Bounded Symbiosis and Upward Reflection

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In mathematics, there are two ways to describe a class of structures:

- **defining** in set theory: \( \{ \mathcal{A} : \Phi(\mathcal{A}) \} \)
- **axiomatizing** by logic: \( \{ \mathcal{A} : \mathcal{A} \models \phi \} \)
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Describe the class of all structures with 3 or more elements:

- In set theory \( \{ A : \Phi(A) \} \) where \( \Phi(x) = \lvert x \rvert \geq 3 \).
- In logic \( \{ A : A \models \phi \} \) where \( \phi \) is

\[
\exists x \exists y \exists z \ ( x \neq y \land x \neq z \land y \neq z )
\]
Defining or Axiomatizing?

Example 2

Describe the class of infinite structures:

\[ A : (A) \]

Impossible in \( L \). But, e.g., in \( L_1 \) we can axiomatize infinite structures:

\[ Q \exists x_0 \ldots \exists x_n : x_0 = x_1 = \cdots = x_n \]

Alternatively, add a generalized quantifier \( Q \) expressing "there are infinitely many".
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\phi \equiv \exists x_0 \exists x_1, \ldots \bigwedge_{i \neq j} x_i \neq x_j
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  $$\phi \equiv \exists x_0 \exists x_1, \cdots \bigwedge_{i \neq j} x_i \neq x_j$$

- Alternatively, add a generalized quantifier $Q_\infty$ expressing “there are infinitely many”.

Yuri Khomskii (UHH & AUC)  
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Describe the class of structures \((A, P)\) such that

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- In set theory: like above.
- Impossible in \(\mathcal{L}_{\omega\omega}, \mathcal{L}_{\omega_1\omega_1}\) or \(\mathcal{L}_{\omega_1}\).
- The **Härtig Quantifier** \(I\) is defined by

\[
A \models lxy\phi(x)\psi(y) \iff |\{a : A \models \phi[a]\}| = |\{b : A \models \psi[b]\}|
\]

Then this class is axiomatizable in \(\mathcal{L}_{\omega\omega}(I)\) by the formula \(\phi \equiv lxyP(x)\neg P(x)\).
Since $||=L^{\forall}$, any first-order axiomatizable model class is $\lambda$-class (closed under isomorphisms) is axiomatizable by the logic $(L^{\forall}(I))$ (definition later).

But not vice versa.

Every $(L^{\forall}(I))$ model class is $\lambda$-class.

But not vice versa.
Since $\models L_{\omega \omega}$ is $\Delta_1$, any first-order-axiomatizable model class is $\Delta_1$. But not vice versa.
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Since $\models L_{\omega} \omega$ is $\Delta_1$, any first-order-axiomatizable model class is $\Delta_1$.

But not vice versa.

Every $\Delta_1$-class (closed under isomorphisms) is axiomatizable by the logic $\Delta(L_{\omega} \omega(I))$ (definition later).
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But not vice versa.
Strength of set-theoretic and model-theoretic definability

- Since $|=\mathcal{L}_\omega \omega$ is $\Delta_1$, any first-order-axiomatizable model class is $\Delta_1$.
- But not vice versa.
- Every $\Delta_1$-class (closed under isomorphisms) is axiomatizable by the logic $\Delta(\mathcal{L}_\omega \omega(I))$ (definition later).
- But not vice versa.
- Every $\Delta(\mathcal{L}_\omega \omega(I))$ model class is $\Delta_2$. 
Since $\models_{\mathcal{L}_{\omega\omega}}$ is $\Delta_1$, any first-order-axiomatizable model class is $\Delta_1$.

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Every $\Delta_1$-class (closed under isomorphisms) is axiomatizable by the logic $\Delta(\mathcal{L}_{\omega\omega}(I))$ (definition later).

But not vice versa.

Every $\Delta(\mathcal{L}_{\omega\omega}(I))$ model class is $\Delta_2$.

But not vice versa.
Relative Strength

\[ \Delta_2 \leftrightarrow \Delta(\mathcal{L}_{\omega\omega}(l)) \]

\[ \Delta_1 \leftrightarrow \mathcal{L}_{\omega\omega} \]

Set Theory \quad Logic
An exact match between set-theory and logic?

