Applying dimension concepts to team semantics

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22 Aug 2022

Dimensions of families

Families of sets arising from team semantics

Consider a formula $\phi(x_0, \cdots, x_{n-1})$ and a structure $\mathfrak{M}$ with a common vocabulary. Supposing ϕ can be interpreted within team semantics, we may associate to ϕ and $\mathfrak{M}$ the family

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Similarly, if $\mathcal{K}$ is a Kripke model with the set of worlds W and θ is a formula of some modal logic with team semantics, we may consider

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$$||\phi||^{\mathcal{K}} = \{T \subseteq W \mid \mathcal{K}, T \vDash \theta\}.$$

In both cases, the expressive power of the corresponding logics reflect to the structure of the arising families of sets. With this in mind, we shall consider some dimensional concepts related to families of sets. Our emphasis will be on the first case.

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on families of sets. In general, each logical construct corresponds to an operation on families of sets.

Brief history of the upper dimension

The *upper dimension* of a family of sets was introduced in Hella, Luosto, Sano and Virtema (2014) to study extensions of the modal dependence logic $\mathcal{MDL}$. Similar ideas were already developed in Ciardelli 2009. Originally, upper dimension was only defined for downwards closed families, but Lück and Vilander 2019 extended the definition for all families of sets. The version of upper dimension that we use in this talk, is a reformulation of Lück's and Vilander's. We also introduce a dual concept, *dual upper dimension*, and what we call *the cylindrical dimension*.

Basic concepts

For sets B and C, we write

$$[B, C] = \{S \mid B \subseteq S \subseteq C\}.$$

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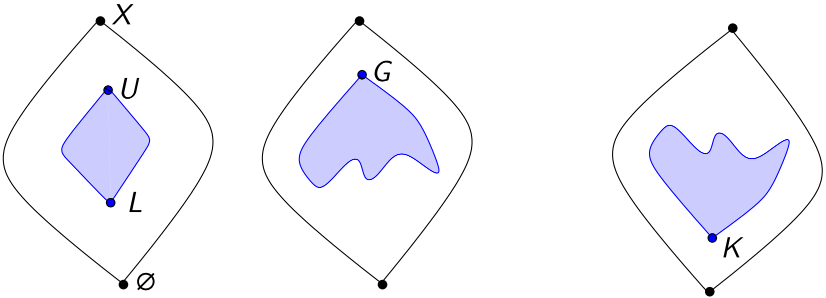
Let $\mathcal{A}$ be a family of sets. The family $\mathcal{A}$ is an *interval or cylinder*, if there exist L and U such that $L \subseteq U$ and $\mathcal{A} = [L, U]$. The family $\mathcal{A}$ is *convex* if for all $S, T \in \mathcal{A}$, we have $[S, T] \subseteq \mathcal{A}$. $\mathcal{A}$ is *downwards closed* if $A \in \mathcal{A}$ and $S \subseteq A$ imply $S \in \mathcal{A}$. A family of sets $\mathcal{A}$ is *dominated (by $\bigcup \mathcal{A}$)* if $\bigcup \mathcal{A} \in \mathcal{A}$. The family $\mathcal{A}$ is *supported* (by $\bigcap \mathcal{A}$) if $\mathcal{A}$ is nonempty and $\bigcap \mathcal{A} \in \mathcal{A}$. Naturally, we say that $\mathcal{A}$ is *dominated convex* if it is dominated and convex. Similarly, $\mathcal{A}$ is *supported convex* if it is supported and convex.



interval dominated convex family supported convex family

Denote $\mathcal{A}^{\mathrm{d}} = \{X \setminus A \mid A \in \mathcal{A}\}$. The following facts are now immediate:

a. The family $\mathcal{A}$ is an interval iff it is dominated, supported and convex.
b. $\mathcal{A}$ is convex iff $\mathcal{A}^{\mathrm{d}}$ is convex.
c. $\mathcal{A}$ is dominated iff $\mathcal{A}^{\mathrm{d}}$ is supported.

