Axiom Selection after Large Cardinals: Maximize and the Question of CH

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Cantor’s Continuum Hypothesis

- Set theory emerges from Cantor’s work on trigonometric series.
  - Sets of points as genuine math objects w/ operations on them.
- Systematic study of sets truly begins with Cantor’s Theorem:
  - There are more real numbers than natural numbers.
- Cantor proposes introducing numbers for the sizes of infinite sets.
  - Cantor proved $|\mathbb{N}| = \aleph_0$
  - Open question whether $|\mathbb{R}| = \aleph_1$ or $|\mathbb{R}| > \aleph_1$. 

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Axiom Selection after Large Cardinals
Early Work on Settling \( CH \)

- Cantor makes many attempts to prove \( CH \).
  - Proposed proofs rely on version of well-ordering principle.
  - “The concept of well-ordered set is fundamental... It is a basic law of thought [Denkgesetz], rich in consequences and particularly remarkable for its general validity, that it is always possible to bring any well-defined set into the form of a well-ordered set.” (Cantor, 1883)
- 1900: Hilbert raises Continuum Problem to prominence.
- In Heidelberg 1904, König presents “proof” that \( CH \) (and well-ordering principle) is false.
- Soon after, Zermelo finds crucial error in König’s proof.
  - Going further, Zermelo publishes proof of the well-ordering principle (using axiom of choice).
Zermelo’s Axiomatization

- Zermelo’s proof of the well-ordering principle faces widespread critique.
  - Type 1: the axiom of choice is suspect.
  - Type 2: risk of invalid proof methods (Burali-Forti).
- In order to defend his proof, Zermelo turned to the ancient notion of rigor.
  - Laid out a collection of axioms including choice (ZC).
  - (Mostly) explicit existence claims.
- In the 1920’s, Fraenkel and Skolem propose two friendly additions (implicitly found in Cantor).
  - This revised system is the set theoretic standard ZFC.
The Cumulative Hierarchy

- As ZFC becomes established, attempts to settle CH continue.
- In the mid-1930’s, Zermelo introduces a heuristic picture inspired by his axioms:
  - Let V stand for the collection of all sets.
  - We can think of V as generated by taking, in discrete stages, every possible subset of what sets already exist.
- Note! The axioms inspire the picture, not vice versa!
  - It is only much later (1970) that Martin suggests the cumulative hierarchy as justifying the axioms of ZFC.
Gödel’s Constructible Universe

- 1938: first big advance on CH from Gödel’s work on choice.
  - Gödel proves the consisty of ZFC from that of ZF.
  - Method: Instead of taking all arbitrary sets at each stage, only take those that are f.o.-definable.
    - This constructible universe (L) is a subclass of V with all ordinal numbers (inner model).
    - Is it proper? Not provable in ZFC.
    - The claim that “V = L” is consistent and definable in language of set theory.
  - Gödel further proves that $ZFC + V = L \models CH$. 
An Early Hope for Resolution...

Gödel briefly considers $ZFC + V = L$ as a “natural completion of the axioms of set theory”. (Gödel, 1938)
- Quickly comes to reject this idea (more to come...)

Instead, Gödel endorses large cardinal axioms.

A large cardinal axiom implies the existence of stages of $V$ higher than can be proven in ZFC.
- Measurable Cardinal: Smallest implying $V \neq L$.*
- Supercompact Cardinal: Very large cardinal, implies projective determinacy (nearly complete theory of reals).

Gödel’s 1944 hope: perhaps some large cardinal could settle the Continuum Problem!
... Dashed on the Rocks of Forcing

- 1963: Cohen introduces the forcing technique.
  - Shows if $\text{ZFC}$ is consistent, then so is $\text{ZFC} + \neg \text{CH}$.
    - Combined with Gödel, shows that CH is independent of ZFC.
    - So the only way to settle the Continuum Problem is with new axioms beyond ZFC.

