Part II: Examples

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Overview

- **Day 1** Computability on Uncountable Structures: Computability on Cantor space, represented spaces and synthetic topology
- **Day 2** Examples: The continuous functions on the unit interval; probability measures; the space of countable ordinals
- **Day 3** Non-computability: An introduction to Weihrauch degrees

Outline

Understanding $\mathcal{O}(\mathbb{N})$

- $\mathcal{O}(\mathbb{N})$ gives just enough information to semidecide membership: Sets are coded by enumerations of its elements
- The computable points in $\mathcal{O}(\mathbb{N})$ are the recursively enumerable sets.
- The continuous functions $f : \mathcal{O}(\mathbb{N}) \to \mathcal{O}(\mathbb{N})$ are the Scott-continuous ones.
- The computable functions $f : \mathcal{O}(\mathbb{N}) \to \mathcal{O}(\mathbb{N})$ are the enumeration operators.

Computable metric spaces

Definition

A computable metric space consists of

1. A metric space (X, d)
2. together with a dense sequence $(a_n)_{n \in \mathbb{N}}$
3. such that $(n, m) \mapsto d(a_n, a_m) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is computable.

The associated representation δ_X lets $p \in \mathbb{N}^{\mathbb{N}}$ code $x \in X$ iff $\forall n \; d(x, a_{p(n)}) < 2^{-n}$.

$\mathcal{C}([0,1],\mathbb{R})$ as computable metric space

1. Equip $\mathcal{C}([0,1],\mathbb{R})$ with the metric $d(f,g) = \max_{x\in[0,1]} d(f(x),g(x))$.
2. and as dense set an enumeration of the rational piecewise linear functions.

Theorem

The resulting computable metric space has the same computability structure as the usual function space $\mathcal{C}([0,1],\mathbb{R})$.

Bisection vs trisection

Definition

Let $\mathrm{mIVT} :\subseteq \mathcal{C}([0,1],\mathbb{R}) \to [0,1]$ map monotone functions f with $f(0) = -1$ and $f(1) = 1$ to their unique root.

Theorem

mIVT *is computable.*

Proof.

While bisection does not work, trisection does. □

More computability on $\mathcal{C}([0,1],\mathbb{R})$

Theorem

1. *Integration is computable on* $\mathcal{C}([0,1],\mathbb{R})$
2. *So are* min *and* max*.*
3. *Differentiation is not.*

The lower reals and valuations

Definition

The represented space $\mathbb{R}_<$ has the reals $\mathbb{R}$ as underlying set, represented as limits of monotonely increasing sequences of rational numbers.

Definition

We obtain the space $\mathcal{P}(\mathbf{X})$ of probability measures over a represented space $\mathbf{X}$ by restricting $\mathcal{C}(\mathcal{O}(\mathbf{X}), \mathbb{R}_<)$ to σ-additive functions μ satisfying $\mu(X) = 1$.

Integration

Theorem

The map $\int : \mathcal{C}(\mathbf{X}, [0,1]) \times \mathcal{P}(\mathbf{X}) \to [0,1]$ *mapping* (f, μ) *to* $\int f d\mu$ *is computable.*

Countable tupling functions

Given $p_0, p_1, \ldots \in \Sigma^{\mathbb{N}}$, let $\langle p_0, p_1, \ldots \rangle \in \Sigma^{\mathbb{N}}$ be defined via $\langle p_0, p_1, \ldots \rangle(\langle n, m \rangle) = p_n(m)$.

Representing the countable ordinals

Definition

We define a representation δ_{nK} of the countable ordinals inductively via:

1. $\delta_{\mathrm{nK}}(0^{\mathbb{N}}) = 0$
2. $\delta_{\mathrm{nK}}(1p) = \delta_{\mathrm{nK}}(p) + 1$
3. $\delta_{\mathrm{nK}}(2p) = \sup_{n \in \mathbb{N}} \delta_{\mathrm{nK}}(p_n)$

We denote the resulting represented space as COrd.

Characterizing the structure

The space COrd is completely characterized by the computability of

1. The constant 0.
2. The map $+1 : \mathrm{COrd} \to \mathrm{COrd}$
3. The map $\mathsf{sup} : \mathrm{COrd}^{\mathbb{N}} \to \mathrm{COrd}$
4. The map $\mathsf{ListLower} : \mathrm{COrd} \rightrightarrows \mathcal{C}(\mathbb{N}, \mathrm{COrd} + \{\mathrm{Skip}\})$ mapping α to some $(x_n)_{n \in \mathbb{N}}$ such that $\{\beta \mid \beta < \alpha\} = \{\beta \in \mathrm{COrd} \mid \exists n\, x_n = \beta\}$.

Computable operations on the countable ordinals

- Ordinal addition, multiplication and exponentiation are computable.
- The only open sets are of the form $\{\alpha \mid \alpha \geq n\}$.
- The map $\mathtt{min} : \mathrm{COrd} \times \mathrm{COrd} \to \mathrm{COrd}$ is computable.
- The map $\mathsf{Bound} : \mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathrm{COrd}) \rightrightarrows \mathrm{COrd}$ mapping f to some α with $\alpha \geq f(p)$ for all $p \in \mathbb{N}^{\mathbb{N}}$ is computable.

Standard text book, very different approach

Klaus Weihrauch:
Computable Analysis.
Springer, 2000.

Probability theory

Pieter Collins.
Computable Stochastic Processes.
arXiv:1409.4667

Countable ordinals

Arno Pauly.
Computability on the space of countable ordinals.
arXiv:1501.00386