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Let $\mathcal{A}$ be a family of sets. We say that a subfamily $\mathcal{G} \subseteq \mathcal{A}$ *dominates* $\mathcal{A}$ if there exist dominated convex families $\mathcal{D}_G$, $G \in \mathcal{G}$, such that $\bigcup_{G \in \mathcal{G}} \mathcal{D}_G = \mathcal{A}$ and $\bigcup \mathcal{D}_G = G$, for each $G \in \mathcal{G}$. The subfamily $\mathcal{K} \subseteq \mathcal{A}$ *supports* $\mathcal{A}$ if there exist supported convex families $\mathcal{S}_K$, $K \in \mathcal{K}$ such that $\bigcup_{K \in \mathcal{K}} \mathcal{S}_K = \mathcal{A}$ ja $\bigcap \mathcal{S}_K = K$, for each $K \in \mathcal{K}$.

Dimensions

Definition

The *upper dimension* of the family is $\mathcal{A}$

$$\mathrm{D}(\mathcal{A}) = \min\{|\mathcal{G}| \mid \mathcal{G} \text{ dominates the family } \mathcal{A}\},$$

the dual upper dimension is

$$\mathrm{D}^{\mathrm{d}}(\mathcal{A}) = \min\{|\mathcal{G}| \mid \mathcal{G} \text{ supports the family } \mathcal{A}\}$$

and the *cylindrical dimension* is

$$\mathrm{CD}(\mathcal{A}) = \min\{|I| \mid (\mathcal{A}_i)_{i\in I} \text{ is an indexed family of intervals with } \bigcup_{i\in I} \mathcal{A}_i = \mathcal{A}\}.$$

Proposition

Let $\mathcal{A}$ be a finite family of sets. Then the following are equivalent:

(a) *For all $B, C \in \mathcal{A}$, if $B \subsetneq C$, then $[B, C] \nsubseteq \mathcal{A}$.*

(b) $\mathrm{D}(\mathcal{A}) = |\mathcal{A}|$.

(c) $\mathrm{D}^{\mathrm{d}}(\mathcal{A}) = |\mathcal{A}|$.

(d) $\mathrm{CD}(\mathcal{A}) = |\mathcal{A}|$.

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Proof.

All the intervals contained in $\mathcal{A}$ are singletons iff the condition is met. The same holds for dominated convex and supported convex families. □

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- b $\mathrm{D}(\mathcal{A}) = |\mathcal{A}|$.
- c $\mathrm{D}^{\mathrm{d}}(\mathcal{A}) = |\mathcal{A}|$.
- d $\mathrm{CD}(\mathcal{A}) = |\mathcal{A}|$.

Proof.

All the intervals contained in $\mathcal{A}$ are singletons iff the condition is met. The same holds for dominated convex and supported convex families. □

Fixing the base set X with $|X| = n$, the family $\mathcal{E}$ of all subsets of X of even size is a maximal family with this property (the other is $\mathcal{O}$), so $\mathrm{D}(\mathcal{E}) = \mathrm{D}^{\mathrm{d}}(\mathcal{E}) = \mathrm{CD}(\mathcal{E}) = 2^{n-1}$. *It can also be shown that for all $\mathcal{A} \subseteq \mathcal{P}(X)$ (regardless if is satisfies the condition or not), then* $\mathrm{CD}(\mathcal{A}) \leq 2^{n-1}$.

Proposition

Let $\mathcal{A}$ be a family of sets. Then

$$\mathrm{D}(\mathcal{A}) \leq \mathrm{CD}(\mathcal{A}) \text{ and } \mathrm{D}^{\mathrm{d}}(\mathcal{A}) \leq \mathrm{CD}(\mathcal{A}).$$

If in addition $\mathcal{A}$ is convex, then

$$\mathrm{CD}(\mathcal{A}) \leq \mathrm{D}(\mathcal{A})\, \mathrm{D}^{\mathrm{d}}(\mathcal{A}).$$

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$$\mathrm{CD}(\mathcal{A}) \leq \mathrm{D}(\mathcal{A})\,\mathrm{D}^{\mathrm{d}}(\mathcal{A}).$$

There are no further restrictions in the following sense: for every $m, n, s \in \mathbb{Z}_+$ satisfying $\max\{m, n\} \leq s \leq mn$, there is a convex family $\mathcal{C}$ with $\mathrm{D}(\mathcal{C}) = m$, $\mathrm{D}^{\mathrm{d}}(\mathcal{C}) = n$ and $\mathrm{CD}(\mathcal{C}) = s$.