- 1967: Lévy and Solovay use forcing to show that CH is independent of $\text{ZFC} + \text{LC}$ for any large cardinal axiom.
  - So settling the Continuum Problem requires axioms even beyond $\text{ZFC} + \text{LCs}$!
The Question of CH Today

- Some set theorists took Lévy and Solovay’s result as settling the matter: there can be no solution to the CP.
- Many more followed Gödel, searching for new axioms that could settle the question.
- Originally, partisans thought CH was either true or radically false (very large continuum).
- The latter falls by the wayside, leaving two alternatives:
  1. The inner model program: CH is true, the continuum is $\aleph_1$.
  2. The forcing axioms program: CH is false, the continuum is $\aleph_2$. 
Origins of the Inner Model Program

- The first program starts from Gödel’s $L$.
- “Living in $L$” has many beneficial features:
  - Tractability, absoluteness, projectively-definable well-ordering of $\mathbb{R}$, fine-structure...
- Nevertheless, set theorists rejected $V = L$ due to the desirability of large cardinal axioms.
  - Scott, 1967: $\text{ZFC} + \text{“there is a measurable cardinal”} \vdash V \neq L$
- But perhaps there would be an inner model retaining (some of?) the desirable features of $L$, but permitting large cardinals.
- The inner model program was born from the search for such models.
The Hunt for the Core Models

- Using a measure, Dodd and Jensen isolated the core model below a measurable ($L[U]$).
- Using instead sequences of measures, in 1984 Mitchell introduced a core model containing a measurable.
- In 1994, iteration trees were used by Steel to obtain a fine-structured inner model of a Woodin cardinal.
  - At each step, new formal machinery was required.
- Given the rapidly increasing difficulties for each stronger large cardinal assumption, the prospect of continuing this process indefinitely was quite daunting...
In 2006, however, Woodin made a shocking discovery:

- An inner model of a supercompact must already be a (coarse-structured) inner model of every consistent large cardinal axiom.

Surprisingly, a possible stopping point for the inner model program, at the level of a supercompact.

- Some found this remarkable property reason to suspect the consistency of an inner model of a supercompact.

This inner model of a supercompact became known as the Ultimate L ($Ult(L)$).
Towards a Theory of $\text{Ult}(L)$

- Though $V = \text{Ult}(L)$ is often written in analogy to $V = L$, there is a key asymmetry:
  - $V = \text{Ult}(L)$ is not directly specifiable in $\mathcal{L}_{\text{ZFC}}$.
- Preliminary work on finding a suitable axiom formulation relies on multiple unproven conjectures.
  1). there is a strong cardinal limit of Woodin’s, and
  2). for $\Sigma_3$ sentence $\varphi$, if $\varphi$ holds in $V$, then there is a universally Baire set of reals $B$ st
     $HOD^{L(B,\mathbb{R})} \cap V_{\Theta L(B,\mathbb{R})} \models \varphi$.
- This formulation may not prove stable.
  - $V = \text{Ult}(L)$ sometimes used as proxy for whatever formulation of the axiom eventually is settled on.
Forcing axioms initially did not arise as genuine contenders for extending $ZFC + LCs$.

In 1971, Solovay and Tennebaum used iterated forcing to prove the independence of $SH$.

This method can be extremely tedious in application.

Reflecting on this proof, Martin realized that a single assumption could circumvent the need for using iterated forcings.

$MA(\aleph_1)$: for any ccc poset and $\aleph_1$-sized collection of dense sets, there is a generic filter hitting each dense set.

Even while suggesting it as a technical expedient, Martin refused to consider it as a genuine axiom.

Referred to only as an “axiom”
Referring to Martin and Solvay’s work, in 1982 Shelah introduced a new class of posets: the proper forcing posets.

- $\mathbb{P}$ is proper if $M[G]$ must preserve stationary collections of countable subsets of ordinals.

