Inner Models from Extended Logics

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1. J. Kennedy, M. Magidor and J. Väänänen, *Inner Models from Extended Logics: Part 1*, Journal of Mathematical Logic vol 21(2), 2021, Available in arXiv.
2. J. Kennedy, M. Magidor and J. Väänänen, *Inner Models from Extended Logics: Part 2*, Submitted, Available in arXiv.

The Löwenheim-Skolem Theorem

- Consider $\varphi \in L_{\omega_1\omega}$ and a model $\mathcal{M}$.
- Kueker game in M: $\begin{array}{c|cccccc} I & a_1 & & a_3 & & \ldots & \\ \hline II & & a_2 & & a_4 & & \ldots \end{array}$
- II wins if $\mathcal{M}\restriction\{a_1, a_2, a_3, \ldots\} \models \varphi$
- Player II has a winning strategy in the Kueker game.
- Equivalently, there is a club of countable submodels that satisfy φ.
- $\mathcal{M} \models \mathrm{aa}\, s\, \varphi^{(s)}$: "almost all" countable submodels satisfy φ

Stationary logic $L(\mathrm{aa})$

- $\mathrm{aa}s\varphi(s, \boldsymbol{a}) \iff \{A \in \mathcal{P}_{\omega_1}(M) : \mathcal{M} \models \varphi(A, \boldsymbol{a})\}$ contains a club of countable subsets M.
- $\mathtt{stat}s\varphi \equiv_{\mathrm{def}} \neg\mathrm{aa}s\neg\varphi$
- $\mathcal{M} \models \forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \varphi(\{x_0, x_1, x_2, \ldots\}, \boldsymbol{a})$
- $Q_1 x\varphi(x, \boldsymbol{a}) \iff |\{b \in M : \mathcal{M} \models \varphi(b, \boldsymbol{a})\}| \geq \aleph_1 \iff \neg\mathrm{aa}s\forall y(\varphi(y) \to s(y))$.
- $Q^{cof}_{\omega} xy\varphi(x, y, \boldsymbol{a}) \iff$ the cofinality of the linear order $\{(b, c) : \mathcal{M} \models \varphi(b, c, \boldsymbol{a})\}$ is ω i.e. $\mathrm{aa}s\forall x(\exists y\varphi(x, y) \to \exists y(\varphi(x, y) \wedge s(y)))$
- Craig($L(Q^{cof}_{\omega})$,$L(\mathrm{aa})$).

Completeness of $L(aa)$

- Axioms:

$$
\begin{array}{ll}
(A0) & \mathtt{aa}s\varphi(s) \leftrightarrow \mathtt{aa}t\varphi(t) \\
(A1) & \neg\mathtt{aa}s(\bot) \\
(A2) & \mathtt{aa}s(x \in s),\ \mathtt{aa}t(s \subseteq t) \\
(A3) & (\mathtt{aa}s\varphi \wedge \mathtt{aa}s\psi) \rightarrow \mathtt{aa}s(\varphi \wedge \psi) \\
(A4) & \mathtt{aa}s(\varphi \rightarrow \psi) \rightarrow (\mathtt{aa}s\varphi \rightarrow \mathtt{aa}s\psi) \\
(A5) & \forall x \mathtt{aa}s\varphi(x,s) \rightarrow \mathtt{aa}s\forall x \in s\varphi(x,s).
\end{array}
$$

- Completeness Theorem.

Proof of the Completeness Theorem

We construct[1]:

- $(M_0, T_0) \preccurlyeq (M_1, T_1) \preccurlyeq \ldots (M_\alpha, T_\alpha) \preccurlyeq \ldots \preccurlyeq (M_{\omega_1}, T_{\omega_1})$
- $\mathrm{aa}\, s \varphi(s, \boldsymbol{a}) \in T_\alpha \Rightarrow \varphi(P_\alpha, \boldsymbol{a}) \in T_{\alpha+1}$, $P_\alpha^{M_{\alpha+1}} = M_\alpha$
- $\mathrm{stat}\, s \varphi(s, \boldsymbol{a}) \in T_\alpha \Rightarrow \varphi(P_\alpha, \boldsymbol{a}) \in T_{\alpha+1}$ for stationary many α, using a different stationary set for each φ.

