mathrm{C}_{\mathbb{N}}$.
- So is translating an analytic function that happens to be a polynomial to the polynomial.
- Translating a smooth function that is Schwartz to a Schwartz function is equivalent to $\lim$.

Gale-Stewart games

Theorem

Finding winning strategies for Gale-Stewart games with winning conditions on the n-th level of the difference hierarchy is equivalent to $\mathrm{C}_{\{0,1\}^{\mathbb{N}}} \star \lim^{(n)}$.

Recent survey

Vasco Brattka, Guido Gherardi & Arno Pauly:
Weihrauch Complexity in Computable Analysis.
arXiv 1707.03202