Shadows

To facilitate dimension calculations, we develop some auxiliary tools.

Let $\mathcal{A}$ be a family of sets and $A \in \mathcal{A}$. The *convex shadow of* A *in the family* $\mathcal{A}$ is the family

$$\partial_A(\mathcal{A}) = \{B \subseteq A \mid [B, A] \subseteq \mathcal{A}\}.$$

Similarly, the *dual convex shadow* of A in $\mathcal{A}$ is

$$\partial^A(\mathcal{A}) = \{B \in \mathcal{A} \mid A \subseteq B, [A, B] \subseteq \mathcal{A}\}.$$

Lemma

Let $\mathcal{A}$ be a family of sets and $A \in \mathcal{A}$.

a. $\partial_A(\mathcal{A})$ *is the largest dominated convex family* $\mathcal{C} \subseteq \mathcal{A}$ *with* $\bigcup \mathcal{C} = A$. *Similarly,* $\partial^A(\mathcal{A})$ *is the largest supported convex family* $\mathcal{C} \subseteq \mathcal{A}$ *with* $\bigcap \mathcal{C} = A$.
b. *A family* $\mathcal{G} \subseteq \mathcal{A}$ *dominates* $\mathcal{A}$ *iff* $\bigcup_{G \in \mathcal{G}} \partial_G(\mathcal{A}) = \mathcal{A}$. *Dually,* $\mathcal{H} \subseteq \mathcal{A}$ *supports* $\mathcal{A}$ *iff* $\bigcup_{H \in \mathcal{H}} \partial^H(\mathcal{A}) = \mathcal{A}$.
c. *If* $\mathcal{G}$ *dominates* $\mathcal{A}$*, then* $\mathrm{Max}(\mathcal{A}) \subseteq \mathcal{G}$*, and if* $\mathcal{H}$ *supports* $\mathcal{A}$*, then* $\mathrm{Min}(\mathcal{A}) \subseteq \mathcal{H}$.

Families related to team-semantic atoms

Let X and Y be non-empty base sets. Consider:

$$\begin{aligned}
\mathcal{F} &= \{f \subseteq X \times Y \mid f \text{ is a mapping}\},\\
\mathcal{I}_{\subseteq} &= \{R \subseteq X \times X \mid \mathrm{dom}(R) \subseteq \mathrm{rg}(R)\},\\
\mathcal{I}_{\perp} &= \{A \times B \mid A \subseteq X,\ B \subseteq Y\},\\
\mathcal{X} &= \{R \subseteq X \times X \mid \mathrm{dom}(R) \cap \mathrm{rg}(R) = \emptyset\}.
\end{aligned}$$

Theorem

Let X and Y be finite sets with $m = |X| \geq 2$ and $n = |Y| \geq 2$. Then:

$$\mathrm{D}(\mathcal{F}) = n^m, \quad \mathrm{D^d}(\mathcal{F}) = 1, \quad \mathrm{CD}(\mathcal{F}) = \mathrm{D}(\mathcal{F}),$$

$$\mathrm{D}(\mathcal{I}_\subseteq) = 2^m - m, \quad \mathrm{D^d}(\mathcal{I}_\subseteq) = 1 + \sum_{k=2}^{m} \binom{m}{k} k^k, \quad \mathrm{CD}(\mathcal{I}_\subseteq) = \mathrm{D^d}(\mathcal{I}_\subseteq),$$

$$\mathrm{D}(\mathcal{I}_\perp) = (2^m - m - 1)(2^n - n - 1) + m + n, \quad \mathrm{D^d}(\mathcal{I}_\perp) = (2^m - m - 1)(2^n - n - 1) + 1, \quad \mathrm{CD}(\mathcal{I}_\perp) = \mathrm{D}(\mathcal{I}_\perp),$$

$$\mathrm{D}(\mathcal{X}) = 2^m - 2, \quad \mathrm{D^d}(\mathcal{X}) = 1, \quad \mathrm{CD}(\mathcal{X}) = \mathrm{D}(\mathcal{X}).$$

