Compactness and Incompactness at small cardinals

Menachem Magidor

Institute of Mathematics
Hebrew University of Jerusalem

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Lecture 1
Compactness and Reflection

Large cardinals and compactness

Compact Small Cardinals
We consider properties of mathematical structures. Compactness property for a given property is a statement of the form *If every "small" substructure has the property then the structure has the property.* Typically "small" means *"Having cardinality less than ....."*
The existence of transversals: A *transversal* for a family of non-empty sets is a 1-1 choice function on the family. Suppose that every smaller cardinality subfamily of the family $\mathcal{F}$ has a transversal. Does $\mathcal{F}$ has a transversal? 
Special case: $\mathcal{F}$ is a family of countable sets.

Disjointifying $\mathcal{F}$ is family of infinite sets. Disjointifying $\mathcal{F}$ means picking for every $X \in \mathcal{F}$ a finite $z_X \subseteq X$ such that the family $\{X - z_X | X \in \mathcal{F}\}$ is a family of mutually disjoint sets. Suppose that every smaller cardinality subfamily of $\mathcal{F}$ can be disjointified. Can $\mathcal{F}$ be disjointified?
More combinatorics

**Chromatic numbers**  $G$ is a graph such that every (induced) subgroup of $G$ has a chromatic numbers $\leq \lambda$. Does $G$ has chromatic number $\leq \lambda$?

**Coloring Number**  The graph $G$ is said to have a coloring number $\lambda$ if the nodes of $G$ can be well-ordered such that every node is connected to less than $\lambda$ nodes appearing before it in the well ordering. Suppose that every smaller cardinality (induced) subgraph of $G$ has coloring number $\lambda$. Does $G$ has coloring number $\lambda$?
Algebraic examples

**Freeness of Abelian groups** An Abelian group $\mathcal{H}$ is free if it can be represented as the direct sum of copies of $\mathbb{Z}$. $\mathcal{H} = \bigoplus_i \mathbb{Z}$. Suppose that very smaller cardinality subgroup of $\mathcal{H}$ is free. Is $\mathcal{H}$ free?

**Free** The Abelian group $\mathcal{H}$ is said to be free* if it is a subgroup of a direct product of copies of $\mathbb{Z}$. $\mathcal{H} \subseteq \bigotimes_i \mathbb{Z}$. Suppose that every smaller cardinality subgroup of $\mathcal{H}$ is free*. Is $\mathcal{H}$ free*?
Topological Examples

**Metrizability**  
\( \mathcal{X} \) is a topological space such that every subspace of smaller cardinality is metric. Is \( \mathcal{X} \) metric? 
In order to avoid simple counter examples assume that \( \mathcal{X} \) is first countable.

**Collection-wise Hausdorff**  
\( \mathcal{X} \) is a topological space. \( Y \subseteq \mathcal{X} \) is a discrete closed set. We say that \( Y \) can be *separated* if there is a family of mutually disjoint open sets \( \{ U_y \mid y \in Y \} \) such that for \( y \in Y \) \( y \in U_y \). Suppose that every smaller cardinality subset of \( Y \) can be separated. Can \( Y \) be separated?
Reflection Principles

A reflection principle is a statement of the form "Suppose that the structure $A$ has a certain property then $A$ has a small substructure with the same property". Reflection principles are dual to compactness principles. Namely a reflection principle for a given property is equivalent to a compactness principle for the negation of the property and vice versa.
Other examples from Set Theory

Stationary set reflection \( \kappa \) a regular cardinal. \( S \subseteq \kappa \) is a stationary subset of \( \kappa \). Is there \( \alpha < \kappa \) such that \( S \cap \alpha \) is a stationary subset of \( \alpha \)?

The tree property \( T \) is a tree with \( \lambda \) many levels such that every level has cardinality less than \( \lambda \). Does \( T \) has a branch of length of length \( \lambda \)?