Shelah then proved an iteration theorem for proper forcings, making iterations of them more tractable.

Shelah briefly considered a generalization of MA to all proper forcing posets... but instead focuses on proving results in infinitary combinatorics.

In 1984, Baumgartner introduces the proper forcing axiom ($PFA$), and was the first to take it seriously as a candidate for extending $ZFC + LCs$. 
PFA as a Genuine Axiom Candidate

- Baumgartner went on to find a surprisingly robust collection of important consequences of PFA.
- In 1989, Todorčević takes this project further, proving the Open Coloring Axiom (OCA) from PFA.
  - With the OCA directly applicable to many active areas of set-theoretic research, the applicability of PFA grew significantly.
- Given its proximity to fruitful and productive application, consideration of PFA as an axiom continued to grow.
- In 1992, Veličković proved $ZFC + PFA \vdash |\mathbb{R}| = \aleph_2$.
  - With this discovery, skeptics of CH began to focus less on large continuum theories.
Completing the Forcing Axioms Program

- As this work on $PFA$ progressed, Shelah continued to search for more general iteration theorems.
- In 1988, Foreman, Shelah, and Magidor isolated the collection of stationary set preserving (ssp) forcings.
  - $\mathbb{P}$ is ssp if every stationary subset of $\aleph_1$ in $M$ must remain stationary in $M[G]$.
- Generalizing $MA(\aleph_1)$ and $PFA$, the trio introduced the axiom Martin’s Maximum ($MM$):
  - $MM$: for an ssp poset and $\aleph_1$-sized collection of dense sets, there is a generic filter hitting each dense set.
- As the name $MM$ suggests, any further generalization of $MA(\aleph_1)$ is inconsistent with $ZFC$. 
How to Settle this Debate?

- Contemporary set theorists face two well-developed alternatives for settling the CP.
  - Both are directly supported with a wide-variety of desirable features.
- So how can we decide between them?
  - Mathematical proof methods obviously can’t help.
  - Conceptual analysis of “concept of set” seems ill-suited to the highly technical nature of this dilemma.
- So where should we look for the proper way to settle this debate?
  - Proposal: The considerations used by set theorists in the case of accepting large cardinal axioms/rejecting $V = L$. 
The Case against $V = L$

- We return to Gödel’s early case against $V = L$:
  - Instead “[Gödel is] thinking of an axiom which... would state some maximum property of the system of all sets, whereas [$V = L$] states a minimum property” (Gödel, 1964).
  - “Only a maximum property would seem to harmonize with the concept of set” (Gödel, 1964).
- Idea: $V$ should permit any (even “random”) possible set to exist, instead of restricting to those f.o.-definable.
  - That is, our theory of sets should maximize the set-theoretic universe.
Maximize After Gödel

- Later set theorists endorsed this same view:
  - “The key argument against $V = L$... is that $[V = L]$ appears to restrict unduly the notion of arbitrary set.” (Moschovakis, 1980).
  - “Most set theorists regard $[V = L]$ as a restriction which may prevent one from taking every subset at each stage, and so reject it.” (Drake, 1974).
  - “Beautiful as they are, his so-called constructable sets are very special being almost minimal... They just do not capture the notion of set in general (and they were not meant to)” (Scott, 1977).
Maximize in Contemporary Set Theory

- ‘Maximize’ motivates axiom selection still today:
  - Forcing axioms: “A set that it’s existence is possible and there is no clear obstruction to its existence does exist.” (Magidor, 2012)
  - $\mathcal{V} = \text{Ult}(\mathcal{L})$: “The key methodological maxim that epistemology can contribute to the search for a stronger foundation of mathematics is: maximize interpretive power.” (Steel, 2012)
  - Multiverse approaches: “We may follow a multiverse analogue of MAXIMIZE, placing no undue limitation on what universes might exist in the multiverse. This is simply a higher-order analogue of the same motivation underlying set-theorists’ ever more expansive vision of set theory.” (Hamkins, 2012).
Maximize as a Methodological Principle

