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Logical Charactarizations of algebraic circuit classes over rings

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1. Introduction
2. Models and Logics
3. Some Characterizations
4. Outlook

Introduction

- Parallel computation over an integral domain R
  - Algebraic circuits (over R)
    - AC_R^0
- Logics & Descriptive complexity
  - $\mathrm{AC}^0 = \mathrm{FO}$ [Im89]
  - $\mathrm{NC}^1 = \mathrm{FO}[\mathrm{BIT}]+\mathrm{GPR}_{\mathrm{bound}}$ [DHV18]
  - $\mathrm{AC}^1 = \mathrm{FO}[\mathrm{BIT}]+\mathrm{GPR}$ [DHV18]
  - $\mathrm{AC}_{\mathbb{R}}^0 = \mathrm{FO}_{\mathbb{R}}[\mathrm{Arb}_{\mathbb{R}}]$ [BV21]

Basic algebraic definitions

Recall:

A ring (with unity) is a set R equipped with two binary operations $+$ and $\times$, such that

- $(R, +)$ is an abelian group,
- $(R, \times)$ is a monoid,
- in particular, there is a $1 \in R$ such that $r \times 1 = 1 \times r = r$, for all $r \in R$.
- multiplication is distributive with respect to addition

Basic algebraic definitions

Recall:

An integral domain is a nonzero commutative ring without zero divisors, i.e. for every $a, b \neq 0 \in R : a \times b \neq 0$

- every field (e.g. $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p$, for a prime p)
- $\mathbb{Z}$, $\mathbb{R}[x]$, $\mathbb{Z}[i] \ldots$

Algebraic Circuits over R

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- Directed Acyclic Graph with node types:
  - input (fan-in 0)
  - constant (fan-in 0)
  - arithmetic (fan-in ≥ 0)
  - one of
    - sign (fan-in 1)
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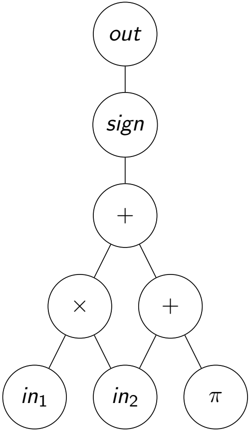

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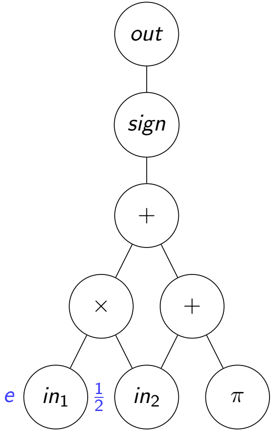

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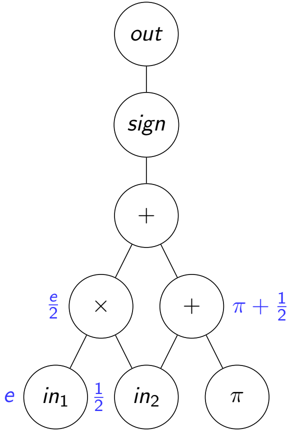

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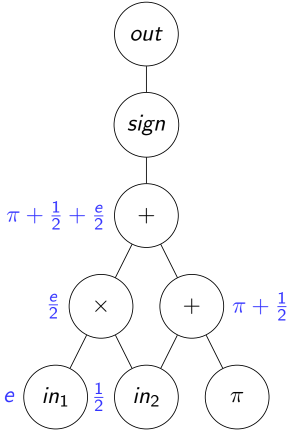

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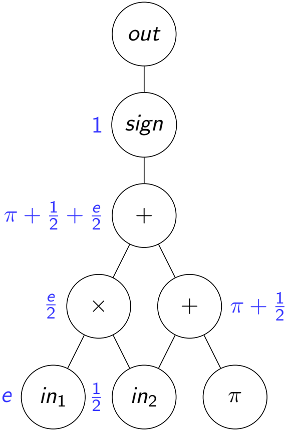