Tensor semantics of connectives

Connectives

There are $2^4 = 16*$ binary operations $\circledast$ on the set $\{0,1\}$, each related to a connective. Given a base set X, for each connective $\circledast$ there is a corresponding set-theoretic operation $*$,

$$A * B = \{x \in X \mid \chi_A(x) \circledast \chi_B(x) = 1\}$$

where χ_C is the characteristic function related to a set C. We lift this operation $*$ on the level of families of sets putting

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In particular,

$$\begin{aligned}
\mathcal{A} \vee \mathcal{B} &= \{A \cup B \mid A \in \mathcal{A},\ B \in \mathcal{B}\},\\
\mathcal{A} \wedge \mathcal{B} &= \{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\},\\
\mathcal{A} - \mathcal{B} &= \{A \setminus B \mid A \in \mathcal{A},\ B \in \mathcal{B}\} \text{ and}\\
\mathcal{A} \oplus \mathcal{B} &= \{A \triangle B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}.
\end{aligned}$$

Characteristic function of families

Employ a new symbol $\Cap \neq 0, 1$ 'unknown'. Let $\circledast$ be a binary operation on $\{0, 1\}$. Write $V_0 = \{0\}$, $V_1 = \{1\}$ and $V_\Cap = \{0, 1\}$ and $A \circledast B = \{u \circledast v \mid a \in A, b \in B\}$, for $A, B \subseteq \{0, 1\}$. We now define *Kleene's extension* $\widetilde{\circledast}$ of $\circledast$ as follows:

$$u \widetilde{\circledast} v = w \text{ if and only if } V_u \circledast V_v = V_w,$$

for $u, v, w \in \{0, 1, \Cap\}$. Overloading the notation, we shall denote also the extension by $\circledast$ instead on $\widetilde{\circledast}$ in the sequel.

Definition

The characteristic function of a family of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ is $\xi_\mathcal{A} \colon X \to \{0, 1, \Cap\}$,

$$\xi_\mathcal{A}(x) = \begin{cases} 1, & \text{for } E(x) = \{1\} \\ \Cap, & \text{for } E(x) = \{0, 1\} \\ 0, & \text{for } E(x) = \{0\} \end{cases}$$

where $E(x) = \{\chi_A(x) \mid A \in \mathcal{A}\}$.

Tables for connectives

$\wedge$	0	∩	1
0	0	0	0
∩	0	∩	∩
1	0	∩	1

$\vee$	0	∩	1
0	0	∩	1
∩	∩	∩	1
1	1	1	1

$-$	0	∩	1
0	0	0	0
∩	∩	∩	0
1	1	∩	0

$\oplus$	0	∩	1
0	0	∩	1
∩	∩	∩	∩
1	1	∩	0

Preserving intervals

Lemma

Let $\circledast$ be a binary operation on $\{0,1\}$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. Then for every $x \in X$, it holds that

$$\xi_{\mathcal{A} \circledast \mathcal{B}}(x) = \xi_{\mathcal{A}}(x) \circledast \xi_{\mathcal{B}}(x).$$

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Proposition

Let $\circledast$ be a tensor operator. Then if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ are intervals, then so is $\mathcal{A} \circledast \mathcal{B}$, too. Indeed, if we write $\xi = \xi_{\mathcal{A} \circledast \mathcal{B}}$, $C_0 = \xi^{-1}[\{1\}]$ and $C_1 = \xi^{-1}[\{\text{ᑎ}, 1\}]$, then $\mathcal{A} \circledast \mathcal{B} = [C_0, C_1]$.

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Corollary

For all families of sets $\mathcal{A}$ and $\mathcal{B}$, we have $\mathrm{CD}(\mathcal{A} \circledast \mathcal{B}) \leq \mathrm{CD}(\mathcal{A})\,\mathrm{CD}(\mathcal{B})$.