- Possible way of settling dispute: ‘Maximize’.
- Initially developed by Maddy in *Naturalism in Mathematics*.
  - Set theory as a foundational discipline must provide a generous arena for mathematical objects, without encumbering its potential development.
- ‘Maximize’ cited by advocates of both $V=Ult(L)$ and forcing axioms.
- Given its historical importance and support from both alternatives, perhaps ‘maximize’ could help settle this debate...
Intuitions on Maximize: A Rope of Sand

- Our intuitions on maximization can be very unclear.
- Consider Suslin’s Hypothesis (SH): every dense, complete, unbounded total order satisfying the c.c.c. is isomorphic to R.
- $SH$ and its negation are independent of $ZFC + LCs$.
- Which does maximize support?
  - $SH$: Asserts that for any two such orders there is an isomorphism, maximizing isomorphisms.
  - $\neg SH$: Asserts that there are many different structures on such orders, maximizing order types.
- Given the even greater complexity of current axiom candidates, intuitions of maximization alone are not sufficient to decide between them.
For this reason, we will instead use formal explications of the concept of ‘maximize’.

Two formal approaches found in the extant literature:

1. Steel’s approach: Based on maximizing the interpretative power of a theory.
2. Maddy’s approach: Based on maximizing interpretative power AND providing novel isomorphism types of mathematical interest.

We shall consider each in turn...
Approach 1: Steel Maximization

- Key idea: many different, even incompatible theories are all of genuine mathematical interest.
  - Ex: choice vs determinacy axioms.
  - A maximizing theory must provide domains for the study of more restrictive theories.
- An interpretation of a theory $T$ in $T'$ is S-Fair if it can be generated through iterations of definable inner models and forcing extensions.
- A theory $T'$ weakly S-maximizes over $T$ when:

$$T \triangleleft_{S} T' \text{ iff there is some } \varphi(x) \text{ st i). for all } \sigma \in T \ T' \vdash \sigma \varphi, \text{ and ii). } T' \text{ proves that } \varphi \text{ is } S\text{-fair.}$$
Approach 2: Maddy Maximization

- We can understand Maddy’s approach as having S-Max as a necessary, but not sufficient, condition of maximization.
- Key idea: a maximizing theory should introduce novel mathematical content not found in restrictive theories.
- An interpretation of a theory $T$ in $T'$ is M-Fair if it is a definable inner model.
- A theory $T'$ weakly M-Maximizes over $T$ when:
  
  $T \trianglelefteq_M T'$ iff there is some $\varphi(x)$ st
  
  i). for all $\sigma \in T$ $T' \vdash \sigma^{\varphi}$, ii). $T'$ proves that $\varphi$ is M-fair, and iii). $T' \vdash \exists X(\neg \varphi(X))$.

- Note: this is a simpler, but equivalent, variant of M-Max.
Comparing the Approaches

- S-Max and M-Max differ in their notions of “fair interpretation” and in the isomorphism-type condition.
- Note: M-Max is a strictly more fine-grained notion of maximization.
  - That is:
    \[ T \triangleleft_M T', \text{ then } T \triangleleft_S T' \]
- There is some disagreement over which best captures the informal notion of “maximize”.
- We will therefore apply both to the contemporary debate, comparing the results.
Applying the Approaches to $V = L$

- How do these formalizations fare on the question of $V = L$ vs large cardinals?
  - Let $T'$ stand for ZFC + there is a measurable cardinal.

- Claim 1: $T'$ has an M-fair interpretation of $ZFC + V = L$.

- Claim 2: If ZFC is consistent, then $ZFC + V = L$ cannot prove any interpretation of $T'$.

- Claim 3: $T' \vdash \exists X(\neg \varphi(X))$
  - Let $X$ be $0^\#$.