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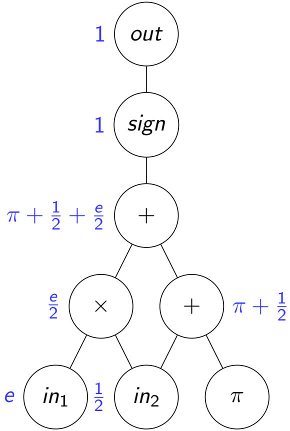

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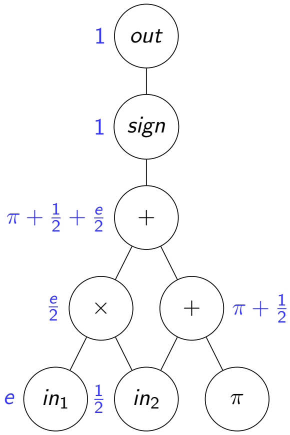
- *size*: number of gates
- *depth*: longest path from input to output



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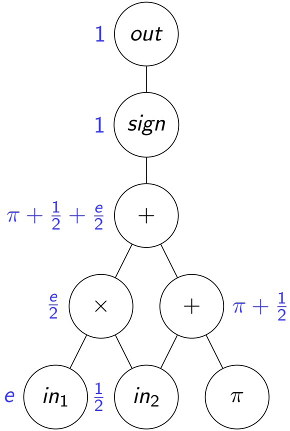
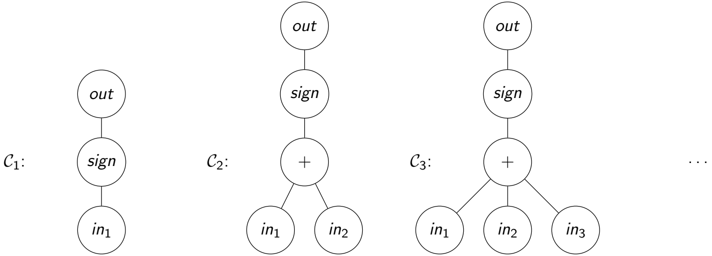
- *size*: number of gates
- *depth*: longest path from input to output
- non-uniform!



Circuit families

Circuit families

- Sequences $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \ldots)$ of circuits where $\mathcal{C}_i$ has i input gates
- If $\mathcal{C}_i$ computes $f_{\mathcal{C}_i}$ for all $i \in \mathbb{N}$, then $\mathcal{C}$ computes $f_{\mathcal{C}}(w) = f_{\mathcal{C}_{|w|}}(w)$.



Circuit families

Circuit families deciding sets

A circuit family $\mathcal{C}$ decides a set $S \subseteq \bigcup_{n \in \mathbb{N}} R^n$ iff $\mathcal{C}$ computes the characteristic function of S.

Circuit classes

- NC^i_R: sets decided by bounded fan-in circuits families of size $\mathcal{O}(n^{\mathcal{O}(1)})$ and depth in $\mathcal{O}(\log(n)^i)$
- AC^i_R: sets decided by unbounded fan-in circuit families of size $\mathcal{O}(n^{\mathcal{O}(1)})$ and depth in $\mathcal{O}(\log(n)^i)$

Uniformity

Uniform circuit families

- There is an R-machine M producing the circuit family.
  - i.e. M computes the function $n \mapsto \mathcal{C}_n$
- M works in polynomial time $\rightarrow$ P-uniform
  - $\mathrm{U_P}$-C is the subclass of C decided by P-uniform families
- M works in logarithmic time $\rightarrow$ L-uniform
  - $\mathrm{U_{LT}}$-C is the subclass of C decided by L-uniform families

First-order Logic over R

$$\varphi := \exists x_1 \exists x_2 \; val(x_1) = val(x_2) + \pi \wedge x_1 = succ(x_2)$$