In the end:

- $\mathrm{aa}\, s \varphi(s, \boldsymbol{a}) \in T_{\omega_1} \Rightarrow M_{\omega_1} \models \mathrm{aa}\, s \varphi(s, \boldsymbol{a})$
- $\mathrm{stat}\, s \varphi(s, \boldsymbol{a}) \in T_\alpha \Rightarrow M_{\omega_1} \models \mathrm{stat}\, s \varphi(s, \boldsymbol{a})$

[1] Abusing notation.

Löwenheim-Skolem-Tarski property LST(L(aa))

- LST(L(aa)): For every $\mathcal{M}$ and every $X \in [M]^{\leq \aleph_1}$ there is $\mathcal{N} \preccurlyeq_{aa} \mathcal{M}$ such that $|N| \leq \aleph_1$ and $X \subseteq N$.
- Violates V=L: Let $\mathcal{M} = (\omega_2, <, S)$, where S is a non-reflecting stationary set.

LST(L(aa)) follows from PFA^{++}

- PFA^{++}: If $\mathbb{P}$ is proper, D_α, $\alpha < \omega_1$, are dense open sets, and t_α, $\alpha < \omega_1$ are $\mathbb{P}$-names such that $\Vdash_{\mathbb{P}}$ "$t_\alpha \subseteq \omega_1$ is stationary", then there is a compatible $G \subseteq \mathbb{P}$ meeting each D_α such that t_α^G is stationary for all $\alpha < \omega_1$.
- PFA^{++} implies $\mathrm{LST}(L(\mathsf{aa}))$.

Proof

- Let $\mathbb{P}$ collapse $|M|$ to $\aleph_1$. $\mathbb{P}$-name $\tilde{F} : \omega_1 \to M$ onto.
- Note: $\mathbb{P} \Vdash$ "$S \subseteq \mathcal{P}_{<\omega_1}(M)$ stationary iff $\{\alpha < \omega_1 : \tilde{F}[\alpha] \in S\}$ stationary".
- If $p \Vdash$ "$\mathcal{M} \models \mathrm{stat}\, s \theta(s, \tilde{F}[\delta_1], \ldots, \tilde{F}[\delta_n], \tilde{F}(\eta_1), \ldots, \tilde{F}(\eta_m))$", then p forces the term $\sigma_{\vec{\delta},\vec{\eta}}$ to be a name for the set of β such that $\mathcal{M} \models \theta(\tilde{F}[\beta], \tilde{F}[\delta_1], \ldots, \tilde{F}[\delta_n], \tilde{F}(\eta_1), \ldots, \tilde{F}(\eta_m))$.
- PFA^{++} gives a compatible $G \subseteq \mathbb{P}$ which meets some obvious dense sets and which renders each $\sigma_{\vec{\delta},\vec{\eta}}$ stationary.
- The model $\mathcal{N}$ is the submodel of $\mathcal{M}$ with $\tilde{F}^G[\omega_1]$ as domain.

Gödel's L-hierarchy

$$
\begin{cases}
L_0 & = & \emptyset \\
L_\nu & = & \bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu \\
L_{\alpha+1} & = & \mathrm{Def}_{FO}(L_\alpha) \\
L & = & \bigcup_\alpha L_\alpha .
\end{cases}
$$

The inner model $C(aa)$, an analogue of L, based on stationary logic

Recall the J-hierarchy: Suppose T is a set or a class.

$$\begin{cases} J_0^T & = & \emptyset \\ J_{\alpha+\omega}^T & = & \mathrm{rud}_T(J_\alpha^T \cup \{J_\alpha^T\}) \\ J_{\omega\nu}^T & = & \bigcup_{\alpha<\omega\nu} J_\alpha^T, \text{ for } \nu = \cup\nu. \end{cases}$$

Here rud_T means the closure under rudimentary functions[2] and the operation $x \mapsto x \cap T$