Dimension functions

Growth classes

As the sizes of finite structures vary, we need study the following *dimension functions*:

$$\mathrm{Dim}_\phi \colon \mathbb{N} \to \mathbb{N}, \qquad \mathrm{Dim}_\phi(n) = \sup\left\{\mathrm{D}(\|\phi\|^{\mathfrak{M}}) \mid \mathrm{card}(\mathfrak{M}) = n\right\},$$

$$\mathrm{Dim}^{\mathrm{d}}{}_\phi \colon \mathbb{N} \to \mathbb{N}, \qquad \mathrm{Dim}^{\mathrm{d}}{}_\phi(n) = \sup\left\{\mathrm{D}^{\mathrm{d}}(\|\phi\|^{\mathfrak{M}}) \mid \mathrm{card}(\mathfrak{M}) = n\right\},$$

$$\mathrm{CDim}_\phi \colon \mathbb{N} \to \mathbb{N}, \qquad \mathrm{CDim}_\phi(n) = \sup\left\{\mathrm{CD}(\|\phi\|^{\mathfrak{M}}) \mid \mathrm{card}(\mathfrak{M}) = n\right\}.$$

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We are interested in the rate of growth:
A set $\mathbb{O}$ of mappings $f : \mathbb{N} \to \mathbb{N}$ is a *growth class* if the following conditions hold for all $f, g : \mathbb{N} \to \mathbb{N}$:

- (a) If $g \in \mathbb{O}$ and $f \leq g$, then $f \in \mathbb{O}$.
- (b) If $f, g \in \mathbb{O}$, then $f + g \in \mathbb{O}$ and $fg \in \mathbb{O}$.

For $k \in \mathbb{N}$, the class $\mathbb{E}_k$ consist all $f\colon \mathbb{N} \to \mathbb{N}$ such that there exists a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ of degree k and with coefficients in $\mathbb{N}$ such that $f \leq 2^p$. In addition, $\mathbb{F}_k$ is the class of functions $f\colon \mathbb{N} \to \mathbb{N}$ such that there exists a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ of degree k and with coefficients in $\mathbb{N}$ such that for every $n \in \mathbb{N} \setminus \{0, 1\}$ we have that

$$f(n) \leq n^{p(n)}.$$

Obviously, every $\mathbb{E}_k$ and $\mathbb{F}_k$ (for $k \in \mathbb{N}$) is a growth class. We shall see that these classes are naturally useful in studying the dependence logic and its friends. It is also important that we have a hierarchy:

$$\mathbb{E}_0 \subsetneq \mathbb{F}_0 \subsetneq \mathbb{E}_1 \subsetneq \mathbb{F}_1 \subsetneq \cdots \subsetneq \mathbb{E}_k \subsetneq \mathbb{F}_k.$$

Growth classes of atoms

Let $\mathfrak{M}$ be a finite structure with domain M. When we do the substitutions indicated in the table, we get families of sets corresponding to the interpretations of some of the key atoms, i.e., $\mathcal{A} = ||\alpha||^{\mathfrak{M}}$ where $\mathcal{A}$ is the family and α is the atom indicated in the row. Fix parameters $k, l \in \mathbb{Z}_+$ related to arities of the atoms and tuples $\vec{x}$ and $\vec{y}$ of variables with $|\vec{x}| = |\vec{y}| = k$ and $|\vec{z}| = l$.

				Growth class of		
family	X	Y	atom α	Dim_α	$\mathrm{Dim}^{\mathrm{d}}_\alpha$	CDim_α
$\mathcal{F}$	M^k	M	$=(\vec{x}, t)$	$\mathbb{F}_k$	$\mathbb{E}_0$	$\mathbb{F}_k$
$\mathcal{I}_\subseteq$	M^k	M^k	$\vec{x} \subseteq \vec{y}$	$\mathbb{E}_k$	$\mathbb{F}_k$	$\mathbb{F}_k$
$\mathcal{I}_\perp$	M^k	M^l	$\vec{x} \perp \vec{z}$	$\mathbb{E}_{k+l}$	$\mathbb{E}_{k+l}$	$\mathbb{E}_{k+l}$
$\mathcal{X}$	M^k	M^k	$\vec{x} \mid \vec{y}$	$\mathbb{E}_k$	$\mathbb{E}_0$	$\mathbb{E}_k$

