Axiomatization of implication for probabilistic independence and unary variants of marginal identity and marginal distribution equivalence

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Helsinki Logic Seminar, March 27, 2024

Overview

Background

- (Probabilistic) team semantics
- Dependencies and independences
- Logical implication and implication problems

PIA+UMI+UMDE implication

- Axiomatization
- A polynomial time algorithm

Team semantics

- For a finite set of variables D and a finite set of values A, an *assignment* is a function $s\colon D \to A$.
- A *team* X is a finite set of assignments $s\colon D \to A$.

x	y	z	u
a	b	c	a
a	b	d	a
b	a	c	a
b	a	d	a

Probabilistic team semantics

- A *probabilistic team* $\mathbb{X}$ is a function $\mathbb{X}\colon X \to (0,1]$ such that $\sum_{s \in X} \mathbb{X}(s) = 1$.

x	y	z	u	#
a	b	c	a	1/4
a	b	d	a	1/4
b	a	c	a	1/4
b	a	d	a	1/4

Probabilistic independence

Let $\bar{x}$ and $\bar{y}$ be (possibly empty) tuples of variables from D.

An atom $\bar{x} \perp\!\!\!\perp \bar{y}$ is a *probabilistic independence atom* (PIA)

For a tuple of variables $\bar{x}$ from D and a tuple of values $\bar{a}$ from A, let

$$|\mathbb{X}_{\bar{x}=\bar{a}}| := \sum_{\substack{s(\bar{x})=\bar{a} \\ s \in X}} \mathbb{X}(s),$$

i.e, $|\mathbb{X}_{\bar{x}=\bar{a}}|$ is the marginal probability of that the variables $\bar{x}$ have the values $\bar{a}$ in probabilistic team $\mathbb{X}$.

$\mathbb{X} \models \bar{x} \perp\!\!\!\perp \bar{y}$ iff $|\mathbb{X}_{\bar{x}=\bar{a}}| \cdot |\mathbb{X}_{\bar{y}=\bar{b}}| = |\mathbb{X}_{\bar{x}\bar{y}=\bar{a}\bar{b}}|$ for all $\bar{a}\bar{b} \in A^{|\bar{x}\bar{y}|}$.

Let $\mathbb{X}$ be the probabilistic team depicted below. Then $\mathbb{X} \models x \perp\!\!\!\perp u$, $\mathbb{X} \models u \perp\!\!\!\perp u$, and $\mathbb{X} \models y \perp\!\!\!\perp z$, but $\mathbb{X} \not\models x \perp\!\!\!\perp y$.

x	y	z	u	#
a	b	c	a	1/4
a	b	d	a	1/4
b	a	c	a	1/4
b	a	d	a	1/4

Unary marginal identity and marginal distribution equivalence

Let x, y be variables from D.

An atom $x \approx y$ is called a *unary marginal identity* (UMI).

An atom $x \approx^* y$ is called a *unary marginal distribution equivalence* (UMDE).

Satisfaction:

(i) $\mathbb{X} \models x \approx y$ iff $|\mathbb{X}_{x=a}| = |\mathbb{X}_{y=a}|$ for all $a \in A$.

(ii) $\mathbb{X} \models x \approx^* y$ iff $\{\{|\mathbb{X}_{x=a}| : a \in A\}\} = \{\{|\mathbb{X}_{y=a}| : a \in A\}\}$.

Example

Let $\mathbb{X}$ be the probabilistic team depicted below. Then $\mathbb{X} \models x \approx y$ and $\mathbb{X} \models y \approx^* z$, but $\mathbb{X} \not\models y \approx z$ and $\mathbb{X} \not\models x \approx^* u$.

x	y	z	u	#
a	b	c	a	1/4
a	b	d	a	1/4
b	a	c	a	1/4
b	a	d	a	1/4

Logical implication and implication problems

A set of atoms Σ is said to *logically imply* another atom σ, if every (probabilistic) team that satisfies the set of atoms Σ also satisfies the atom σ.

We write $\Sigma \models \sigma$ for "Σ logically implies σ".

An *implication problem* is the task of deciding whether a given set of atoms Σ logically implies another given atom σ.