Is there a correspondence between the expressive power of some strong logics \( \mathcal{L} \) and a certain set-theoretic complexity level?
An exact match between set-theory and logic?

Is there a correspondence between the expressive power of some strong logics $\mathcal{L}$ and a certain set-theoretic complexity level?

**Definition**

Let $R$ be a predicate in the language of set theory. We can classify formulas as $\Sigma_1(R)$, $\Pi_1(R)$, $\Delta_1(R)$ etc., if they have that complexity level with $R$ treated as a new symbol.
An exact match between set-theory and logic?

Is there a correspondence between the expressive power of some strong logics $\mathcal{L}$ and a certain set-theoretic complexity level?

**Definition**

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**Example:**

- $\text{Cd}(x) \leftrightarrow "x \text{ is a cardinal}".$
- $\text{PwSt}(x, y) \leftrightarrow "y \text{ is the power set of } x".$

\[
\text{If } R \text{ true, then } \Delta_i \leq \Delta_i(R) \leq \Delta_2
\]
Symbiosis (Väänänen, 1976)

**Idea:** a logic $\mathcal{L}$ is **symbiotic** with a predicate $R$, if $\mathcal{L}$ and $\Delta_1(R)$ have the same expressive power in terms of characterizing a class of structures in a fixed vocabulary.

Applications of Symbiosis

Why is this useful?

1. Set theoretic principles (e.g. reflection) corresponding to model-theoretic principles (e.g., Löwenheim-Skolem).

2. Large cardinal strength of model-theoretic principles.

3. Large cardinal strength of set-theoretic principles (e.g., Vopěnka-type principles)
Definition

The (downwards) Löwenheim-Skolem number $\text{LST}(\mathcal{L})$ is the least $\kappa$ (assuming it exists) such that if $\mathcal{A} \models_\mathcal{L} \phi$ then there is a sub-structure $\mathcal{B} \subseteq \mathcal{A}$ s.t. $|\mathcal{B}| < \kappa$ and $\mathcal{B} \models_\mathcal{L} \phi$.

Definition

The (downwards) structural reflection number $\text{SR}(R)$ for a predicate $R$ is the least $\kappa$ (assuming it exists) such that if $\mathcal{K}$ is a $\Sigma_1(R)$-class of $\tau$-structures, then for every $\mathcal{A} \in \mathcal{K}$ there is $\mathcal{B} \in \mathcal{K}$ with $|\mathcal{B}| < \kappa$ and $\mathcal{B} \preceq \mathcal{A}$.

Theorem (Bagaria-Väänänen 2015)

Suppose $\mathcal{L}$ and $R$ is symbiotic. Then $\text{LST}(\mathcal{L}) = \kappa$ iff $\text{SR}(R) = \kappa$. 
Definitions

- We consider extensions of first-order logic: \( \mathcal{L} \) may refer to any logic with a definable satisfaction relation \( \models \).

- E.g.: second-order logic with full semantics, first order logic \( \mathcal{L}_{\omega \omega} \) enhanced with generalized quantifiers.

- A class \( \mathcal{K} \) of \( \tau \)-structures is \( \mathcal{L} \)-axiomatizable if \( \mathcal{K} = \text{Mod}(\phi) = \{ A : A \models \mathcal{L} \phi \} \) for some \( \phi \) in \( \mathcal{L} \).

- We will always consider many-sorted languages.

- If \( \tau \subseteq \tau' \) and \( A \) is a \( \tau' \)-structure, then the \( \tau \)-reduct \( A \upharpoonright \tau \) is defined by ignoring all symbols not in \( \tau' \) and restricting the domain to the sorts in \( \tau \).

- In particular, \( A \upharpoonright \tau \) can have a smaller domain than \( A \).
The $\Delta$-operator

Definition
A class $\mathcal{K}$ of $\tau$-structures is $\Sigma(\mathcal{L})$-axiomatizable if $\mathcal{K} = \{ A|_\tau : A \models _\mathcal{L} \phi \}$ for $\phi$ in some finite extension $\tau'$ of $\tau$. 

