Inverse problems course, spring 2013 Exercise 3 (8.2.2013)
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Note that this exercise has more than one page.

Please complete the theoretical exercises (marked with T) before the exercise session and be prepared to present your solution there.

T1. Consider the inverse problem defined by the measurement model $m = Ax$ in the cases

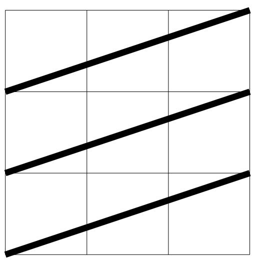
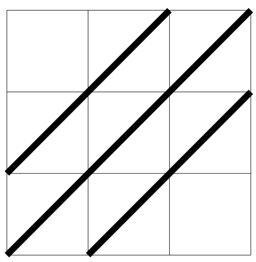
$$\text{(a) } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{(b) } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 13 & 31 \end{bmatrix}, \; m = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here the *input* is the measurement m, and the *solution* (also called *output*) is any $\widetilde{x}$ such that $m = A\widetilde{x}$.

Which of Hadamard's conditions is violated, if any? (Hadamard's conditions are (i) solution must exist, (ii) solution must be unique, and (iii) the output should depend continuously on the input.)

T2. Assume that the $n \times n$ matrix $U : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal: $UU^T = I = U^T U$. Show that $\|U^T y\| = \|y\|$ for any $y \in \mathbb{R}^n$.

T3. Thin lines depict pixels and thick lines X-rays in this image:

Give a numbering to the nine pixels ($f \in \mathbb{R}^9$) and to the six X-rays ($m \in \mathbb{R}^6$), and construct the matrix A for the measurement model $m = Af$. The length of the side of a pixel is one.

You can work on these Matlab exercises (marked with M) in the exercise session.

M1. Run the file `DC1_cont_data_comp.m`. Set the number of discretization points to 100 by writing `n = 100;` on line 12 in the file `DC2_discretedata_comp.m`. Then run the file `DC2_discretedata_comp.m` to produce discrete convolution data.

Compute the singular value decomposition of the 100×100 measurement matrix A using the code in the file `DC4_truncSVD_comp.m`. Determine the *condition number* of A, defined as the first (largest) singular value divided by the last (smallest) singular value.

Repeat the above with resolutions $n = 200, 300, 400$. What happens to the condition number when n grows?

M2. Find out the minimum relative error achievable by choosing the optimal number of singular values in the truncated SVD routine `DC4_truncSVD_comp.m`. Compute the minimal error for all four resolution values $n = 100, 200, 300, 400$.

You can work on these LaTeX exercises (marked with L) in the exercise session, or you can complete them beforehand.

L1. Include in your LaTeX document all the plots produced by Exercise M2.

L2. Add the following list of references (actually a list with only one entry) to your LaTeX document:

```
\begin{thebibliography}{99}
\bibitem{kurssikirja}
Mueller, J.L. and Siltanen, S: {\em Linear and Nonlinear Inverse
Problems with Practical Applications.}
SIAM 2012. ISBN 978-1-611972-33-7
\end{thebibliography}
```

Include in the text a reference to the book using the command

```
\cite{kurssikirja}
```