Some related implication problems

Axiomatization and polynomial time algorithm for disjoint PIAs (Geiger et al., 1991). Note that a PIA $\bar{x} \perp \bar{y}$ is disjoint if $\mathrm{Var}(\bar{x}) \cap \mathrm{Var}(\bar{y}) = \emptyset$.

Axiomatization for (general) marginal identity (Hannula et al. 2022).

Axiomatization and polynomial time algorithm for FDs+UMIs+UMDEs (Hirvonen, 2022).

PIA+UMI+UMDE implication

Sound and complete axiomatization (Hirvonen, 2024) for PIA+UMI+UMDE:

UMI1: $x \approx x$

UMI2: If $x \approx y$, then $y \approx x$.

UMI3: If $x \approx y$ and $y \approx z$, then $x \approx z$.

UMDE1: $x \approx^* x$

UMDE2: If $x \approx^* y$, then $y \approx^* x$.

UMDE3: If $x \approx^* y$ and $y \approx^* z$, then $x \approx^* z$.

UMI & UMDE: If $x \approx y$, then $x \approx^* y$.

PIA1: $\emptyset \perp\!\!\!\perp \bar{x}$

PIA2: If $\bar{x} \perp\!\!\!\perp \bar{y}$, then $\bar{y} \perp\!\!\!\perp \bar{x}$.

PIA3: If $\bar{x} \perp\!\!\!\perp \bar{y}$, then $\bar{x}' \perp\!\!\!\perp \bar{y}'$ for all $\mathrm{Var}(\bar{x}') \subseteq \mathrm{Var}(\bar{x})$, $\mathrm{Var}(\bar{y}') \subseteq \mathrm{Var}(\bar{y})$.

PIA4: If $\bar{x} \perp\!\!\!\perp \bar{y}$ and $\bar{x}\bar{y} \perp\!\!\!\perp \bar{z}$, then $\bar{x} \perp\!\!\!\perp \bar{y}\bar{z}$.

PIA5: If $\bar{x} \perp\!\!\!\perp \bar{y}$ and $\bar{z} \perp\!\!\!\perp \bar{z}$, then $\bar{x} \perp\!\!\!\perp \bar{y}\bar{z}$.

PIA & UMDE 1: If $x \approx^* y$ and $y \perp\!\!\!\perp y$, then $x \perp\!\!\!\perp x$.

PIA & UMDE 2: If $x \perp\!\!\!\perp x$ and $y \perp\!\!\!\perp y$, then $x \approx^* y$.

Sound and complete axiomatizations

We write $\Sigma \vdash \sigma$, if σ can be deduced from Σ by using the axioms introduced above.

Theorem (Hirvonen, 2024)

For any set $\Sigma \cup \{\sigma\}$ of PIA, UMI, and UMDE atoms,

$$\Sigma \models \sigma \text{ if and only if } \Sigma \vdash \sigma.$$

Since it is straightforward to check the soundness of the axioms, our axiomatization for PIA+UMI+UMDE implication is sound.

Theorem (Soundness)

For any set $\Sigma \cup \{\sigma\}$ of PIA, UMI, and UMDE atoms,

$$\textit{If } \Sigma \vdash \sigma, \ \textit{then } \Sigma \models \sigma.$$

Theorem (Completeness)

For any set $\Sigma \cup \{\sigma\}$ of PIA, UMI, and UMDE atoms,

$$\text{If } \Sigma \models \sigma, \text{ then } \Sigma \vdash \sigma.$$

Completeness proof is a modified version of the completeness proof of the axiomatization for disjoint PIAs (Geiger et al., 1991)

The idea of the proof is to show the contraposition: we assume that $\Sigma \nvdash \sigma$, and construct a probabilistic team that witnesses $\Sigma \not\models \sigma$. We will handle the cases where σ is UMI, UMDE, or PIA separately in the following three lemmas. In the below, D is assumed to be the set of variables that appear in Σ.

UMI

Suppose that $\Sigma \nvdash x \approx y$. We show that $\Sigma \not\models x \approx y$.