Example: $\mathcal{K} = \{ (A, B) : |A| = |B| \}$ is $\mathcal{L}$-axiomatizable. Add a new function symbol $F$ and write expressing that $F$ is a bijection between the two sorts. Then $\mathcal{K}$ is the projection of $\text{Mod}(\mathcal{L})$ to the original language.
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**Example:** $\mathcal{K} = \{(A, B) : |A| = |B|\}$ is $\Sigma(\mathcal{L}_{\omega\omega})$. Add a new function symbol $F$ and write $\phi$ expressing that $F$ is a bijection between the two sorts. Then $\mathcal{K}$ is the projection of $\text{Mod}(\phi)$ to the original language.
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**Definition**

A class $\mathcal{K}$ of $\tau$-structures is **$\Delta(\mathcal{L})$-axiomatizable** if both $\mathcal{K}$ and its complement are $\Sigma(\mathcal{L})$-axiomatizable.
The $\Delta$-operator

Some remarks about the $\Delta$-operator.

1. $\Delta(L_{\omega \omega}) = L_{\omega \omega}$ and $\Delta(L^2) = L^2$.

2. In general: if $\mathcal{L}$ satisfies the **Craig Interpolation Theorem** then $\Delta(\mathcal{L}) = \mathcal{L}$.

3. The $\Delta$-operation preserves many properties of $\mathcal{L}$, e.g., downward Löwenheim-Skolem.

4. It is convenient to regard $\Delta(\mathcal{L})$ itself as an **abstract logic** (without syntax but with the model classes corresponding to $\Delta(\mathcal{L})$-axiomatizable classes).
Definition of Symbiosis

Definition (Väänänen)

\( \mathcal{L} \) and \( R \) are \textbf{symbiotic} if

1. \( \models_\mathcal{L} \) is \( \Delta_1(R) \), and

Converse?

Every \( \Delta_1(R) \)-class is \( \text{Mod}(\phi) \)

\( \leq - \text{closed} \)
Definition of Symbiosis

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1. \( \models_\mathcal{L} \) is \( \Delta_1(R) \), and

2. Every \( \Delta_1(R) \)-definable class of \( \tau \)-structures closed under isomorphisms is \( \Delta(\mathcal{L}) \)-axiomatizable.

\[ \mathcal{L} \models \Delta(\mathcal{L}) \]
$L^2$ and PwSt

Example

$L^2$ and PwSt are symbiotic.

Proof.

Note: $\forall \phi \in L^2$ is absolute for models of set theory which correctly interpret $P(x) = y$.
**Proof.**

**DEF:** M is PwSt-correct if

\[(\forall x \in \mathbb{L}^2 \phi) \iff \exists M (M \models \exists x \phi \land M \models (x =_{\mathbb{L}^2} \phi))\]

\[\vdash_{\Delta_1} (\forall x \in \mathbb{L}^2 \phi) \iff \exists M (M \models \exists x \phi \land M \models (x =_{\mathbb{L}^2} \phi))\]

\[\Sigma_1 (\text{PwSt}) \iff \forall M (M \models (x =_{\mathbb{L}^2} \phi) \rightarrow M \models (x =_{\mathbb{L}^2} \phi))\]

\[\Pi_1 (\text{PwSt}) \iff \Delta_1 (\text{PwSt})\]
$L^2$ and PwSt

Proof.

2. $K$ is $\Delta^1_1(PwSt)$ class. First look at $\exists_1(R)$ formula $\Phi$. $\psi \Pi^1_1(R)$

Now: You want $K = \text{proj of } \text{Mod}(\forall)$ to original lang.

Extend: $I + \text{lang of set theory } (E)$

Idea: $\forall$ express "$(N,E,c \ldots) \text{ is such that}$ is PwSt-correct"

$\Delta(L^2)$ $\Delta(L^2)$ $\Delta(L^2)$ $\Delta(L^2)$

$(N,E) \equiv (M,E) \vDash ZFC$ which

and $\Gamma \vDash \Phi(c)$ $\psi(c)$ $\vDash \Phi(c)$ $\psi(c)$

expr in $L^2$
$\mathcal{L}^2$ and PwSt

Essence of the proof:

1. $\models_{\mathcal{L}^2}$ is absolute for PwSt-correct models of set theory, and

2. Being (isomorphic to) a PwSt-correct model of set theory can be expressed in a $\Delta(\mathcal{L}^2)$ way.
In fact, this is the essence of Symbiosis.

