Introduction to algebraic topology, fall 2013

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Lecturer

Aleksandr Pasharin

Scope

10 sp.

Type

Advanced studies

About the course

The course is intended to be the first introduction to the singular homology theory and homological methods in algebraic topology.

The main idea of algebraic topology is to study topological problems using suitable **algebraic invariants**. These invariants enable one to convert a given difficult topological problem into algebraic problem, which is easier to solve than the original problem. For instance consider a very natural question - is plane R^2 homeomorphic to a space R^3? More generally can R^n be homeomorphic to R^m when n is not equal to m? Intuitively it is clear that the answer should be 'no', but it is surprisingly difficult to actually prove this precisely. Algebraic methods, such as homology, turn out to be extremely efficient in the course of studying this and other similar questions.

We start off with the brief excursion to the world of simplices, simplicial methods and Delta-complexes. Then we move on to the main subject of the course - construction of the singular homology theory. We go through all the essential properties of the singular homology and apply them in order to prove the classical topological results such as the Invariance of Domain, Brouwer's fixed-point theorem, Brouwer-Jordan separation theorem, the main theorem of Algrebra and others. We define and study the notion of the degree of the mapping. If schedule permits, we will also talk about CW-complexes, cellular homology and the classification of compact 2-manifolds.

For more detailed exposure, see Foreword.

Prerequisites

Linear algebra I, Algebra I, Topology I or corresponding courses. Topology II is useful, but not necessary. However, the notion of a general topological space and basic notions of topology -such as compactness, connectedness, quotient space et cetera, should be familiar. We will briefly revisit the basic definitions and results during the lectures.

Contents

1. Simplices, simplicial complexes and Delta-complexes.
2. Homological algebra - abelian free groups, chain complexes, homology, long exact homology sequence.
3. Singular homology theory - definition, basic properties, long exact homology sequences of pairs and triples, homotopy axiom, excision, mayer-viatoris sequence, computational devices, homology of the spheres, the degree of the map.
4. Applications - Brouwer fixed-point theorem, Jordan-Brouwer separation theorem, Invariance of Domain, generalizations to the theory of manifolds, Hairy Ball Theorem, Fundamental Theory of Algebra.
   If time permits:
5. CW - complexes and cellular homology. Classification of compact 2-manifolds.

Lecture material

Foreword

Part I - Simplices and simplicial methods

Part II - Chain complexes and homology groups

Part III - Singular homology

Notes - Parts I-III last updated on 8.12.

Bonus (won't be asked in exam):

CW-complexes and cellular homology

Summaries of the topics of the second part of the course:

Homological algebra

Singular homology

Simplicial homology

List of applications

Exercises

Exercises 1 Solutions 1

Exercises 2 Solutions 2

Exercises 3 Solutions 3

Exercises 4 Solutions 4

Exercises 5 Solutions 5

Exercises 6 Solutions 6

Exercises 7 Solutions 7

Exercises 8 Solutions 8

Exercises 9 Solutions 9

Exercises 10 Solutions 10

Exercises 11 Solutions 11

Exercises 12 Solutions 12

Exercises 13 Solutions 13

Brief Summary of the first part of the course

Middle Exam 1 - SOLUTIONS

Middle exam 2 - SOLUTIONS

Lectures

Weeks 36-42 and 44-50, Tuesday 12-14 in room B321 and Wednesday 12-14 in room C122. Two hours of exercise classes per week.

Exams

Two middle exams during fall.

Second exam of the course is December 13 at 12.30-16.30 in Exactum, CK111. Contact the lecturer for any information and questions.
You do not have to enroll.

First middle exam is TU 22.10 at 12-16 at A111 or B123, during official general exam of the department.

Enroll at the office latest 14.10! Contact the lecturer if you cannot attend first middle exam.

Contents of the first exam - Part I and Algebra part of Part II - everything up to the page 122, simplicial chains is the first topic NOT in the area of the 1st exam.

Exercise wise - everything covered in exercises 1-6

It is also possible to complete the course via general exam in the end of the course.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.

Bibliography

`Hatcher, A.: Algebraic Topology, available online at http://www.math.cornell.edu/~hatcher/AT/AT.pdf`

Maunder, C.R.F: Algebraic topology - Van Nostrand Reinhold Company, 1970.