Let $z_1, \ldots, z_n$ be a list of those variables $z_i \in D$ for which $\Sigma \vdash x \approx^* z_i$, and let $u_1, \ldots, u_m$ be a list of those variables $u_j \in D$ for which $\Sigma \nvdash x \approx^* u_j$. These lists are clearly disjoint, and $D = \{z_1, \ldots, z_n\} \cup \{u_1, \ldots, u_m\}$.

Let team $X = \{s\}$, where

$$s(v) = \begin{cases} 0, & \text{if } v \in \{z_1, \ldots, z_n\} \\ 1, & \text{if } v \in \{u_1, \ldots, u_m\}. \end{cases}$$

Define then $\mathbb{X} \colon X \to (0, 1]$ such that $\mathbb{X}(s) = 1$.

Since $\Sigma \vdash x \approx x$ and $\Sigma \nvdash x \approx y$, we have $x \in \{z_1, \ldots, z_n\}$ and $y \in \{u_1, \ldots, u_m\}$. Hence, by the construction, $\mathbb{X} \not\models x \approx y$.

Suppose that $\Sigma \vdash v \approx v'$. Now, because of the transitivity axiom UMI3, either $v, v' \in \{z_1, \ldots, z_n\}$ or $v, v' \in \{u_1, \ldots, u_m\}$. This means that $\mathbb{X} \models v \approx v'$.

It is easy to see that all UMDEs and PIAs are satisfied by $\mathbb{X}$, so $\mathbb{X} \models \Sigma$.

UMDE

Suppose that $\Sigma \nvdash x \approx^* y$. We show that $\Sigma \not\models x \approx^* y$.

First, note that $\Sigma \vdash x \perp\!\!\!\perp x$ and $\Sigma \vdash y \perp\!\!\!\perp y$ imply $\Sigma \vdash x \approx^* y$, so either $\Sigma \nvdash x \perp\!\!\!\perp x$ or $\Sigma \nvdash y \perp\!\!\!\perp y$. Without loss of generality, assume that $\Sigma \nvdash x \perp\!\!\!\perp x$.

Let $z_1, \ldots, z_n$ be a list of those variables $z_i \in D$ for which $\Sigma \vdash x \approx^* z_i$, and let $u_1, \ldots, u_m$ be a list of those variables $u_j \in D$ for which $\Sigma \nvdash x \approx^* u_j$. These lists are clearly disjoint, and $D = \{z_1, \ldots, z_n\} \cup \{u_1, \ldots, u_m\}$.

Let team X consist of all the tuples from the set

$$Z_1 \times \cdots \times Z_n \times U_1 \times \cdots \times U_m,$$

where $Z_i = \{0, 1\}$ and $U_j = \{0\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Define then $\mathbb{X}$ as the uniform distribution over X.

It is easy to see that $\mathbb{X} \not\models x \approx^* y$. Suppose that $\Sigma \vdash v \approx v'$. Now, since $v \approx v'$ implies $v \approx^* v'$, because of the transitivity axiom UMDE3, either $v, v' \in \{z_1, \ldots, z_n\}$ or $v, v' \in \{u_1, \ldots, u_m\}$. This means that $\mathbb{X} \models v \approx v'$. The case $\Sigma \vdash v \approx^* v'$ is analogous.

Suppose that $\Sigma \vdash v \perp\!\!\!\perp v$. Suppose for a contradiction that $\mathbb{X} \not\models v \perp\!\!\!\perp v$. Then by the construction of $\mathbb{X}$, $v \in \{z_1, \ldots, z_n\}$. This means that $\Sigma \vdash x \approx^* v$, and thus by applying the axiom PIA & UMDE 1, we obtain $\Sigma \vdash x \perp\!\!\!\perp x$. This contradicts the assumption that $\Sigma \not\vdash x \perp\!\!\!\perp x$.

Suppose then that $\Sigma \vdash \bar{w} \perp\!\!\!\perp \bar{w}'$. If $\mathrm{Var}(\bar{w}) \cap \mathrm{Var}(\bar{w}') \neq \emptyset$, then by the decomposition axiom PIA3, $\Sigma \vdash v \perp\!\!\!\perp v$ for all $v \in \mathrm{Var}(\bar{w}) \cap \mathrm{Var}(\bar{w}')$. By the previous case, we know that then $\mathbb{X} \models v \perp\!\!\!\perp v$. Thus, by the constancy axiom PIA5, we may assume that $\mathrm{Var}(\bar{w}) \cap \mathrm{Var}(\bar{w}') = \emptyset$. But then it is easy to see that by the construction, $\mathbb{X} \models \bar{w} \perp\!\!\!\perp \bar{w}'$.