Call a model \( M \) of set theory \( R\)-correct if for all \( x_1, \ldots, x_n \in M \) we have 
\[ M \models R(x_1, \ldots, x_n) \text{ iff } R(x_1, \ldots, x_n). \]

**Lemma**

\( \mathcal{L} \) and \( R \) are **symbiotic** if

1. \( \models_\mathcal{L} \) is absolute for \( R\)-correct models of set theory, and
2. being (isomorphic to) an \( R\)-correct model of set theory is \( \Delta(\mathcal{L}) \).
Symbiosis

\[
\Delta_2 = \Delta_1(PwSt) \quad \Delta(L^2)
\]

\[
\Delta_1(Cd) \quad \Delta(L_{\omega\omega}(I))
\]

\[
\Delta_1 \quad \Delta(L_{WF})
\]

Set Theory \quad Logic
Definition

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Theorem (Bagaria-Väänänen 2015)

Suppose $\mathcal{L}$ and $R$ is symbiotic. Then $\text{LST}(\mathcal{L}) = \kappa$ iff $\text{SR}(R) = \kappa$. 
About the proof

\[ \text{SR}(R) \subseteq K \quad \Rightarrow \quad \text{LSI}(L) \subseteq K \]

Other directions:

- Direct:
  - All of this can be expressed in \( \Delta(L) \)

- Symbiosis (2) is \( \Delta(R) \)

\[ \text{R symb. in } L \]

A satisfies \( \phi(\mathfrak{M}) \)

\( \Theta \text{ R-correct} \)

\[ H \Theta = \phi(\mathfrak{M}) \]

\( K = \{ \mu \mid \phi(\mathfrak{M}) \} \)
By Löwenheim–Skolem

\[ N \models \phi(B) \]

\[ N \text{ also } R\text{-correct} \]

\[ B \]

\[ \text{Lö-Sk} \]

\[ N \leq H\theta \]

\[ B \models A \]

\[ \text{coll} \downarrow N \]
What about other properties?

We were interested in other properties of $\mathcal{L}$, in particular $\kappa$-compactness?
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**Question**

*Which set-theoretic principle for $\Sigma_1(R)$-classes corresponds to $\kappa$-compactness of $\mathcal{L}$, for symbiotic $\mathcal{L}$ and $R$?*
What about other properties?

We were interested in other properties of $\mathcal{L}$, in particular $\kappa$-\textbf{compactness}?

**Question**

\textit{Which set-theoretic principle for $\Sigma_1(R)$-classes corresponds to $\kappa$-compactness of $\mathcal{L}$, for symbiotic $\mathcal{L}$ and $R$?}

Compactness is related to \textbf{upwards} Löwenheim-Skolem principles, so it is natural to look at them first.
Upwards Löwenheim-Skolem

Definition

The **upwards Löwenheim-Skolem number** $\text{ULST}(\mathcal{L})$ is the least $\kappa$ such that if $\mathcal{A} \models \mathcal{L} \phi$ and $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$ there is a super-structure $\mathcal{B} \supseteq \mathcal{A}$ with $|\mathcal{B}| \geq \kappa'$ and $\mathcal{B} \models \mathcal{L} \phi$. 
Definition

The **upwards Löwenheim-Skolem number** $\text{ULST}(\mathcal{L})$ is the least $\kappa$ such that if $\mathcal{A} \models \phi$ and $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$ there is a super-structure $\mathcal{B} \supseteq \mathcal{A}$ with $|\mathcal{B}| \geq \kappa'$ and $\mathcal{B} \models \phi$.

- The **Hanf number** of $\mathcal{L}$ is defined identically but without the assumption “super-structure”. Hanf numbers always exist in ZFC but $\text{ULST}(\mathcal{L})$ usually implies large cardinals.

