Hyperfine Structure:

An easy route to the fine-structural properties of L

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Hyperfine Structure

- Introduced by Friedman and Koepke in "An elementary approach to the fine structure of L", Bulletin of Symbolic Logic, 1997
- An alternative to Jensen's fine structure theory for looking at L.
- Uses simple model theory
- makes the combinatorial aspects of proofs more accessible

The Constructible Hierarchy

L is constructed by iterating the definable power set through the ordinals:

$$L_0 = \emptyset$$
$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \text{ for limit ordinals } \lambda$$
$$L_{\alpha+1} = Def(L_\alpha)$$

We will look more closely at *how* a set first appears in L.

$$x \in Def(L_\alpha) \leftrightarrow x = \{z \in L_\alpha : L_\alpha \models \varphi(z, \vec{y})\}$$

for some sentence φ and parameters $\vec{y}$ from L_α.

- the triple $(\alpha, \varphi, \vec{y})$ can be thought of as *a name* for the constructible set x
- we will also see such triples as *locations* in the hyperfine hierarchy

Interpretation

We call a triple $(\gamma, \varphi, \vec{y})$ a *location* if γ is an ordinal, φ is a formula with n free variables and $\vec{y}$ is an $n-1$-tuple of sets from L_γ.

We have seen that such a location can be seen as a name for a set in $L_{\gamma+1}$; the interpretation operator, I, formalises this, maping locations to constructible sets:

$$I(\gamma, \varphi, \vec{y}) = \{z \in L_\gamma : L_\gamma \vDash \varphi(z, \vec{y})\}$$

We also want to be able to give a unique name for a set (as there will be many with the same interpretation). We will take the first name that works - but for that we need to well-order the locations.

$\tilde{<}$, $<_L$ and Naming

- First order locations by the level of L.
- Fix an ordering of formulae $\langle \varphi_n : n \in \omega \rangle$ in order type ω such that sub-formulae appear earlier
- Use the canonical well-ordering of L to order the parameter sequences lexicographically.

$$(\alpha, \varphi_n, \vec{y}) \tilde{<} (\beta, \varphi_m, \vec{x}) \text{ iff } \alpha < \beta \quad \text{or}$$
$$\alpha = \beta \wedge n < m \quad \text{or}$$
$$\alpha = \beta \wedge n = m \wedge x <_L^{lex} y$$

For a constructible set y set

$$N(y) = \tilde{<} \text{ least location } r \text{ such that } y = I(r)$$

Note: $<_L$ can be inductively defined together with $\tilde{<}$ by taking the $\tilde{<}$ ordering of names to order the next level of L

Skolem functions

Define a Skolem operator S with locations as domain so that

$$S(\gamma, \varphi, \vec{y}) = <_L \text{ least } z \in L_\gamma \text{ such that } L_\gamma \vDash \varphi(z, \vec{y})$$

(if such exists - otherwise set $S(\gamma, \varphi, \vec{y}) = 0$.)

We say $X \subseteq L$ is constructible closed if X is closed under I,N and S, i.e. if a location $r \in X$ then $I(r), S(r) \in X$ and for any set $x \in X$, $N(x) \in X$.

Constructible closure condensation:
If $X \subseteq L_\alpha$ is constructible closed then $M \cong L_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$. Moreover, the collapsing map preserves I, N, S and $<_L$.

Hulls and intermediate Skolem functions

We can extend this condensation result to the whole hierarchy of locations, by defining an intermediate notion of Skolem functions associated with locations.

Let $S_\varphi^\gamma(\vec{x}) := S(\gamma, \varphi, \vec{x})$.

A location $r = (\gamma, \varphi_n, \vec{x})$ corresponds to the structure:

$$L_r := \langle L_\gamma, \in, <_L, I, N, S, S_{\varphi_0}^\gamma, \ldots, S_{\varphi_n}^\gamma \restriction \vec{x} \rangle$$

We define a hull operation associated with this location: For $X \subseteq L_\gamma$, $L_r\{X\}$ is the algebraic closure of X as a substructure of L_r.