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$\mathcal{I}_\perp$	M^k	M^l	$\vec{x} \perp \vec{z}$	$\mathbb{E}_{k+l}$	$\mathbb{E}_{k+l}$	$\mathbb{E}_{k+l}$
$\mathcal{X}$	M^k	M^k	$\vec{x} \mid \vec{y}$	$\mathbb{E}_k$	$\mathbb{E}_0$	$\mathbb{E}_k$

Note: The growth class in the table is the least appropriate class in the hierarchy. Thus, we actually have $\mathrm{Dim}_{=(\vec{x},t)} \in \mathbb{F}_k \setminus \mathbb{E}_k$, $\mathrm{Dim}_{\vec{x}\subseteq\vec{y}} \in \mathbb{E}_k \setminus \mathbb{F}_{k-1}$ and so forth.

Operators

Logical operators

We are interested in the behaviour of dimensions under various operators. Our goal is to find natural criteria for operators to preserve growth classes.

In general, an *operator* on a set X is a function $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ for some $n \in \mathbb{N}$. Now let $\mathcal{R} \subseteq \mathcal{P}(X)^{n+1}$ be an $(n+1)$-ary relation. The corresponding operator $\Delta_{\mathcal{R}}\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ is defined by

$$A \in \Delta_{\mathcal{R}}(\mathcal{A}_0, \dots, \mathcal{A}_{n-1}) \iff \exists A_0 \in \mathcal{A}_0 \dots \exists A_{n-1} \in \mathcal{A}_{n-1} : (A, A_0, \dots, A_{n-1}) \in \mathcal{R}.$$

Definition

Let X be a set. A function $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ is a *(second-order) Kripke-operator*, if there is a relation $\mathcal{R} \subseteq \mathcal{P}(X)^{n+1}$ such that $\Delta = \Delta_{\mathcal{R}}$.

Logical operators

We are interested in the behaviour of dimensions under various operators. Our goal is to find natural criteria for operators to preserve growth classes.

In general, an *operator* on a set X is a function $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ for some $n \in \mathbb{N}$. Now let $\mathcal{R} \subseteq \mathcal{P}(X)^{n+1}$ be an $(n+1)$-ary relation. The corresponding operator $\Delta_{\mathcal{R}}\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ is defined by

$$A \in \Delta_{\mathcal{R}}(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) \iff \exists A_0 \in \mathcal{A}_0 \ldots \exists A_{n-1} \in \mathcal{A}_{n-1} : (A, A_0, \ldots, A_{n-1}) \in \mathcal{R}.$$

Definition

Let X be a set. A function $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ is a *(second-order) Kripke-operator*, if there is a relation $\mathcal{R} \subseteq \mathcal{P}(X)^{n+1}$ such that $\Delta = \Delta_{\mathcal{R}}$.

For example, $\cap$ and $\vee$ are Kripke-operators.

Dimension inequalities

The following are desired properties of logical operators.

Definition

Let $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ be an operator. We say that Δ *(weakly) preserves dominated (supported, resp.) convexity* if $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$ is dominated (supported, resp.) and convex or $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) = \varnothing$, whenever $\mathcal{A}_i$ is dominated (supported, resp.) and convex for each $i < n$.

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It can be shown, e.g., that $\cap$ and $\vee$ weakly preserve dominated and supported convexity.

We can now prove the following dimension inequalities for Kripke-operators that preserve dominated and/or supported convexity:

Theorem

Let $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(X))$ *be a Kripke-operator, and* $\mathcal{A} = \Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$.

a. *If* Δ *preserves dominated convexity, then* $\mathrm{D}(\mathcal{A}) \leq \mathrm{D}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}(\mathcal{A}_{n-1})$.
b. *If* Δ *preserves supported convexity, then* $\mathrm{D}^{\mathrm{d}}(\mathcal{A}) \leq \mathrm{D}^{\mathrm{d}}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}^{\mathrm{d}}(\mathcal{A}_{n-1})$.
c. *If* Δ *has both preservation properties, then* $\mathrm{CD}(\mathcal{A}) \leq \mathrm{CD}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{CD}(\mathcal{A}_{n-1})$.