- A suitable version of $\kappa$-compactness implies $\text{ULST}(\mathcal{L})$, but it is not clear whether (for which logics) the converse implication holds.
Problems

However, if we define an “upwards” version of Structural Reflection for $\Sigma_1(R)$-classes and try to carry over the same proof, we are faced with some difficulties:
Dealing with the problems of size

- Not just a problem with the proof: while the $\Delta$-operator preserves downwards Löwenheim-Skolem properties of a logic, it may not preserve the Hanf number (J. Väänänen, $\Delta$-extensions and Hanf-numbers, 1983).

- We need the bounded $\Delta$-operator.

- Also need some definable way of “linking” the size of a model $\mathcal{A}$ with the size of the larger model of set theory $M$ containing it.

- Need a bounded version of $\Sigma_1(R)$ formulas.

- We need to adapt the concept of symbiosis: Bounded Symbiosis.
Dealing with the problems of size
The Bounded $\Delta$-operator

Definition (Väänänen, 1983)

A class $\mathcal{K}$ of $\tau$-structures is $\Sigma^B(\mathcal{L})$-axiomatizable if there is $\phi$ is a finite extension $\tau'$ of $\tau$ such that

1. $\mathcal{K} = \{B|_\tau : B \models \mathcal{L} \phi\}$, and
2. For all $\mathcal{A}$ there is a cardinal $\lambda_{\mathcal{A}}$ such that for any $\tau'$-structure $B$, if $B \models \phi$ and $\mathcal{A} = B|_\tau$, then $|B| \leq \lambda_{\mathcal{A}}$.

A class is $\Delta^B(\mathcal{L})$-axiomatizable if both $\mathcal{K}$ and its complement are $\Sigma^B(\mathcal{L})$-axiomatizable.
For many logics, $\Delta(\mathcal{L}) = \Delta^B(\mathcal{L})$.

(Väänänen, 1983) It is consistent that for some logics $\Delta(\mathcal{L}) \neq \Delta^B(\mathcal{L})$.

(Väänänen, 1983) $\Delta^B$ preserves the Hanf number of a logic.
Definably Bounding Functions

Now we turn to set theory:

Definition

A class function \( F : \text{Card} \to \text{Card} \) is **definably bounding** if the class of structures

\[
\mathcal{K} := \{(A, B) : |B| \leq F(|A|)\}
\]

is \( \Sigma^B (\mathcal{L}_{\omega \omega}) \)-axiomatizable.

**Idea:** the **witness** \( B \) may be larger than \( A \), but not too much. This can be expressed in first-order logic with the help of additional sorts (but with a bound on the domain).
Lemma

$F(\kappa) = 2^\kappa$ is definably bounding.

Proof.

Extend the vocabulary with a new relation symbol $\dot{E} \subseteq \dot{A} \times \dot{B}$ and consider the first-order formula

$$\phi \equiv \forall b \forall b' (\forall a (a \dot{E} b \iff a \dot{E} b') \rightarrow b = b')$$

If $(A, B, E) \models \phi$ then $b \mapsto \{a \in A : a \dot{E} b\}$ is an injection from $B$ to $\mathcal{P}(A)$.

Also: if $F, G$ definably bounding then $F \circ G$ definably bounding. Hence $F(\kappa) = 2^{2^\kappa}$, $F(\kappa) = 2^{2^\cdot\cdot\cdot\kappa}$ etc. are definably bounding.
$\Sigma_1^F(R)$ and $\Delta_1^F(R)$ formulas

**Definition**

1. The **H-rank** of a set $x$, denoted by $\rho_H(x)$, is the least infinite $\kappa$ such that $x \in H_{\kappa^+}$.
2. Let $F$ be **definably bounding**. A formula $\phi(x)$ is $\Sigma_1^F$ if there exists a $\Delta_0$ formula $\psi(x, y)$ such that

   $$\forall x (\phi(x) \iff \exists y (\rho_H(y) \leq F(\rho_H(x)) \land \psi(x, y)))$$

3. $\Pi_1^F$ and $\Delta_1^F$ are defined as usual, and $\Delta_1^F(R)$ etc., if we add a predicate $R$.