[2] projection, difference, unordered pair, general composition, general union

The definition of $C(\text{aa})$

$$
\begin{cases}
J'_0 & = & \emptyset \\
J'_{\omega\nu} & = & \bigcup_{\alpha<\nu} J'_\alpha \text{ for limit } \nu \\
J'_{\alpha+\omega} & = & \text{rud}_P(J'_\alpha \cup \{J'_\alpha\}), \text{ where} \\
 & & P = Tr \restriction (\alpha+\omega) \\
 & & Tr \restriction \beta = \bigcup_{\gamma<\beta}(\{\gamma\} \times Tr_\gamma) \\
 & & Tr_\gamma = \{\varphi(\vec{b}) : (J'_\gamma, \in, Tr \restriction \beta) \models \varphi(\vec{b}), \\
 & & \qquad\qquad \varphi(\vec{x}) \in L(\text{aa}), \vec{b} \in J'_\gamma\} \\
C(\text{aa}) & = & \bigcup_\alpha J'_\alpha
\end{cases}
$$

It follows: $C(\text{aa}) \models$ ZFC and there is a canonical well-order $\prec$ of $C(aa)$.

The older definition[3] of $C(aa)$

Compare with

$$\begin{cases} L'_0 & = & \emptyset \\ L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \\ L'_{\alpha+1} & = & \mathrm{Def}_{L(\mathtt{aa})}(L'_\alpha) \\ C_o(\mathtt{aa}) & = & \bigcup_\alpha L'_\alpha. \end{cases}$$

For most logics the two definitions agree, but not always. We shall give an example at the end of this talk.

[3] In the arXiv version.

A crucial property of $C(aa)$: Club Determinacy

- For all α:

$$(J'_\alpha, Tr \restriction \alpha) \models \forall \bar{x}[\mathrm{aa}s\varphi(\bar{x}, \bar{t}, s) \vee \mathrm{aa}s\neg\varphi(\bar{x}, \bar{t}, s)],$$

  where $\varphi(\bar{x}, \bar{t}, s)$ is any formula in $L(\mathrm{aa})$ and $\bar{t}$ is a finite sequence of countable subsets of J'_α.
- Follows from a proper class of Woodin cardinals. (See below)
- Follows from PFA^{++}. (See below)

Stationary tower forcing

- Suppose λ is Woodin (i.e. for every $f : \lambda \to \lambda$ there is $\kappa < \lambda$ closed under f and $j : V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subseteq M$.)
- There is a po-set $Q_{<\lambda}$ such that if G is $Q_{<\lambda}$-generic then there is $j : V \to M \subseteq V[G]$ such that $j(\omega_1) = \delta$,
- ${}^{\omega}M \subseteq M$

The main technical tools:

Proposition

Suppose λ is Woodin and G is $Q_{<\lambda}$-generic over V. For every set A in V, if $S \subseteq \mathcal{P}_\lambda(A)$ is stationary in V, then it is a stationary subset of $\mathcal{P}_\lambda(A)$ in $V[G]$.

Club Determinacy from a proper class of Woodin cardinals.

- Assume Club Determinacy fails.
- After some preliminary forcing we still have a Woodin cardinal δ and a measurable above.
- Now we use stationary tower forcing and obtain $j : V \to M$.
- We compare $C(\mathrm{aa}_\delta)^V$, $C(\mathrm{aa})^M$, level by level, and show that they are the same model.
- As we do this, we establish Club Determinacy in V.

Club Detreminacy from PFA^{++}

- After some preliminary steps, involving $\mathrm{LST}(L(aa))$...
- ...we use the result of Jensen, Schimmerling, Schindler and Steel (2009) to the effect that PFA implies that for any set X there is an inner model with a proper class of Woodin cardinals, containing X.
- Then we proceed as above.

Theorem

Assuming Club Determinacy, every regular $\kappa \geq \aleph_1$ is measurable in $C(\mathrm{aa})$.

Proof.