PIA

Suppose first that $\Sigma \nvdash x \perp\!\!\!\perp x$. Then the construction from the UMDE case shows that $\Sigma \not\models x \perp\!\!\!\perp x$.

Suppose then that $\Sigma \nvdash \bar{x} \perp\!\!\!\perp \bar{y}$. We show that $\Sigma \not\models \bar{x} \perp\!\!\!\perp \bar{y}$.

First, assume that $\mathrm{Var}(\bar{x}) \cap \mathrm{Var}(\bar{y}) \neq \emptyset$. Let $v \in \mathrm{Var}(\bar{x}) \cap \mathrm{Var}(\bar{y})$. If $\Sigma \nvdash v \perp\!\!\!\perp v$, then $\Sigma \not\models v \perp\!\!\!\perp v$ already follows from the previous case, and thus by the decomposition axiom PIA3, we also have $\Sigma \not\models \bar{x} \perp\!\!\!\perp \bar{y}$. If $\Sigma \vdash v \perp\!\!\!\perp v$, we can use the constancy axiom PIA5 to infer $\Sigma \nvdash \bar{x}' \perp\!\!\!\perp \bar{y}'$ where $\bar{x}'$ and $\bar{y}'$ are obtained from $\bar{x}$ and $\bar{y}$ by removing v. Hence, we may assume that $\mathrm{Var}(\bar{x}) \cap \mathrm{Var}(\bar{y}) = \emptyset$.

We may additionally assume that the atom $\bar{x} \perp\!\!\!\perp \bar{y}$ is minimal in the sense that $\Sigma \vdash \bar{x}' \perp\!\!\!\perp \bar{y}'$ for all $\bar{x}'$, $\bar{y}'$ such that $\mathrm{Var}(\bar{x}') \subseteq \mathrm{Var}(\bar{x})$, $\mathrm{Var}(\bar{y}') \subseteq \mathrm{Var}(\bar{y})$, and $\mathrm{Var}(\bar{x}'\bar{y}') \neq \mathrm{Var}(\bar{x}\bar{y})$.

If not, we can remove elements from $\bar{x}$ and $\bar{y}$ until this holds. By the decomposition axiom PIA3, it suffices to show the claim for the minimal atom.

Note that due to the trivial independence axiom PIA1 both $\bar{x}$ and $\bar{y}$ are at least of length one.

Let $\bar{x} = x_1 \dots x_n$ and $\bar{y} = y_1 \dots y_m$. Note that by the minimality of $\bar{x} \perp\!\!\!\perp \bar{y}$, we have $\Sigma \nvdash x_i \perp\!\!\!\perp x_i$ and $\Sigma \nvdash y_j \perp\!\!\!\perp y_j$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

Let $\{u_1, \dots, u_k\} \subseteq D$ be the set of those variables for which $\Sigma \vdash u_i \perp\!\!\!\perp u_i$. Let

$$\{z_1, \dots, z_l\} = D \setminus (\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_m\} \cup \{u_1, \dots, u_k\}).$$

Define a team X_0 over $D \setminus \{x_1\}$ such that it consists of all the tuples from the set

$$X_2 \times \dots \times X_n \times Y_1 \times \dots Y_m \times Z_1 \times \dots \times Z_l \times U_1 \times \dots \times U_k,$$

where $X_2 = \dots = X_n = Y_1 = \dots Y_m = Z_1 = \dots = Z_l = \{0, 1\}$ and $U_1 = \dots = U_k = \{0\}$.

Let then $X = \{s \cup \{(x_1, a)\} \mid s \in X_0\}$, where

$$a = \sum_{i=2}^{n} s(x_i) + \sum_{j=1}^{m} s(y_j) \pmod 2.$$

Define then $\mathbb{X}$ as the uniform distribution over X.