Spanier, E.H: Algebraic Topology - McGraw-Hill, 1966.

Eilenberg, S., Steenrod, N.: Foundations of Algebraic Topology - Princeton University Press, 1952.

Short video on Klein's bottle - http://www.youtube.com/watch?v=E8rifKlq5hc

Registration

Did you forget to register? What to do?

Exercise session:

Group	Weekday	Time	Place	Teacher
1.	Tuesday	14-16	B321	Aleksandr Pasharin

Lecture log

3. 9 - Introductory speech " what is algebraic topology and why we need it ". Recollection of the elementary linear algebra (Part I, section 1). Definitions of affine and convex subsets.

4.9 - Affine and convex sets. Affine and convex hull. Affinely independent subsets, simplices.

10.9 - Standard simplices. Affine mappings. Boundary and interior points of a simplex. Topology - basic definitions, continuous mappings, homeomorphisms. Subspaces. Product topology. Metric spaces.

11.9 - Normed spaces. Standard topology on finite-dimensional vector spaces and simplices. Compactness and connectedness.

17.9 - Closure, interior and boundary. Theorem 3.19. Simplicial complexes - definition and basic properties. Weak topology.

18.9 - More on weak topology of simpicial complexes. Subdivisions. Barycentric subdivision.

24.9 - Essential properties of the barycentric subdivision. Proposition 4.21. Continuous mappings between polyhedra. Simplicial mappings. The concept of homotopy and its basic properties.

25.9 - Homotopy equivalences and classification of spaces up to a homotopy type. Lemma 5.8 and Lemma 5.9. Simplicial approximation theorem and its consequences.

1.10 - Introduction to Delta-complexes. Quotient spaces and quotient mappings.

2.10 - Delta-complexes - formalities, construction of the polyhedron, examples. "Cut and glue" technique (example 6.18). Short video about Klein's bottle. Algebra - concepts and examples of abelian groups and homomorphisms of groups.

8.10 - Quotient groups, factorization and isomorphism theorems for abelian groups. Linear combinations, linearly independent sets and basis in abelian groups. Finitely supported families. Construction of free abelian groups with given basis.

9.10 - Direct product and direct sum, their essential properties. Simplicial chains and singular chains.

15.10 - Boundary operator. Motivation for it through the notion of orientation. The essential property of boundary operator - Theorem 9.11. Chain complexes in general. Cycles, boundary and homology groups for chain complexes. Simplicial and singular homology groups.

16.10 - Calculation of simplicial homologies - Examples 9.4. - 9.7.

29.10 - Chain mappings. Mappings induced in homology. Topological invariance of singular homology

30.10 - Subcomplex and quotient complexes. Factorization and isomorphism theorems for chain complexes. Relative homology. Exact sequences. Short exact sequences - definition and Lemma 11.4

05.11 - Long exact homology sequence induced by a short exact sequence of chain complexes. Boundary homomoprhism of this exact sequence and its naturality.

06.11 - Long exact homology sequence of the pair of Delta-complexes (K, L). Long exact homology sequence of the pair of topological spaces (X, A) and its naturality with respect to continuous mappings of pairs. Five Lemma. Splitting exact sequences.

12.11 - Chapter 12 (from Part III). Singular homology and path components. Zeroth homology group. Augmentation. Reduced homology groups. Homology of a singleton space.

13.11 - Chapter 13 - Homotopy axiom of singular homology and its consequences. Excision property - formulation and calculation of homology groups of S^n assuming excision

19.11 - Proof of excision property through Theorem 14.6. Corollary 14.5.

20.11. - Equivalence of singular and simplicial homology theories

26.11 - Examples 15.3-15.5. Homology and composition of pathes (Lemma 9.8. and its consequences). Mayer-Vietoris Sequences. NOTE Example 16.10 and Proposition 16.12 are skipped for now on, we might come back to them later.

27.11 - Brouwer's fixed point Theorem 17.1, Jordan Separation Theorem 17.5 (and technical lemmas in between).

3.12 - Invariance of Domain. Manifolds. Degree of a mapping. Hairy Ball Theorem

4.12 - Suspension. Fundamental Theorem of Algebra. About CW complexes