Let $\mathcal{F}_{aa} = \mathcal{F}^{\omega}(\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}) \cap C(\mathrm{aa})$. This is a normal filter on κ in $C(\mathrm{aa})$. Suppose $X \subseteq \kappa$ is in $C(\mathrm{aa})$. There are $\alpha > \kappa$ and $\varphi(x, \boldsymbol{y})$ such that

$$X = \{\beta < \kappa : \boldsymbol{J}'_{\alpha} \models \varphi(\beta, \boldsymbol{b})\}$$

for some $\boldsymbol{b} \in J'_{\alpha}$. Since $\boldsymbol{J}'_{\alpha}$ is club determined,

$$\boldsymbol{J}'_{\alpha} \models \mathrm{aa}s \exists x(x = \sup(s) \wedge x < \kappa \wedge \varphi(x, \boldsymbol{b})) \text{ or}$$

$$\boldsymbol{J}'_{\alpha} \models \mathrm{aa}s \neg \exists x(x = \sup(s) \wedge x < \kappa \wedge \varphi(x, \boldsymbol{b})).$$

In the first case $X \in \mathcal{F}_{aa}$. In the second case $\kappa \setminus X \in \mathcal{F}_{aa}$. □

Proposition

Suppose Club Determinacy holds, δ is Woodin, $G \subseteq Q_{<\delta}$ is generic and M is as above. Then

$$C(\mathtt{aa})^M = C(\mathtt{aa}_\delta)^V.$$

Proof sketch.

The proof is, as in the proof of Club Determinacy, an inductive argument, level by level, based on Club Determinacy and properties of stationary sets in stationary tower forcing. □

Theorem

Suppose there are a proper class of Woodin cardinals. Then the first order theory of $C(\mathtt{aa})$ is (set) forcing absolute.

Proof.

Suppose $\mathbb{P}$ is a forcing notion and δ a Woodin cardinal $> |\mathbb{P}|$. Let $j : V \to M$ be the (generic) associated elementary embedding. Now

$$C(\mathtt{aa}) \equiv (C(\mathtt{aa}))^M = C(aa_\delta).$$

Let $H \subseteq \mathbb{P}$ be generic over V and $j' : V[H] \to M'$. Again:

$$(C(\mathtt{aa}))^{V[H]} \equiv (C(\mathtt{aa}))^{M'} = (C(aa_\delta))^{V[H]}.$$

But $(C(aa_\delta))^{V[H]} = C(aa_\delta)$, since $|\mathbb{P}| < \delta$. □

Now we proceed towards proving CH in $C(\mathtt{aa})$.

An *aa-premouse* is a structure

$$\boldsymbol{J}^T_\alpha = (J^T_\alpha, \in, T, T^*)$$

with possibly some unary predicates $\vec{P}$ ("expanded aa-premouse"), such that

(1) $T \subseteq \alpha \times L(aa)$, and for all $\beta < \alpha$, $\{\varphi(\vec{a}) : (\beta, \varphi(\vec{a})) \in T, \vec{a} \in J^T_\beta\}$ is a complete consistent $L(aa)$-theory extending the first order theory of $(J^T_\beta, \in, T{\restriction}\beta)$.

(2) T^* is a complete consistent $L(aa)$-theory extending the first order theory of $(J^T_\alpha, \in, T{\restriction}\alpha)$.

(3) $\varphi(\vec{a})^{(J^T_\beta)} \in T^* \iff (\beta, \varphi(\vec{a})) \in T$, when $\vec{a} \in J^T_\beta$.

(4) If $\exists x \varphi(x, \vec{a}) \in T^*$, then there is $b \in J^T_\alpha$ such that $\varphi(b, \vec{a}) \in T^*$.

(5) $\mathrm{aa}s \exists x \varphi(x, s, \vec{a}) \to \mathrm{aa}s \exists x (\varphi(x, s, \vec{a}) \wedge \forall y \prec x \neg \varphi(y, s, \vec{a})) \in T^*$

(6) The Club Determinacy schema is contained in T^*.