**Example**

The satisfaction relation $\models_{\omega_1}$ is $\Delta_1^{id}$. 
\[ \Sigma^F_1(R) \text{ and } \Delta^F_1(R) \text{ formulas} \]

\[ \phi(x) \iff \exists y \psi(x,y) \]

\[ \rho(y) \leq F(\rho(x)) \]

\[ \Delta^0 \]
Definition (Galeotti-K-Väänänen)

\( \mathcal{L} \) and \( R \) are **bounded-symbiotic** if

1. \( \models_{\mathcal{L}} F(R) \) for some definably bounding \( F \), and
2. Every \( \Delta^E (R) \) class closed under isomorphisms is \( \Delta^B (\mathcal{L}) \)-axiomatizable.
Bounded Symbiosis

**Definition (Galeotti-K-Väänänen)**

$\mathcal{L}$ and $R$ are **bounded-symbiotic** if

1. $\models\mathcal{L}$ is $\Delta_1^F(R)$ for some definably bounding $F$, and
2. Every $\Delta_1^F(R)$ class closed under isomorphisms is $\Delta_1^B(\mathcal{L})$-axiomatizable.

**Lemma (Galeotti-K-Väänänen)**

All examples of $\mathcal{L}$ and $R$ which were known to be **symbiotic**, are in fact **bounded-symbiotic**.

In particular

- $\mathcal{L}_{WF}$ and $\emptyset$ ✓
- $\mathcal{L}_{\omega}(I)$ and Cd ✓
- $\mathcal{L}^2$ and PwSt ✓

are **bounded-symbiotic**.
Upwards Structural Reflection Principle

Definition

Let $R$ be a $\Pi_1$ predicate. The **bounded upwards structural reflection number** $\mathcal{USR}(R)$ is the least $\kappa$ (if it exists) such that:

For every definably bounding function $F$, and every $\Sigma_1^F(R)$-definable class of $\tau$-structures in a fixed vocabulary $\tau$ closed under isomorphisms:

If there is $A \in \mathcal{K}$ with $|A| \geq \kappa$, then for every $\kappa' > \kappa$ there is a $B \in \mathcal{K}$ with $|B| \geq \kappa'$ and an elementary embedding $e : A \preceq B$. 

$\Sigma_1^F(R)$
**Main Result**

Theorem (Galeotti-K-Väänänen)

Let $\mathcal{L}$ be a logic with $\Delta_0$-definable syntax and dependence number $= \omega$, and let $R$ be a $\Pi_1$ predicate. Assume that $\mathcal{L}$ and $R$ are **boundedly symbiotic**.

Then $\text{ULST}(\mathcal{L}) = \kappa$ iff $\text{USR}(R) = \kappa$. 

---

**0-definable syntax**

$L$ has $\Delta_0$-definable syntax if every formula (as a syntactic object) is $\Delta_0$-definable with parameter $\dd$. 

$L$ has dependence number $\omega$ if every $L$-formula uses at most finitely many symbols from the vocabulary. 

All logics obtained by adding finitely many generalized quantifiers to $L$ or $L^2$ satisfy these conditions. This covers most of the logics we are interested in.
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Let $\mathcal{L}$ be a logic with $\Delta_0$-definable syntax and dependence number $= \omega$, and let $R$ be a $\Pi_1$ predicate. Assume that $\mathcal{L}$ and $R$ are boundedly symbiotic.

Then $\text{ULST}(\mathcal{L}) = \kappa$ iff $\text{USR}(R) = \kappa$.

- $\mathcal{L}$ has $\Delta_0$-definable syntax if every formula $\phi$ (as a syntactic object) is $\Delta_0$-definable with parameter $\tau$.

- $\mathcal{L}$ has dependence number $\omega$ if every $\mathcal{L}$-formula $\phi$ uses at most finitely many symbols from the vocabulary.

All logics obtained by adding finitely many generalized quantifiers to $\mathcal{L}_{\omega\omega}$ or $\mathcal{L}^2$, satisfy these conditions. This covers most of the logics we are interested in.
Main Result

About the proof

\[ |M| \geq \ell (2^2 F(K')) \]

\[ |d| > k' \]

**upw. Low-Sk**

\[ M \leq N \]

\[ A \trianglelefteq_{\text{low}} B \]

|B| as large as needed

|B| > k'
Main Result

About the proof
Applications: \textit{USR} from large cardinals

We look at the \textbf{large cardinal strength} of \textit{ULST}(L^2) and \textit{USR}(PwSt).
Applications: \textit{USR} from large cardinals

We look at the \textbf{large cardinal strength} of ULST($\mathcal{L}^2$) and \textit{USR}(PwSt).