If the answer is always "Yes" then \( \lambda \) is said to have the tree property. \( \lambda \) is weakly compact if it is inaccessible and has the tree property.
Definition
A structure $\mathcal{A}$, whose signature contains a distinguished unary predicate $R$ is said to be of type $(\kappa, \lambda)$ if $|\mathcal{A}| = \kappa$ and $|R^\mathcal{A}| = \lambda$. We say the Chang’s conjecture $(\kappa, \lambda) \Rightarrow (\tilde{\kappa}, \tilde{\lambda})$ holds if every structure of countable signature of type $(\kappa, \lambda)$ has a substructure of type $(\tilde{\kappa}, \tilde{\lambda})$.

Question (Chang’s question)

*For which cardinals $\lambda, \tilde{\lambda}$ we can have $(\lambda^+, \lambda) \Rightarrow (\tilde{\lambda}^+, \tilde{\lambda})$?*

A related question: Suppose that the structure $\mathcal{A}$ is well ordered where the order type is a regular cardinal, Can we find a proper substructure whose order type is also a regular cardinal?
Reflection cardinals

Definition

1. Given a property of mathematical structure, we say that a cardinal $\kappa$ is a reflection cardinal for this property if every structure of cardinality $\kappa$ has a substructure of cardinality less than $\kappa$ having the given property.

2. $\kappa$ is a strong reflection cardinal if the every structure (no restriction on the cardinality) having this property has a substructure of cardinality less than $\kappa$ having the given property.
Compactness and Reflection

Compactness and Reflection

Compactness cardinals

Definition

1. A cardinal $\kappa$ is a (weakly) compact for the given property if a structure of cardinality $\kappa$ has the given property, given that every substructure of cardinality less than $\kappa$ has the property.

2. $\kappa$ is a strongly compact cardinal for the property if every structure (no restriction on the cardinality) has the property given that every substructure of cardinality less than $\kappa$ has the property.

The duality of reflection and compactness: $\kappa$ is a reflection cardinal for a certain property iff it is a compactness cardinal for the negation of the property.

Fact

If $\lambda$ is weakly compact then $\lambda$ is compact for any of the properties considered above.
Large large cardinals

A typical large cardinals definition is
$
\kappa
$ is ...... if there is a transitive class $M$ and an elementary embedding $j : V \rightarrow M$ having $\kappa$ as its critical point such that $M$ is closed under...

Examples

Measurability $\kappa$ is measurable iff there is $j : V \rightarrow M$ with $\kappa$ the critical point of $j$.

Strongness $\kappa$ is $\lambda$ strong if there is $j : V \rightarrow M$, $\kappa$ the critical point of $j$ and $V_{\kappa+\lambda} \subseteq M$. $\kappa$ is strong if it is $\lambda$ strong for all $\lambda$.

Supercompactness $\kappa$ is $\lambda$ supercompact if there is $j : V \rightarrow M$ with $\kappa$ its critical point $\kappa = \text{crit}(j)$ and such that $M^\lambda \subseteq M$. (Without loss of generality we can assume $j(\kappa) > \lambda$). $\kappa$ is supercompact if it is $\lambda$ supercompact for all $\lambda$. 
Supercompact is the ultimate strongly compact

Theorem

1. A cardinal $\kappa$ is supercompact iff $\kappa$ is a strongly compact cardinal for every property which can be expressed in the $\mathcal{L}^2_\kappa$ where $\mathcal{L}^2_\kappa$ is a Second order logic with conjunctions and disjunctions of size less than $\kappa$).

2. There is a supercompact cardinal iff there is a strongly compact cardinal for every second order property and the first supercompact is the first cardinal which strongly compact for any second order property.

All the properties we considered can be expressed in an appropriate infinitary version of second order logic.
Sketch of Proof

Let κ be supercompact, $\mathcal{A}$ a structure of signature of cardinality less than κ. Assume that every substructure of cardinality less than κ satisfy a certain property expressed by a second order sentence $\Psi$.

Let λ be the cardinality of $\mathcal{A}$. By the supercompactness of κ let $j : V \rightarrow M$ be an elementary embedding, crit($j$) = κ, $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$.