Definition

Suppose $\boldsymbol{J}_\alpha^T = (J_\alpha^T, \in, T, T^*)$ is an aa-premouse with vocabulary τ. A mapping $\pi : J_\beta^T \to J_\beta^S$ is called an *elementary embedding* of $\boldsymbol{J}_\alpha^T$ into the aa-premouse $\boldsymbol{J}_\beta^S = (J_\beta^S, \in, S, S^*)$, in symbols

$$\pi : \boldsymbol{J}_\alpha^T \to \boldsymbol{J}_\beta^S,$$

if π is an elementary embedding

$$(J_\alpha^T, \in, T, T^*) \to (J_\beta^S, \in, S, S^*) \restriction \tau$$

and for all $\varphi(x_{a_1}, \ldots, x_{a_n}) \in L(\mathrm{aa})$ with vocabulary τ and all $a_1, \ldots, a_n \in J_\alpha^T$,

$$\varphi(c_{a_1}, \ldots, c_{a_n}) \in T^* \iff \varphi(c_{\pi(a_1)}, \ldots, c_{\pi(a_n)}) \in S^*.$$

Example

The canonical example of an aa-premouse is

$$\mathcal{N} = (J'_\alpha, \in, Tr \restriction \alpha, Tr_\alpha),$$

where Tr_β is the $L(\mathtt{aa})$-theory of $(J'_\beta, \in, Tr \restriction \beta)$ for $\beta \leq \alpha$. Note that $\mathcal{N} \in C(\mathtt{aa})$. We obtain other examples of aa-premice by taking elementary substructures of $\mathcal{N}$. Since $\mathcal{N} \in C(\mathtt{aa})$, we can take such also inside $C(\mathtt{aa})$.

The aa-ultrapower of an aa-premouse

If $\mathtt{aa}s\exists x\varphi(s, x, \boldsymbol{a}) \in T^*$, we use the term

$$f_{\varphi(s,x,\boldsymbol{a})}(s)$$

to denote the $\prec$-minimal x satisfying $\varphi(s, x, \boldsymbol{a})$. In particular, the sentence

$$\mathtt{aa}s\exists x\varphi(s, x, \boldsymbol{a}) \to \mathtt{aa}s\varphi(s, f_{\varphi(s,x,\boldsymbol{a})}(s), \boldsymbol{a})$$

is in T^*.

Suppose $(J_\alpha^T, \in, T, T^*)$ is an aa-premouse. We define

$$\varphi(s, x, \boldsymbol{a}) \sim \varphi'(s, x, \boldsymbol{a}')$$

if and only if

$$\mathrm{aa}s(f_{\varphi(s,x,\boldsymbol{a})}(s) = f_{\varphi'(s,x,\boldsymbol{a}')}(s)) \in T^*.$$

The aa-ultrapower of $(J_\alpha^T, \in, T, T^*)$ has the set M^* of $\sim$-equivalence classes as its domain. The *canonical embedding* $j : J'_\alpha \to M^*$ is defined by $j(a) = [x = a]$. For predicates R we define:

$$R^{M^*}([\varphi_1(s, x, \boldsymbol{a}_1)], \ldots, [\varphi_n(s, x, \boldsymbol{a}_n)]) \iff$$

$$\mathrm{aa}sR(f_{\varphi_1(s,x,\boldsymbol{a}_1)}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a}_n)}(s)) \in T^*.$$

Definition

Let $\tau^* = \tau \cup \{P^*\}$, where P^* is a new unary predicate symbol. We make M a τ^*-structure by defining $(P^*)^M = \{j(a) : a \in J_\alpha^T\}$. We let $S = R_T^M$ and let S^* consist of

$$\psi(P^*, [\varphi_1(s, x, \boldsymbol{a})], \ldots, [\varphi_n(s, x, \boldsymbol{a})]),$$

where $\psi(s, x_1, \ldots, x_n)$ is a $\tau \setminus \{R_{T^*}\}$-formula of $L(\mathrm{aa})$ such that

$$\mathrm{aa}\, s\, \psi(s, f_{\varphi_1(s,x,\boldsymbol{a})}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a})}(s)) \in T^*.$$

We obtain a structure $(M, R_\in^M, S, S^*, (P^*)^M)$ that we call the *aa-ultrapower* of $(J_\alpha^T, \in, T, T^*)$ and denote it by

$$Ult(J_\alpha^T, \in, T, T^*).$$

Lemma

The aa-ultrapower $(M, R_{\in}^{M}, S, S^{*}, (P^{*})^{M})$*, if well-founded, collapses to an aa-premouse* $(J_{\beta}^{E}, \in, E, E^{*}, E^{**})$ *with vocabulary* τ^{*}*. The mapping j (composed with the collapse function) is an elementary embedding* $(J_{\alpha}^{T}, \in, T, T^{*}) \to (J_{\beta}^{E}, \in, E, E^{*})$ *(in the sense of Definition* **??***).*