- So $ZFC + V = L \triangleleft_M T'$.
  - It follows immediately that $ZFC + V = L \triangleleft_S T'$.
The Task at Hand

- We now have two candidates for strong theories of sets.
- We also have two formal tools to use to try to adjudicate this debate.
- In the remainder of the talk, we will apply these tools to our theories.
- But first: one further complication.
  - Recall, Unlike $V = L$, $V = \text{Ult}(L)$ is not directly statable in the language of ZFC.
  - So we will need to first figure out how to handle the inner model program precisely.
Complication: Inner Model Program in Flux

- ZFC+LCs+MM provides a ready theory of forcing axioms.
- It takes a bit more work (and some unproven assumptions) to find a suitable theory of Ult(L).
  - Multiple conjectures have been proposed that would guarantee an axiom of $\text{Ult}(L)$.
  - We will focus on the weakest: the HOD hypothesis.
- HOD hypothesis: there is a regular cardinal $\lambda$ above an extendible which is not $\omega$-strongly measurable in HOD.
  - Implies HOD is a weak-extender model for the supercompactness of an extendible $\kappa$.
- Collegial assumption: the HOD hypothesis is true.
  - Thus $\text{ZFC} + \text{LCs} + V = \text{Ult}(L)$ will be used as short-hand for $\text{ZFC} + \text{LCs} + V = \text{HOD}$ (or stronger).
Contemporary Axioms and S-Max: Equivalence

- We now evaluate these two theories with the formal explications of ‘maximize’.
- First, we use the more coarse-grained S-Max.
  - \( ZFC + LCs + V = \text{Ult}(L) \trianglelefteq_S ZFC + LCs + MM \): follows directly from \( HOD \) hypothesis.
  - \( ZFC + LCS + MM \trianglelefteq_S ZFC + LCs + V = \text{Ult}(L) \): follows from the standard forcing poset used to prove the consistency of \( ZFC + MM \) relative to a supercompact.
  - Thus, \( ZFC + LCs + V = \text{Ult}(L) \equiv_S ZFC + LCs + MM \)
- S-Max is therefore too coarse-grained to distinguish these theories on the basis of their maximization power.
M-Max: The “Easy” Direction

- Recall: for \( \text{ZFC} + \text{LCs} + V = \text{Ult}(L) \trianglelefteq_M \text{ZFC} + \text{LCs} + \text{MM} \) it suffices that \( \text{ZFC} + \text{LCs} + \text{MM} \) has proper inner model interpretation of \( 
\text{ZFC} + \text{LCs} + V = \text{Ult}(L) \).

- Given the \textit{HOD} hypothesis, this must be the case:
  - \textit{HOD} hypothesis directly states that there is a definable inner model of \( \text{ZFC} + \text{LCs} + V = \text{Ult}(L) \) in \( \text{ZFC} + \text{LCs} \).
  - Since \( V = \text{Ult}(L) \) and \text{MM} disagree on the truth of \textit{CH}, this inner model must be proper in \( \text{ZFC} + \text{LCs} + \text{MM} \).

- Thus, assuming only a conjecture crucial to the successful justification of the axiom candidate \( V = \text{Ult}(L) \):
  \( \text{ZFC} + \text{LCs} + V = \text{Ult}(L) \trianglelefteq_M \text{ZFC} + \text{LCs} + \text{MM} \).
M-Max: The “Hard” Direction

- Can \( ZFC + LCs + V = Ult(L) \) prove the existence of a proper inner model interpretation of \( ZFC + LCs + MM \)?
- Currently unknown.
  - The only known M-Fair interpretation of \( ZFC + MM \) within \( ZFC + V = Ult(L) \) is of severely restricted large cardinal strength.
    - The \( P \)-max extension of \( L(\mathbb{R}) \).
  - No seeming route to generalize this interpretation to contexts with greater large cardinal strength.
- Any final resolution of this question awaits further development of the theory of \( Ult(L) \). But...
The Interpretation Non-Existence Conjecture

- Interpretation Non-Existence Conjecture (INEC): There is no M-Fair interpretation of $\mathsf{ZFC} + \mathsf{LCs} + \mathsf{MM}$ within $\mathsf{ZFC} + \mathsf{LCs} + V = \mathsf{Ult}(L)$.