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Definition: metafinite R-structures

- R-structure $\mathcal{D} = (\mathcal{A}, \mathcal{F})$ of signature $\sigma = (L_s, L_f)$
  - $\mathcal{A}$: finite structure of L_s with universe A
    - the *skeleton* of $\mathcal{D}$
  - $\mathcal{F}$: finite set of functions $X : A^k \to R$ interpreting symbols in L_f
    - the *arithmetic part* of $\mathcal{D}$

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Definition: FO_R

- formulas and terms over signature $\sigma = (L_s, L_f)$ for variables $x_1, x_2, \ldots$
  - *index terms*: variables, functions $f \in L_s$
  - *number terms*: ring elements, functions $g \in L_f$, $t_1 + t_2$, $t_1 \times t_2$, $sign(t_1)$
  - formulas
    - atomic: $t_1 = t_2$, $t_1 \leq t_2$, predicates $P \in L_s$
    - non-atomic: closure of atomic formulas under Boolean connectives and quantification $(\exists, \forall)$

First-order Logic over R - Example

$$\sigma = (\{succ\}, \{val\})$$
$$\varphi := \exists x_1 \exists x_2 \; val(x_1) = val(x_2) + \pi \land x_1 = succ(x_2)$$

First-order Logic over R - Example

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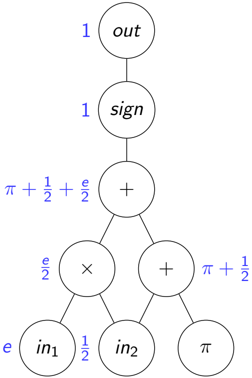
Structure $\mathcal{D}$:

$$A = \{v \mid v \in C\},$$

$$val(v) = \text{the value of } v \text{ on inputs } e \text{ and } \frac{1}{2},$$

$$succ(v) = \text{the successor gate of } v$$

Circuit C:



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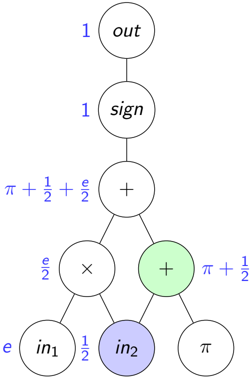

Extensions to FO_R

Additional functions / relations

- $FO_R[S]$ for a set S of functions and relations

Additional constructions

- the *sum*, *product* and *maximization* rules for creating number terms
  - to use $\sum_{i \in A} t(i)$, $\prod_{i \in A} t(i)$ and $\max_{i \in A}(t(i))$ in formulas

$$\varphi = \exists v_1 \; \underset{v_2}{sum}(g(v_2)) > g(v_1) \times 2$$

$$A = \{\diamondsuit, \spadesuit\}, \; g = \{\diamondsuit \mapsto \pi, \spadesuit \mapsto 42\}$$

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  - to use $\sum_{i \in A} t(i)$, $\prod_{i \in A} t(i)$ and $\max_{i \in A}(t(i))$ in formulas

$$\varphi = \exists v_1 \; \overbrace{\underset{v_2}{sum}(g(v_2))}^{42+\pi} > g(v_1) \times 2$$

$$A = \{\diamondsuit, \spadesuit\}, \; g = \{\diamondsuit \mapsto \pi, \spadesuit \mapsto 42\}$$

Logical Characterizations

Non-uniform AC^0_R

$$\mathrm{AC}^0_R = \mathsf{FO}_R[\mathsf{Arb}_R]$$

Polynomial-time uniform AC^0_R

$$\mathrm{U_P}\text{-}\mathrm{AC}^0_R = \mathsf{FO}_R[\mathsf{FTIME}_R(n^{\mathcal{O}(1)})]$$