Example: collapse of an ordinal

What is the first location where we can see α is not a cardinal?

Suppose $f : \beta \to \alpha$ onto. Suppose we can define f over L_γ by $f(x) = y \leftrightarrow L_\gamma \models \varphi(y, \vec{p}, x)$, where $\vec{p}$ are parameters from L_γ. Then we could say f appears at the location $(\gamma, \varphi, \vec{p}^\frown\beta)$. As f is onto we have $S_\varphi^\gamma{}''\{\vec{p}^\frown\delta : \delta < \beta\} = \alpha$.

Claim: α is not a cardinal (in L) iff there exists a location $s = (\gamma, \varphi, \vec{x})$ and a finite set $p \subset L_\gamma$ such that

$$\{\beta < \alpha : \alpha \neq \alpha \cap L_s\{\beta \cup p\}\}$$

is bounded in α.

So we can say α is first collapsed at the $\tilde{<}$ least location such location.

Key properties: Condensation

We have condensation for this finer hierarchy:

Fine Condensation:
Let $r = (\gamma, \varphi_n, \vec{x})$ be a location and $X = L_r\{X\}$. Then there is a unique isomorphism

$$\pi : \langle X, \in, <_L, I, N, S, S^{\gamma}_{\varphi_0}, \ldots, S^{\gamma}_{\varphi_n} \restriction \vec{x}\rangle \\ \cong L_{\bar{r}} = \langle L_{\bar{\gamma}}, \in, <_L, I, N, S, S^{\gamma}_{\varphi_0}, \ldots, S^{\gamma}_{\varphi_{\bar{n}}} \restriction \vec{\bar{x}}\rangle$$

To prove this we use the Condensation result for constructible closure.
Note :

- $\bar{r}$ is the least upper bound of locations s such that $\pi^{-1}(s) \tilde{<} r$
- $\bar{n} = n$ or $\bar{n} = n + 1$ with $\vec{\bar{x}} = \emptyset$.

Key properties: Growth

Let $r = (\gamma, \varphi, \vec{x})$ and $X \subseteq L_\gamma$.

Monotonicity:
(a) If r' is also a γ location with $r' \tilde{>} r$ then

$$L_r\{X\} \subseteq L'_r\{X\}$$

(b) If r' is a β location with $\beta > \gamma$ then

$$L_r\{X\} \subseteq L_{r'}\{X \cup \{\gamma\}\}$$

Successor stages: Finiteness
If r^+ the successor of r under $\tilde{<}$ there is some $z \in L_\gamma$ such that for any $X \subseteq L_\gamma$ we have

$$L_{r^+}\{X\} = L_r\{X \cup \{z\}\}$$

Limit stages: Continuity

(a) If r is a limit location but $r \neq (\gamma, \varphi_0, \emptyset)$ then

$$L_r\{X\} = \bigcup\{L_s\{X\} : s \tilde{<} r \text{ and } s \text{ is a } \gamma \text{ location}\}$$

(b) If $r = (\gamma, \varphi_0, \emptyset)$ and γ is a limit ordinal then

$$L_r\{X\} = \bigcup_{\beta<\gamma} L_{(\beta,\varphi_0,\emptyset)}\{X \cap L_\beta\}$$

(c) If $r = (\alpha + 1, \varphi_0, \emptyset)$ and $X \subseteq L_\alpha$ then

$$L_r\{X \cup \{\alpha\}\} \cap L_\alpha = \bigcup\{L_s\{X\} : \; s \text{ is a } \alpha \text{ location}\}$$

uses and limitations

So far hyperfine structure has been used to prove the following hold within L:

- Global Square
- Morasses
- Equivalence of stationary reflection and weak compactness
- and the Covering Lemma for L

This seems to only work in L, not extending to Core models.

Thank you!