- We argue that the INEC is quite likely true:
  - Known inner model interpretations of restrictions of forcing axioms in $\mathsf{Ult}(L)$ cannot suffice.
  - Falsity would represent inexplicable break with established patterns of inner/outer models at the level of a supercompact.
  - Falsity would pose a challenge to desirability of $\mathsf{Ult}(L)$:
    - Descending chain of $\mathsf{Ult}(L)$s would reveal arbitrariness of sets in any given model of $V = \mathsf{Ult}(L)$. 

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Axiom Selection after Large Cardinals
The Maximality Result

- Combining these facts, we find that M-Max plausibly separates $V = Ult(L)$ and forcing axioms:
  - If $HOD$ hypothesis holds, then
    $$ZFC + LCs + V = Ult(L) \sqsupseteq_M ZFC + LCs + MM.$$  
  - If the $INEC$ holds, then
    $$ZFC + LCs + MM \not\sqsubseteq_M ZFC + LCs + V = Ult(L).$$  
  - If both hold, then
    $$ZFC + LCs + V = Ult(L) \triangleleft_M ZFC + LCs + MM.$$  
- That is, if the inner model program is successful by their own standards, then forcing axioms are better supported by maximization considerations.
Implications: The Inner Model Program

- How can the advocate of $V = \text{Ult}(L)$ respond to this justificatory challenge?
- Option 1: Reject the HOD hypothesis or the INEC.
  - Raises more serious justificatory challenge for $V = \text{Ult}(L)$.
- Option 2: Accept the informal maxim ‘maximize’, but reject M-Max as its proper formalization.
  - What alternative?
- Option 3: Question the importance of ‘maximize’ in the face of other theoretical virtues of $V = \text{Ult}(L)$.
  - Possible alternatives: arguments from “$L$-likeness” or arguments from “provable closeness to $V$”.
  - Burden of proof on advocates of $V = \text{Ult}(L)$.
Implications: Forcing Axioms Program

- Do these results justify forcing axioms?
- Doubt: actual content of forcing axioms do no work in establishing our maximization result.
  - $\text{ZFC} + \text{LCs} + V = \text{Ult}(L) \triangleq M \text{ZFC} + \text{LCs} + \neg(V = \text{Ult}(L))$.
- ‘Maximize’ seems to justify $\text{ZFC} + \text{LCs} + \text{MM}$ solely because it is not the restrictive theory of $\text{Ult}(L)$.
- A new, third option to extend $\text{ZFC} + \text{LCs}$ could pose a serious challenge to this preliminary justification of forcing axioms.
  - We believe that there may well be such a third option in the near future...
Question 1: Prospects for a Large Continuum?

- Recall: original alternative to $CH$ was a very large continuum.
- Why did this approach drop away?
  - Lack of a clear axiom candidate.
- Such a theory might provide a useful test case to compare the maximization potential of two strong theories of $\neg CH$.
  - Recent work on cardinal characteristics provides one possible route for a “large continuum” theory.
- Question: is there a justifiable theory of the large continuum?
Question 2: Prospect for Alternative Notions of ‘Maximize’

- Possible response: the informal notion of maximize is well justified, but M-Max is not the correct formalization.
  - Support: false positives and negatives of M-Max (as developed by Löwe and Incurvati).

- Question: is there a well-justified alternative formalization of ‘maximize’?
  - Intriguing options: Permit forcing extensions / allow parameters in definitions.
  - Perhaps such an alteration of M-Max could provide inner model advocates with a maximization defense of $V = Ult(L)$. 
Thanks for Listening!