Logarithmic-time uniform AC^0_R

$$\mathrm{U_{LT}}\text{-}\mathrm{AC}^0_R = \mathsf{FO}_R[\mathsf{FTIME}_R(\log(n))] + \mathsf{SUM}_R + \mathsf{PROD}_R$$

Guarded predicative/functional recursion

Definition: [Durand, Haak, Vollmer, 2018]

A formula φ is in $\mathrm{FO} + \mathrm{GPR}^1$ if it has the form

$$\varphi ::= [P(\overline{x}, \overline{y}) \equiv \theta(\overline{x}, \overline{y}, P)]\varphi(P) | \psi,$$

where ψ and θ are FO formulae with free variables $\overline{x}, \overline{y}$ such that each atomic sub-formula involving the symbol P

1. is of the form $P(\overline{x}, \overline{z})$, where $\overline{z}$ is in the scope of a guarded quantification $Q\overline{z}.((\overline{z} \leq \overline{y}/2) \wedge \xi(\overline{y}, \overline{z}))$ with $Q \in \{\forall, \exists\}$, $\xi \in \mathrm{FO}$ and
2. never occur in the scope of any quantification not guarded this way.

Results

Former results [DHV, 2018]

1. $\mathrm{NC}^1 = \mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^1_{\mathsf{bound}}$
2. $\mathrm{SAC}^1 = \mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^1_{\mathsf{semi}}$
3. $\mathrm{AC}^1 = \mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^1$
4. $\#\mathrm{AC}^0 = \#\mathsf{Win}\text{-}\mathrm{FO}[\mathrm{BIT}]$
5. $\#\mathrm{NC}^1 = \#\mathsf{Win}\text{-}\mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^1_{\mathsf{bound}}$
6. $\#\mathrm{SAC}^1 = \#\mathsf{Win}\text{-}\mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^1_{\mathsf{semi}}$
7. $\#\mathrm{AC}^1 = \#\mathsf{Win}\text{-}\mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^1$

Guarded predicative logic - ideas of adaptations

ideas

- Extend GPR in order to characterize the whole AC, NC and SAC -hierachies
  - Substitute the factor $\frac{1}{2}$ by $2^{\log_2(n)/\log_{2^i}(n)}$ for AC^i etc.
    - But exponantiation is not part of the logic

  $\Rightarrow$ GPR^i_R by tuples
- Add a similar construction to logics over metafinite structures
  - probably functional instead of predicative recursion

Further results

Goal

1. $\mathrm{NC}^i = \mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^i_{\text{bound}}$
2. $\mathrm{SAC}^i = \mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^i_{\text{semi}}$
3. $\mathrm{AC}^i = \mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^i$
4. $\#\mathrm{NC}^i = \#\text{Win-}\mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^i_{\text{bound}}$
5. $\#\mathrm{SAC}^i = \#\text{Win-}\mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^i_{\text{semi}}$
6. $\#\mathrm{AC}^i = \#\text{Win-}\mathrm{FO}[\mathrm{BIT}] + \mathrm{GPR}^i$
7. $\mathrm{NC}^i_R = \mathrm{FO}_R[\mathrm{Arb}_R] + \mathrm{GPR}^i_{R,\text{bound}}$
8. $\mathrm{AC}^i_R = \mathrm{FO}_R[\mathrm{Arb}_R] + \mathrm{GPR}^i_R$

Relationship between versions of AC^0_R over different rings – canonical maps

Definition:

A canonical ring map from a ring R_1 to a ring R_2 is a fixed injective function $f_{R_1,R_2}\colon R_1 \to R_2^k$ for some $k \in \mathbb{N}$.
f_{R_1,R_1} is the identity function.