Now $\mathbb{X} \not\models \bar{x} \perp\!\!\!\perp \bar{y}$. Let $s, s' \in X$ be such that $s(x_1) = 1$, $s(x_i) = 0$, and $s'(y_j) = 0$ for all $2 \leq i \leq n$ and $1 \leq j \leq m$. Then there is no $s'' \in X$ such that $s''(\bar{x}) = s(\bar{x})$ and $s''(\bar{y}) = s'(\bar{y})$.

Suppose that $\Sigma \vdash v \approx v'$. Suppose for a contradiction that $\Sigma \not\models v \approx v'$. Then one of v and v' must be in $\{u_1, \ldots, u_k\}$ and one in $D \backslash \{u_1, \ldots, u_k\}$[1]. Assume that $v' \in \{u_1, \ldots, u_k\}$. This means that $\Sigma \vdash v' \perp\!\!\!\perp v'$. Since $\Sigma \vdash v \approx v'$, by applying UMI & UMDE and PIA & UMDE 1, we obtain $\Sigma \vdash v \perp\!\!\!\perp v$. But then $v \in \{u_1, \ldots, u_k\}$, which is a contradiction.

The case $\Sigma \vdash v \approx^* v'$. is analogous.

[1] Note that clearly $|\mathbb{X}_{w=0}| = 1$ for all $w \in \{u_1, \ldots, u_k\}$, and $|\mathbb{X}_{w=a}| = 1/2$ for all $a \in \{0, 1\}$ and $w \in D \backslash \{x_1, u_1, \ldots, u_k\}$. An easy induction proof shows that $|\mathbb{X}_{x_1=a}| = 1/2$ for all $a \in \{0, 1\}$.

Suppose then that $\Sigma \vdash \bar{w} \perp\!\!\!\perp \bar{w}'$.

If $\mathrm{Var}(\bar{w}) \cap \mathrm{Var}(\bar{w}') \neq \emptyset$, then by the decomposition axiom PIA3, $\Sigma \vdash v \perp\!\!\!\perp v$ for all $v \in \mathrm{Var}(\bar{w}) \cap \mathrm{Var}(\bar{w}')$. Then $v \in \{u_1, \ldots, u_k\}$, and thus $\mathbb{X} \models v \perp\!\!\!\perp v$. Thus, by the constancy axiom PIA5, we may assume that $\mathrm{Var}(\bar{w}) \cap \mathrm{Var}(\bar{w}') = \emptyset$.

Note that by a similar reasoning, we may more generally assume that $u_i \notin \mathrm{Var}(\bar{w}\bar{w}')$ for all $1 \leq i \leq l$.

Assume first that $\mathrm{Var}(\bar{w}\bar{w}') \cap \mathrm{Var}(\bar{x}\bar{y}) = \emptyset$.

Then $\mathrm{Var}(\bar{w}\bar{w}') \subseteq \mathrm{Var}(\bar{z})$, where $\bar{z} = z_1 \ldots z_l$. It is clear by the definition of $\mathbb{X}$ that $\mathbb{X} \models \bar{w} \perp\!\!\!\perp \bar{w}'$.

Assume then that $\mathrm{Var}(\bar{w}\bar{w}') \cap \mathrm{Var}(\bar{x}\bar{y}) \neq \emptyset$, but $\mathrm{Var}(\bar{x}\bar{y}) \not\subseteq \mathrm{Var}(\bar{w}\bar{w}')$.

Then $\mathbb{X} \models \bar{w} \perp\!\!\!\perp \bar{w}'$ because $|\mathbb{X}_{\bar{w}=\bar{a}}| = (1/2)^{|\bar{w}|}$ for all $a \in \{0,1\}^{|\bar{w}|}$ and $|\mathbb{X}_{\bar{w}'=\bar{a}}| = (1/2)^{|\bar{w}'|}$ for all $a \in \{0,1\}^{|\bar{w}'|}$.

Assume lastly that $\mathrm{Var}(\bar{x}\bar{y}) \subseteq \mathrm{Var}(\bar{w}\bar{w}')$. We show that this case is not possible.