$j'' \mathcal{A} \in M$ is a substructure of $j(\mathcal{A})$. By the closure of $M$

$M \models |j'' \mathcal{A}| = \lambda < j(\kappa)$, hence by elementarity of $j$

$M \models "j'' \mathcal{A} \models \Psi"$.

But by the closure of $M$ every subset of $j'' \mathcal{A}$ is in $M$. Hence in $V$

$j'' \mathcal{A} \models \Psi$. $j'' \mathcal{A}$ is isomorphic to $\mathcal{A}$.

Hence $\mathcal{A} \models \Psi$. 
Failure of stationary reflection implies general failure of compactness

Fact
Suppose that $\kappa$ is a regular cardinal $\kappa$, $S \subseteq \kappa$ is a stationary set of $\kappa$ of such that for $\alpha \in S \text{ cof}(\alpha) = \omega$. Suppose that $S$ does not reflect. ($S \cap \alpha$ is not stationary in $\alpha$ for $\alpha < \kappa$). Then for most of the properties the compactness property fails. For all the above properties compactness fails for $\aleph_1$. 
Example of the Fact

Theorem
Suppose that $\kappa$ is a regular cardinal, $S \subseteq \kappa$ is a stationary set of $\kappa$ of such that for $\alpha \in S \text{ cof}(\alpha) = \omega$. Suppose that $S$ does not reflect. Then there is a first countable space $\mathcal{X}$ of cardinality $\kappa$ such that every subspace of smaller cardinality is metric, but $\mathcal{X}$ is not metric.

Proof.
For each limit $\alpha \in S$ pick an $\omega$-sequence $\langle \beta^\alpha_n | n < \omega \rangle$ cofinal in $\alpha$ made up of successor ordinals. We define a topology $\tau$ on $\kappa$ by specifying the discrete topology on $\kappa - S$. If $\alpha \in S$ then we take $U^\alpha_n = \{ \beta^\alpha_k | k \geq n \}$ for $n < \omega$ be a neighborhood basis for $\alpha$. □
Conclusion of the proof

Lemma

For every $\alpha < \kappa$ the space $\langle \alpha, \tau \rangle$ is metric.

Claim

Let $\alpha < \kappa$. Then the collection $\alpha \cap$ can be separated. More specifically for every $\beta < \alpha$ there is a family of mutually disjoint open sets $\langle U_\gamma | \gamma \in (\beta, \alpha) \cap S \rangle$ such that for $\gamma \in (\beta, \alpha) \cap S$ $\gamma \in U_\gamma$ and $U_\gamma \cap \beta = \emptyset$.

The proof is by induction on $\alpha$ using that $S$ does not reflect.

Lemma

The space $\langle \kappa, \tau \rangle$ is not metric.
Corollary

1. For all the properties above $\omega_1$ is not compact with respect to them.

2. Let $\kappa$ be regular. $S \subseteq \kappa$ is a stationary in $\kappa$ such that $\alpha \in S \Rightarrow \text{cof}(\alpha) = \omega$ and it does not reflect. Then for most of the properties above there is no strongly compact cardinal for this property which is $\leq \kappa$.

Since by a theorem of Jensen, in the constructible universe $L$ every regular $\kappa$ which is not weakly compact has a non reflecting stationary subset of points of cofinality $\omega$ we get:

Corollary (V=L)

For the most of the properties above:

1. A regular $\kappa$ is weakly compact with respect to the property iff it is weakly compact

2. There is no strongly compact cardinal for the property.
Compactness requires large cardinals

If $\kappa$ is strongly compact with respect to any the above properties. Then for every regular $\lambda \geq \kappa$ every stationary subset $\lambda$ of points of cofinality $\omega$ reflects. So some very large cardinals are consistent.