Lemma

$$\psi(P^*, [\varphi_1(s, x, \boldsymbol{a})], \ldots, [\varphi_n(s, x, \boldsymbol{a})]) \in S^*$$

if and only if

$$\mathrm{aa}\, s\, \psi(s, f_{\varphi_1(s,x,\boldsymbol{a})}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a})}(s)) \in T^*.$$

Lemma (Łoś Lemma)

Suppose $(J_\alpha^T, \in, T, T^*)$ *is an aa-premouse and*

$$(M, E, S, S^*) = Ult(J_\alpha^T, \in, T, T^*).$$

The following holds for $L(\mathtt{aa})$*-formulas* $\psi(x_1, \ldots, x_n)$ *in the vocabulary* τ^**:*

$$\mathtt{aa} s \psi(f_{\varphi_1(s,x,\boldsymbol{a}_1)}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a}_n)}(\bar{s})) \in T^* \text{ if and only if}$$

$$\psi([\varphi_1(s, x, \boldsymbol{a}_1)], \ldots, [\varphi_n(s, x, \boldsymbol{a}_n)]) \in S^*.$$

Lemma

Suppose $(J_\alpha^T, \in, T, T^*)$ *is a countable (expanded) aa-premouse with vocabulary* τ *and*

$$\pi : (J_\alpha^T, \in, T, T^*) \to N$$

is an elementary embedding, where N *is an expansion (by unary predicates) of* $(J'_\beta, \in, Tr \restriction \beta, Tr_\beta)$ *to a* τ*-structure. There are* $P^+ \subseteq J'_\beta$ *and an elementary*

$$\pi^* : Ult(J_\alpha^T, \in, T, T^*) \to (N, P^+)$$

such that $\pi^*(j(a)) = \pi(a)$ *for all* $a \in J_\alpha^T$.

Lemma

- *Let us fix* $(J'_\zeta, Tr\restriction\zeta, Tr_\zeta)$.
- *Suppose* $((M_\beta, T_\beta, T^*_\beta), j_{\beta\gamma})$, $\beta \leq \gamma \leq \delta$, *where* $\delta = \cup\delta \leq \omega_1$, *is a directed system of* $\mathtt{aa}$*-premice (mod* $\cong$*). Let* τ_β *be the vocabulary of* $(M_\beta, T_\beta, T^*_\beta)$.
- *Suppose the mappings* $\pi_\beta : (M_\beta, T_\beta, T^*_\beta) \to N_\beta$, $\beta < \delta$, *are elementary, where* N_β *is an expansion of* $(J'_\zeta, Tr\restriction\zeta, Tr_\zeta)$ *to a* τ_β*-structure, such that* $N_\beta = N_\gamma \restriction \tau_\beta$ *and* $\pi_\beta(x) = \pi_\gamma(j_{\beta\gamma}(x))$ *for all* $x \in M_\beta$ *and all* $\beta < \gamma < \delta$.
- *Then there is an expansion* N_δ *of* $(J'_\zeta, Tr\restriction\zeta, Tr_\zeta)$ *to a* τ_δ*-structure and an elementary* $\pi_\delta : (M_\delta, T_\delta, T^*_\delta) \to N_\delta$ *such that* $N_\beta = N_\delta \restriction \tau_\beta$ *and* $\pi_\beta(x) = \pi_\delta(j_{\beta\delta}(x))$ *for all* $x \in M_\beta$ *and all* $\beta < \delta$.

Lemma

The (images of the) interpretations of the "new" unary predicates $\{P_\alpha : \alpha < \omega_1\}$ in M_{ω_1} form a club in $\mathcal{P}_{\omega_1}(M_{\omega_1})$.