We may assume that $\bar{w} = \bar{x}'\bar{y}'\bar{z}'$ and $\bar{w}' = \bar{x}''\bar{y}''\bar{z}''$, where $\mathrm{Var}(\bar{x}) = \mathrm{Var}(\bar{x}'\bar{x}'')$, $\mathrm{Var}(\bar{y}) = \mathrm{Var}(\bar{y}'\bar{y}'')$, and $\mathrm{Var}(\bar{z}'\bar{z}'') \subseteq \mathrm{Var}(\bar{z})$. By the axiom PIA3, we have $\Sigma \vdash \bar{x}'\bar{y}' \perp\!\!\!\perp \bar{x}''\bar{y}''$.

Note that by the minimality of $\bar{x} \perp\!\!\!\perp \bar{y}$, we have $\Sigma \vdash \bar{x}' \perp\!\!\!\perp \bar{y}'$.

By using the exchange axiom PIA4 to $\bar{x}' \perp\!\!\!\perp \bar{y}'$ and $\bar{x}'\bar{y}' \perp\!\!\!\perp \bar{x}''\bar{y}''$, we obtain $\Sigma \vdash \bar{x}' \perp\!\!\!\perp \bar{y}'\bar{x}''\bar{y}''$.

So now (by PIA3), $\Sigma \vdash \bar{x}' \perp\!\!\!\perp \bar{y}\bar{x}''$, and by the symmetry axiom PIA2, $\Sigma \vdash \bar{y}\bar{x}'' \perp\!\!\!\perp \bar{x}'$.

Again, by the minimality of $\bar{x} \perp\!\!\!\perp \bar{y}$ and the symmetry axiom PIA2, we have $\Sigma \vdash \bar{y} \perp\!\!\!\perp \bar{x}''$.

Then by using the exchange axiom PIA4 again, this time to $\bar{y} \perp\!\!\!\perp \bar{x}''$ and $\bar{y}\bar{x}'' \perp\!\!\!\perp \bar{x}'$, we obtain $\Sigma \vdash \bar{y} \perp\!\!\!\perp \bar{x}''\bar{x}'$. By (PIA3 and) the symmetry axiom PIA2, we have $\Sigma \vdash \bar{x} \perp\!\!\!\perp \bar{y}$, which is a contradiction.

A polynomial time algorithm

The polynomial-time algorithm for disjoint PIA implication (Geiger et al., 1991) can also be used to construct an algorithm for PIA+UMI+UMDE implication.

Theorem (Hirvonen, 2024)

The implication problem for PIA+UMI+UMDE has a polynomial-time algorithm.

Let D be the set of the variables that appear in $\Sigma \cup \{\sigma\}$.

We partition Σ into the sets of PIAs, UMIs, and UMDEs, and denote these sets by Σ_{PIA}, Σ_{UMI}, and Σ_{UMDE}, respectively.

For a UMI atom $\sigma := v \approx w$, it suffices to check whether $\Sigma_{\mathrm{UMI}} \models \sigma$ because no new UMIs can be obtained by using the inference rules for PIAs and UMDEs.

The set Σ_{UMI} can be viewed as an undirected graph $G(\Sigma_{\mathrm{UMI}}) = (D, \approx)$ such that each $x \approx y \in \Sigma_{\mathrm{UMI}}$ corresponds to an undirected edge between x and y.

Then $\Sigma_{\mathrm{UMI}} \models v \approx w$ iff w is reachable from v in $G(\Sigma_{\mathrm{UMI}})$. This can be checked in linear-time by using a breadth-first search.

Consider then the case for PIA or UMDE. We use ideas similar to (Hannula et al., 2018) in order to isolate constancy atoms.

Close the set Σ_{UMDE} under UMI & UMDE rule, i.e., define $\Sigma'_{\mathsf{UMDE}} := \Sigma_{\mathsf{UMDE}} \cup \{x \approx^* y \mid x \approx y \in \Sigma_{\mathsf{UMI}}\}$.

Construct then the undirected graph $G(\Sigma_{\mathsf{UMDE}'}) = (D, \approx^*)$ for Σ'_{UMDE} analogously to the case Σ_{UMI} above. Let $\mathrm{Const}(\Sigma_{\mathsf{PIA}}) := \{v \in D \mid v \in \mathrm{Var}(\bar{x}) \cap \mathrm{Var}(\bar{y})$ for some $\bar{x} \perp\!\!\!\perp \bar{y} \in \Sigma_{\mathsf{PIA}}\}$.