**Theorem**

Suppose that $\kappa$ is supercompact. Then there is a forcing extension of the universe in which the set:

$$\{ \lambda < \kappa \mid \lambda \text{ is regular and there is a non reflecting stationary subset of } \lambda \}$$

is unbounded in $\kappa$. Hence in the resulting model for most of the properties above the minimal strongly compact cardinal with respect to the property is the first supercompact.
Theorem (Shelah)

A singular cardinal $\lambda$ is weakly compact with respect to many properties. Including:

1. Existence of transversals for a family of countable sets.
2. Disjointifying a family of countable sets
3. Coloring number being less than $\rho$ where $\rho < \lambda$
4. Freeness of Abelian groups.
5. The property of being collection-wise Hausdorff for spaces which are locally of cardinality less than $\lambda$. 
Question

*Can a small regular cardinal be a compact cardinal for interesting properties?*
Theorem
Assume that the existence of a supercompact cardinal is consistent. Then it is consistent that $\omega_2$ is a strongly compact cardinal with respect to the property of disjointifying families of countable sets.

Proof.
Let $F$ be a family of countable sets. We say that $X \in P_{\omega_1}(F)$ is bad if there is $A \in (F - X)$ such that $A \cap \bigcup X$ is infinite.

Lemma
Assume that $|F| = \lambda$ where $\lambda$ is regular. Suppose that every smaller cardinality subset can be disjointified. Then $F$ can be disjointified iff

$$\{ X \in P_{\omega_1}(F) \mid X \text{ is bad} \}$$

is not stationary in $P_{\omega_1}(F)$. 
Continuation of the proof

Let $\kappa$ be supercompact. One obtains the model by forcing with the Levy collapse of all the cardinals in the interval $(\omega_1, \kappa)$ to $\omega_1$, $Col(\omega_1, < \kappa)$ which is a $\sigma$ closed forcing notion. Let $V_1$ be the resulting model.

The theorem will be proved if we can show that in $V_1$ if $\mathcal{F}$ is a family of countable sets such that every subfamily of cardinality $\leq \omega_1$ can be disjointified, then $\mathcal{F}$ can be disjointified. We prove it by induction on $\lambda = |\mathcal{F}|$.

$\lambda \leq \omega_1$ is trivial. The case $\lambda$ singular follows from Shelah’s singular compactness. So we are left with the case $\lambda$ regular. By the induction assumption, every subfamily of $\mathcal{F}$ of smaller cardinality can be disjointified.

Suppose it fails. Then the set $\{ X \in P_{\omega_1}(\mathcal{F}) | X \text{ is bad} \}$ is stationary in $P_{\omega_1}(\mathcal{F})$. 
Proof Concluded

The fact that $\kappa$ was supercompact in $V$ is used to show:

**Claim**

In $V_1$ there is $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality $\leq \omega_1$ such that

$$\{X \in P_{\omega_1}(\mathcal{F}') | X \text{ is bad for } \mathcal{F}' \}$$

By the lemma $\mathcal{F}'$ of the claim can not be disjointified. A contradiction.
Sketch of proof for the claim

Let $G \subseteq Col(\omega_1, < \kappa)$ be the generic filter such that $V_1 = V[G]$. $G$ can be extended to a generic filter $G^* \subseteq Col(\omega_1, < j(\kappa))$. Note that the forcing for doing it is a $\sigma$ closed forcing over $V_1$.

Fact

In $V[G^*]$ there is an elementary embedding $j^* : V[G] \rightarrow M[G^*]$ extending $j$.

$j^*([F] \subseteq j^*([F])$ which in $M[G^*]$ has cardinality $\omega_1$ and it is a witness to the truth of the claim in $M[G^*]$ for the family $j^*([F])$. By the elementarity of $j^*$ we can the truth of the claim in $V_1$ for $[F]$. 
$\omega_2$ can be strongly compact for many other properties

Let $V_1$ be as above the universe one gets by Levy collapsing all cardinals in the interval $(\omega_1, \kappa)$ to $\omega_1$, where $\kappa$ is supercompact. As in the previous proof one gets:

**Theorem (Shelah)**

1. In $V_1$ $\omega_2$ is strongly compact for the property of collection-wise Hausdorff in a topological space which locally countable.

2. In $V_1$ $\omega_2$ is strongly compact for the property of a graph having coloring number $\aleph_0$.

**Question**

*For the other properties we listed, how small the first strongly compact cardinal can be?*