Definition

We call the aa-premice $(M_\alpha, T_\alpha, T^*_\alpha)$ *iterates* of $(J^T_\alpha, \in, T, T^*)$. An aa-premouse $(J^T_\alpha, \in, T, T^*)$ is an **aa-mouse** if its iterates $(M_\alpha, T_\alpha, T^*_\alpha)$, $\alpha < \omega_1$, are all well-founded. In this case we say that the aa-premouse $(J^T_\alpha, \in, T, T^*)$ is *iterable*.

Lemma

Suppose M is countable and

$$(M, T, T^*) \preccurlyeq (J'_\alpha, Tr, Tr_\alpha).$$

Then (M, T, T^*) is an aa-mouse.

Proposition

Suppose (J_α^T, T, T^*) *is a countable aa-mouse and* $(M_{\omega_1}, T_{\omega_1}, T^*_{\omega_1})$ *is its* ω_1*'st iterate. Then*

$$\mathrm{aa}\, s\, \varphi(s, \boldsymbol{a}) \in T^*_{\omega_1} \iff (M_{\omega_1}, T_{\omega_1}) \models \mathrm{aa}\, s\, \varphi(s, \boldsymbol{a}).$$

Lemma

Suppose Club Determinacy and $(M_0, T_0, T_0^*) \prec (J'_\alpha, Tr \restriction \alpha, Tr_\alpha)$, α *limit. Then* $(M_{\omega_1}, T_{\omega_1}, T^*_{\omega_1})$ *does not have new reals over those in* (M_0, T_0, T_0^*).

Proof.

We use the facts that there are only countably many reals in $C(aa)$ to start with and, by Club Determinacy, any definable stationary set contains a club. □

Theorem

If Club Determinacy holds, then CH holds in $C(\mathtt{aa})$.

1. Suppose J'_α is a stage where a new real r of $C(\mathrm{aa})$ is constructed, i.e. $r \in J'_{\alpha+\omega} \setminus J'_\alpha$.
2. We show that $J'_\alpha \cap 2^\omega$ is countable in $C(\mathrm{aa})$.
3. W.l.o.g., $|\alpha| = \aleph_1$.
4. Let $(M_0, T_0, T_0^*) \in C(\mathrm{aa})$ be an aa-mouse (mod $\cong$) countable in $C(\mathrm{aa})$ such that

$$\{r, \alpha, J'_\alpha\} \subseteq (M_0, T_0, T_0^*) \preccurlyeq (J'_{\aleph_2^V}, Tr \restriction \aleph_2^V, Tr_{\aleph_2^V}).$$

5. So now $r \in (J'_{\alpha+\omega})^{M_0} \setminus (J'_\alpha)^{M_0}$.
6. We iterate (M_0, T_0, T_0^*) until we obtain $(M_{\omega_1}, T_{\omega_1}, T^*_{\omega_1})$. We show that $(J'_\alpha)^{M_{\omega_1}} = J'_\alpha$, whence $J'_\alpha \cap 2^\omega \subseteq M_{\omega_1}$.
7. Lemma above implies $M_{\omega_1} \cap 2^\omega \subseteq M_0$.
8. It follows that $J'_\alpha \cap 2^\omega$ is countable, as we wished to demonstrate.

Theorem

If club determinacy holds, there is a Δ^1_3 *well-ordering of the reals in* $C(\mathtt{aa})$.

Proof.

The canonical well-order $\prec$ of $C(\mathtt{aa})$ satisfies:

$$x \prec y \iff \exists z \subseteq \omega(\ z \text{ codes an aa-mouse } M \text{ such that}$$

$$x, y \in M \text{ and } M \models \text{“}x \prec y\text{”}).$$

The right hand side of the equivalence is Σ^1_3 and the claim follows. □

Recall $C_o(L^*)$

$$\begin{cases} L'_0 & = & \emptyset \\ L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \\ L'_{\alpha+1} & = & \mathrm{Def}_{L^*}(L'_\alpha) \\ C_o(L^*) & = & \bigcup_\alpha L'_\alpha . \end{cases}$$

AC may fail in $C_o(L^*)$

- AC holds in $C_o(L^*)$ if L^* is "adequate to truth in itself" as logics of the form $L(Q)$, Q a Lindström quantifier, are.
- It is not known whether $L(\mathrm{aa})$ is adequate to truth in itself.
- It is not known whether $C_o(aa)$ satisfies AC.
- We give an example of a logic L^* for which AC may consistently fail in $C_o(L^*)$.