- Magidor 1971: $\kappa$ is the least \underline{extendible cardinal} iff $\mathcal{L}^2$ is $\kappa$-compact.
Applications: \( \text{USR} \) from large cardinals

We look at the **large cardinal strength** of ULST\((\mathcal{L}^2)\) and \( \text{USR}(\text{PwSt}) \).

- Magidor 1971: \( \kappa \) is the least **extendible cardinal** iff \( \mathcal{L}^2 \) is \( \kappa \)-compact.

**Corollary**

*If \( \kappa \) is the least extendible cardinal then ULST\((\mathcal{L}^2) \leq \kappa \).*
Applications: $USR$ from large cardinals

We look at the **large cardinal strength** of $ULST(L^2)$ and $USR(PwSt)$.

- Magidor 1971: $\kappa$ is the least **extendible cardinal** iff $L^2$ is $\kappa$-compact.

**Corollary**

If $\kappa$ is the least extendible cardinal then $ULST(L^2) \leq \kappa$.

**Corollary**

If $\kappa$ is the least extendible cardinal then $USR(PwSt) \leq \kappa$. 
Applications: $USR$ from large cardinals

We look at the large cardinal strength of $ULST(L^2)$ and $USR(PwSt)$.

- Magidor 1971: $\kappa$ is the least **extendible cardinal** iff $L^2$ is $\kappa$-compact.

**Corollary**

*If $\kappa$ is the least extendible cardinal then $ULST(L^2) \leq \kappa$.***

**Corollary**

*If $\kappa$ is the least extendible cardinal then $USR(PwSt) \leq \kappa$.***

**Corollary**

*If $\kappa$ is the least extendible cardinal and $R$ is any $\Pi_1$ predicate then $USR(R) \leq \kappa$.***

\[ \text{because} \quad \Delta_1(R) \leq \Delta_1(PwSt) \leq \Delta_2 \]
What about the converse? How much large cardinals can we obtain from $\text{USR}(\text{PwSt})$?

**Theorem**

If $\text{USR}(\text{PwSt})$ is defined, then there exists an $n$-extendible cardinal for every $n$. 

\[ \eta \text{- ext.} \]
Large cardinals from $USR(PwSt)$

What about the converse? How much large cardinals can we obtain from $USR(PwSt)$?

**Theorem**

If $USR(PwSt)$ is defined, then there exists an $n$-extendible cardinal for every $n$.

**Corollary**

If $ULST(L^2)$ is defined, then there exists an $n$-extendible cardinal for every $n$. 
What about the converse? How much large cardinals can we obtain from $\text{USR}(\text{PwSt})$?

**Theorem**

If $\text{USR}(\text{PwSt})$ is defined, then there exists an $n$-extendible cardinal for every $n$.

**Corollary**

If $\text{ULST}(\mathcal{L}^2)$ is defined, then there exists an $n$-extendible cardinal for every $n$.

**Conjecture**

$\text{USR}(\text{PwSt})$ is defined iff there exists an extendible cardinal.

\[ \forall \alpha \ (\alpha \text{- ext.}) \]
Open questions and future direction

Question
What is the exact large cardinal strength of $\mathcal{USR}(PwSt)$ and ULST($\mathcal{L}^2$)?

Question
What is the large cardinal strength of $\mathcal{USR}(R)$ and ULST($\mathcal{L}$) for other pairs of (boundedly) symbiotic $\mathcal{L}$ and $R$?

Question
Is there a set-theoretic principle for $\Sigma_1(R)$ classes which corresponds to $\kappa$-compactness of $\mathcal{L}$, for (boundedly) symbiotic $\mathcal{L}$ and $R$? Can this be used to compute large cardinal strength of $\kappa$-compactness for various logics $\mathcal{L}$?
Thank You!

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