Definition:

For two complexity classes $\mathcal{C}_{R_1}, \mathcal{C}_{R_2}$ over the respective rings R_1, R_2, we write

$$\mathcal{C}_{R_1} \subseteq_c \mathcal{C}_{R_2},$$

if for all languages $A \in \mathcal{C}_{R_1}$ the following holds:

$$\{f_{R_1,R_2}(x) \mid x \in A\} \in \mathcal{C}_{R_2^*}.$$

The relations $=_c$ and $\subset_c$ are defined analogously.

special cases of AC^0_R

Theorem

- $\mathrm{AC}^0_{\mathbb{Q}} =_c \mathrm{AC}^0_{\mathbb{Z}} =_c \mathrm{AC}^0_{\mathbb{Z}[i]} =_c \mathrm{AC}^0_{\mathbb{Z}[x]} =_c \mathrm{AC}^0_{\mathbb{Z}[x,y]}$
- $\mathrm{AC}^0_{\mathbb{C}} =_c \mathrm{AC}^0_{\mathbb{R}[i]} =_c \mathrm{AC}^0_{\mathbb{R}} =_c \mathrm{AC}^0_{\mathbb{R}[x]} =_c \mathrm{AC}^0_{\mathbb{R}[x,y]}$
- $\mathrm{AC}^0_{\mathbb{Z}} \subset_c \mathrm{AC}^0_{\mathbb{R}}$

idea:

Write a number $z = \frac{a}{b} \in \mathbb{Q}$ (resp. $z = a + bi \in \mathbb{C}$) as tuple (a, b) and use a node per tuple.

Future/Current Research

- How can these classes be contextualized?
  - separate AC^0_R and NC^1_R?
- Can we find a meaningful analogue of TC_R?
  - investigate then in particular the question, whether $\mathrm{TC}^0_R = \mathrm{NC}^1_R$?
- How are $\mathrm{AC}^0_{\mathbb{F}_p}$ and $\mathrm{AC}^0_{\mathbb{F}_q}$ for $p \neq q$ (prime?) related?
- Define SAC^i_R classes
  - Is it important which gate type we bound?
- Is there a connection between GPR-variations and fixed point logics like $\mathrm{FO(LFP)}$?

Sources

[Im89] Neil Immerman. Expressibility and Parallel Complexity. SIAM J. Comput. 18(3), 625-638 (1989)

[CM99] Felipe Cucker and Klaus Meer. Logics which capture complexity classes over the reals. J. Symb. Log., 64(1):363-390, 1999.

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[DHV18] Arnaud Durand, Anselm Haak, and Heribert Vollmer. Model-Theoretic Characterization of Boolean and Arithmetic Circuit Classes of Small Depth. LICS 2018: 354-363.

[BV21] Timon Barlag and Heribert Vollmer. A Logical Characterization of Constant-Depth Circuits over the Reals. WoLLIC 2021: 16-30

Definition (GPR^i_R)

For $i \geq 0$, a formula φ is in $\mathrm{FO}_R(\mathrm{GPR}^i_R)$ if it has the form

$$\varphi ::= [P(\overline{x}, \overline{y_1}, \ldots, \overline{y_i}) \equiv \theta(\overline{x}, \overline{y_1}, \ldots, \overline{y_i}, P)]\varphi(P)]|\psi,$$

where ψ and θ are FO_Rformulae with free variables $\overline{x}, \overline{y_1}, \ldots, \overline{y_i}$ such that each atomic sub-formula involving the symbol P

1. is of the form $P(\overline{x}, \overline{y_1}, \ldots, \overline{y_i})$, the $\overline{y_1}, \ldots, \overline{y_i}$ are in the scope of a guarded aggregation

$$A_{\overline{y_1}, \ldots, \overline{y_i}}.(\bigvee_{j=i}^{1} \overline{z_j} \leq \overline{y_j}/2 \wedge \bigwedge_{k=j-1}^{1} \overline{z_k} \leq \overline{y_k} \wedge \xi(\overline{y_1}, \ldots, \overline{y_i}, \overline{z_1}, \ldots, \overline{z_i}))$$

   with $A \in \{max, sum, prod\}$ and
2. never occur in the scope of any aggregation (or quantification) not guarded this way.