A failure of AC in $C_o(L^*)$

Consider the quantifier

$$M \models Q_n^{ST} xyz\varphi(x,\vec{a})\psi(y,z,\vec{a}) \iff \psi(\cdot,\cdot,\vec{a}) \text{ has order-type } \aleph_{n+1}$$
$$\text{and } \varphi(\cdot,\vec{a}) \text{ is a stationary set of points}$$
$$\text{of cofinality } \aleph_n \text{ in } \psi(\cdot,\cdot,\vec{a}).$$

We let L^* be the extension of first order logic by the infinitely many quantifiers Q_n^{ST}, $n < \omega$. Note that $C_o(L^*) \models ZF$.

Proposition

Relative to the consistency of ZF, it is consistent that the Axiom of Choice fails in the inner model $C_o(L^*)$.

- Assume $V = L$. Let $S_{n,m}$, $j < \omega$, be disjoint stationary sets of ordinals $< \aleph_{n+1}$ of cofinality $\aleph_n$ such that the set $\{\alpha < \aleph_{n+1} : \mathrm{cf}(\alpha) = \aleph_n\} \setminus \bigcup_m S_{n,m}$ is also stationary.
- We force mutually generic Cohen-reals a_n, $n < \omega$. Let us call the po-set of this forcing $\mathbb{Q}$.
- Let $\mathbb{P}_n(a_n)$ force an $\aleph_n$-closed unbounded set C_{n+1} into the set $\bigcup_{m \in a_n} S_{n,m}$. Let $\vec{a} = \langle a_n : n < \omega \rangle$ and let $\mathbb{P}(\vec{a})$ be the product of $\mathbb{P}_n(a_n)$, $n < \omega$. In the extension by $\mathbb{P}(\vec{a})$ we have for all $m, n < \omega$:

$$m \in a_n \iff S_{n,m} \text{ is stationary},$$

  whence $\vec{a} \in C(L^*)$.
- Assume now $V = L[\vec{a}][\langle C_{n+1} : n < \omega \rangle]$.

Claim A: $C_o(L^*) \subseteq L[\vec{a}]$.

For a proof by induction, suppose $L'_\alpha \in L[\vec{a}]$. Suppose $Z \in L'_{\alpha+1}$ is an L^*-definable (with parameters) subset of L'_α. We shall show that Z is in $L[\vec{a}]$. Since we proceed by induction, this boils down to showing that if $X \in L[\vec{a}]$ is set of ordinals $< \aleph_{n+1} \leq \alpha$ of cofinality $\aleph_n$, then we can decide in $L[\vec{a}]$ whether X is stationary in V or not. To this end we shall show that the following conditions are equivalent:

($*$) X is stationary in V.

($**$) There are $m \in a_n$ and $Y \subseteq X$ such that $Y \in L$ and $Y \cap S_{n,m}$ is stationary in L.

Claim A proved.

Let us fix $n^* < \omega$ and form a new sequence

$$\vec{a}^* = \langle a_l : l < n^* \rangle^\frown \langle a^*_{n^*} \rangle^\frown \langle a_l : l > n^* \rangle,$$

where $a^*_{n^*}$ is a finite modification of a_{n^*}. Obviously, $L[\vec{a}] = L[\vec{a}^*]$. Let M_1 be obtained from $L[\vec{a}]$ by forcing with the po-set $\mathbb{P}(\vec{a})$ and M_2 from $L[\vec{a}^*]$ (i.e. $L[\vec{a}]$) by forcing with $\mathbb{P}(\vec{a}^*)$.

Claim B: $(C_o(L^*))^{M_1} = (C_o(L^*))^{M_2}$.

It follows that AC fails in $C_o(L^*)$ in the model M_1.

Summary

- We gave a new definition of $C(aa)$.
- A similar inner model $C(L^*)$ can be built for any logic L^*. Several have been considered.
- A new family of inner models of set theory arises, one emphasising the role of strong logics.
- Forcing absoluteness, large cardinals, CH, etc,...

Thank you!