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- (a) *If* Δ *preserves dominated convexity, then* $\mathrm{D}(\mathcal{A}) \le \mathrm{D}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}(\mathcal{A}_{n-1})$.
- (b) *If* Δ *preserves supported convexity, then* $\mathrm{D}^{\mathrm{d}}(\mathcal{A}) \le \mathrm{D}^{\mathrm{d}}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}^{\mathrm{d}}(\mathcal{A}_{n-1})$.
- (c) *If* Δ *has both preservation properties, then* $\mathrm{CD}(\mathcal{A}) \le \mathrm{CD}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{CD}(\mathcal{A}_{n-1})$.

Corollary

For all families of sets $\mathcal{A}$ *and* $\mathcal{B}$ *and any of the dimension above* $\mathfrak{d}$, *we have*

$$\mathfrak{d}(\mathcal{A} \cap \mathcal{B}) \le \mathfrak{d}(\mathcal{A}) \cdot \mathfrak{d}(\mathcal{B}) \text{ and } \mathfrak{d}(\mathcal{A} \vee \mathcal{B}) \le \mathfrak{d}(\mathcal{A}) \cdot \mathfrak{d}(\mathcal{B}).$$

Preserving growth classes

As a direct consequence to the dimension inequalities we get the following application to logics with team semantics:

Theorem

Let L be a logic, whose formulas are built from atomic formulas using connectives and quantifiers, and let $\mathbb{O}$ *be a growth class. Assume that* $\mathrm{Dim}_\alpha \in \mathbb{O}$ *for every atomic formula* α *of L, and* $\Delta_\odot$ *is a Kripke-operator that preserves dominated convexity for every connective and quantifier* $\odot$ *in L. Then* $\mathrm{Dim}_\phi \in \mathbb{O}$ *for every formula* ϕ *in L.*

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The theorem above can be used for proving inexpressibility results: If ψ is a formula in some other logic such that $\mathrm{Dim}_\psi \notin \mathbb{O}$, then ψ is not expressible in L.

Naturally, a similar result can be formulated for the dual upper dimension D^d and the cylindrical dimension CD.

Hierarchy result for dependency atoms

We get the following strong arity hierarchy theorem for dependency atoms:

Theorem

Let L be the extension of first-order logic with all dependence, inclusion, exclusion and independence atoms of arity at most $k-1$*. Then the* k*-ary dependence, inclusion, exclusion and independence atoms are not expressible in L.*

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Let L be the extension of first-order logic with all dependence, inclusion, exclusion and independence atoms of arity at most $k-1$. Then the k-ary dependence, inclusion, exclusion and independence atoms are not expressible in L.

Remark

The arity hierarchy result above can be strengthened by extending L with

1. tensor conjunction (and other similar tensor connectives),
2. the Boolean disjunction, and
3. arbitrary generalized quantifiers.

Conclusion

- We defined three notions of dimension for families of sets $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$: upper, dual upper, and cylindrical dimensions $\mathrm{D}(\mathcal{A})$, $\mathrm{D}^{\mathrm{d}}(\mathcal{A})$, and $\mathrm{CD}(\mathcal{A})$.
- For each formula ϕ of a team semantical logic L we defined the corresponding dimensional function $\mathrm{Dim}_\phi \colon \mathbb{N} \to \mathbb{N}$ by setting $\mathrm{Dim}_\phi(n) = \sup\{\mathrm{D}(\|\phi\|^{\mathfrak{M}}) \mid |M| = n\}$ (and similarly $\mathrm{Dim}^{\mathrm{d}}{}_\phi$ and CDim_ϕ).

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- We developed a systematic method for proving that certain well-behaved logical operators preserve the growth class of dimensional functions: If $\mathrm{Dim}_\alpha \in \mathbb{O}$ for all atomic formulas α of L, and all logical operators Δ in L are local (and separable), then $\mathrm{Dim}_\phi \in \mathbb{O}$ for every formula ϕ of L.
- Since k-ary dependence, inclusion, exclusion and independence atoms are in a higher growth class than any lower arity atoms, as a corollary we obtained a strong arity hierarchy result for dependent atoms.