Now, we can compute a set of variables Const and a graph G in polynomial-time by using Algorithm 1.

Algorithm 1 Comp(Σ)

Require: Σ_{PIA}, Σ_{UMI}, and Σ_{UMDE}

Ensure: Const and $G := (D, E)$

$\mathsf{Const} \leftarrow \mathsf{Const}(\Sigma_{\mathrm{PIA}})$ $\quad (= \{v \in D \mid v \in \mathsf{var}(\bar{x}) \cap \mathsf{var}(\bar{y})$ for some $\bar{x} \perp\!\!\!\perp \bar{y} \in \Sigma_{\mathrm{PIA}}\})$

$G \leftarrow G(\Sigma'_{\mathrm{UMDE}})$

while $E' := E \cap ((D \setminus \mathsf{Const}) \times \mathsf{Const}) \neq \emptyset$ **do**

$\quad \mathsf{Const} \leftarrow \mathsf{Const} \cup \{x \mid \exists y : (x, y) \in E'\}$

for all $x, y \in \mathsf{Const}$, and $(x, y) \notin E$ **do** add (x, y) to E

For $A \subseteq D$, denote by $\sigma(A)$, the restriction of a PIA σ to the variables in A.

For example, if $\sigma := xyz \perp\!\!\!\perp uw$ and $A = \{x, y, u\}$, then $\sigma(A) = xy \perp\!\!\!\perp u$. Analogously, for set Σ of PIAs, we define $\Sigma(A) := \{\sigma(A) \mid \sigma \in \Sigma\}$.

Consider then $\Sigma_{\mathrm{PIA}}(D')$ and $\sigma(D')$ for $D' = D \setminus \mathrm{Const}$. We will use the following lemma:

Lemma

$\Sigma \models \sigma$ if and only if $\Sigma_{PIA}(D') \models \sigma(D')$.

Now, we show how to use Const and G to decide whether $\Sigma \models \sigma$.

Suppose that $\sigma := v \approx^* w$. Then it suffices to check whether w is reachable from v in G (by using a breadth-first search).

Suppose then that $\sigma := \bar{v} \perp\!\!\!\perp \bar{w}$. If $\mathrm{Var}(\bar{v}) \cap \mathrm{Var}(\bar{w}) \cap D' \neq \emptyset$, then $\Sigma \not\models \sigma$.

Otherwise, by the lemma, it suffices to check whether $\Sigma_{\mathrm{PIA}}(D') \models \sigma(D')$, which is now an instance of the implication problem for disjoint probabilistic independence that was shown to be in polynomial-time in (Geiger et al., 1991).

Let G be a graph and v and w two of its vertices. We write $\mathrm{Reach}(G, v, w)$ for the linear breadth-first search algorithm that outputs true if w is reachable from v in G and false otherwise.

Algorithm 2 PIA+UMI+UMDE-Implication(Σ, σ)

Require: Σ, σ
Ensure: true if and only if $\Sigma \models \sigma$

- **if** $\sigma = v \approx w$ **then return** $\mathrm{Reach}(G(\Sigma_{\mathrm{UMI}}), v, w)$
- **else**
  - construct Const and G using $\mathrm{Comp}(\Sigma)$
  - **if** $\sigma = v \approx^* w$ **then return** $\mathrm{Reach}(G, v, w)$
  - **else if** $\sigma = \bar{v} \perp \bar{w}$ **then**
    - $D' \leftarrow D \setminus \mathsf{Const}$
    - **if** $\mathsf{var}(\bar{v}) \cap \mathsf{var}(\bar{w}) \cap D' \neq \emptyset$ **then return** false
    - **else**
      - **return** true if $\Sigma_{\mathrm{PIA}}(D') \models \sigma(D')$ and false otherwise

Conclusion

The implication problem for PIAs, UMIs, and UMDEs has a sound and complete axiomatization and a polynomial-time algorithm.

Some open questions:

- Can we extend the axiomatization to nonunary marginal identity and marginal distribution equivalence?
- Can we find an axiomatization for unary functional dependency, probabilistic independence, (unary) marginal identity and marginal distribution equivalence?

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