On Toeplitz products on Bergman space and two-weighted inequalities for the Bergman projection

Sandra Pott, Lund
joint work with Alexandru Aleman and Maria Carmen Reguera, Lund

Lunds Universitet

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1 Toeplitz products on the Hardy space

2 Toeplitz products on the Bergman space

3 A dyadic model and a two-weight result for $P + B$

4 Counterexamples
Outline

1. Toeplitz products on the Hardy space
2. Toeplitz products on the Bergman space
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3. A dyadic model and a two-weight result for $P_B^+$
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4. Counterexamples
The Riesz Projection

Let $\mathbb{D}$ be the unit disk. Recall the Hardy space

$$H^p(\mathbb{T}) = \left\{ f \in \text{Hol}(\mathbb{D}) : \sup_{r < 1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(re^{it})|^p \, dt < \infty \right\}, \quad (1 \leq p \leq \infty)$$

and the Riesz projection

$$P_R : L^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}),$$

$$P_Rf(z) = \int_{\mathbb{T}} \frac{f(\xi)}{1 - z\bar{\xi}} \, d|z|.$$
Let \( f \in H^2 \). It is well-known that the Toeplitz operator

\[
T_f : H^2 \rightarrow H^2, \quad h \mapsto P_R fh \quad (h \in H^2)
\]
is bounded, if and only if \( f \in L^\infty \).

**Question, Sarason 1990**

For which \( f, g \in H^2 \) is the Toeplitz product

\[
T_g T_f^* : H^2 \rightarrow H^2
\]
bounded?
Cruz-Uribe 1992:

\[
\begin{array}{cccc}
M_f & H^2(\mathbb{T}) & T_g T_f^* & H^2 \\
\downarrow & & \\
L^2(\frac{1}{|f|^2}, \mathbb{T}) & P_R & H^2(|g|^2, \mathbb{T}) & M_g
\end{array}
\]
Cruz-Uribe 1992:

\[ M_f \downarrow \quad H^2(\mathbb{T}) \xrightarrow{T_g T^*_f} \quad H^2 \quad \uparrow \quad M_g \]

\[ L^2\left(\frac{1}{|f|^2}, \mathbb{T}\right) \xrightarrow{P_R} \quad H^2(|g|^2, \mathbb{T}) \]

Two-weighted Hilbert transform/Riesz projection

When is

\[ P_R : L^2\left(\frac{1}{|f|^2}, \mathbb{T}\right) \longrightarrow L^2(|g|^2, \mathbb{T}) \]

bounded?
Weighted Riesz projection

For which weights \( w \) is \( P_R : L_w^p(\mathbb{T}) \rightarrow H_w^p(\mathbb{T}) \) bounded?

Definition \((A_p\text{-condition})\):

Let \( 1 < p < \infty \). Then \( w \in A_p \), if

\[
A_p(w) := \sup_{I \subseteq \mathbb{T}} \left( \frac{1}{|I|} \int_I w(t) \, dt \right) \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}}(t) \, dt \right)^{\frac{1}{p-1}} < \infty,
\]

where \( I \) is an interval in \( \mathbb{T} \).

For \( p = 2 \), this is equivalent to

The invariant \( A_2 \) condition \( w \in A_2 \iff \sup_{z \in D} P_w(z) (z) < \infty \),

where \( P \) denotes the Poisson extension.
Weighted Riesz projection

For which weights \( w \) is \( P_R : L^p_w(\mathbb{T}) \to H^p_w(\mathbb{T}) \) bounded?

**Definition (\( A_p \)-condition)**

Let \( 1 < p < \infty \). Then \( w \in A_p \), if

\[
A_p(w) := \sup_{I \subseteq \mathbb{T}} \left( \frac{1}{|I|} \int_I w(t) dt \right) \left( \frac{1}{|I|} \int_I w^{1-p'}(t) dt \right)^{\frac{1}{p'-1}} < \infty,
\]

where \( I \) is an interval in \( \mathbb{T} \).

For \( p = 2 \), this is equivalent to

**The invariant \( A_2 \) condition**

\[
w \in A_2 \iff \sup_{z \in \mathbb{D}} \mathcal{P}(w)(z) \mathcal{P}(w^{-1})(z) < \infty,
\]

where \( \mathcal{P} \) denotes the Poisson extension.
Theorem (Hunt, Muckenhoupt, Wheeden)

Let $1 < p < \infty$. Then

\[ P_R : L_w^p(\mathbb{T}) \rightarrow L_w^p(\mathbb{T}) \text{ bounded} \]

$\iff w \in A_p$. 
Joint $A_2$ conjecture

$P_R : L^2_w(\mathbb{T}) \rightarrow L^2_v(\mathbb{T})$ bounded $\iff \sup_{z \in \mathbb{D}} \mathcal{P}(v)(z) \mathcal{P}(w^{-1})(z) < \infty$?

Sarason Conjecture

$T_g T^*_f : H^2 \rightarrow H^2$ bounded $\iff \sup_{z \in \mathbb{D}} \mathcal{P}(|f|^2)(z) \mathcal{P}(|g|^2)(z) < \infty$?
Sarason conjecture for Hardy space

Joint $A_2$ conjecture

$$P_R : L^2_w(\mathbb{T}) \to L^2_v(\mathbb{T}) \text{ bounded} \iff \sup_{z \in \mathbb{D}} \mathcal{P}(v)(z) \mathcal{P}(w^{-1})(z) < \infty ?$$

Sarason Conjecture

$$T_g T_f^* : H^2 \to H^2 \text{ bounded} \iff \sup_{z \in \mathbb{D}} \mathcal{P}(|f|^2)(z) \mathcal{P}(|g|^2)(z) < \infty ?$$

Answer: No! (F. Nazarov, around 2000)

Extensive history of the question (Cotlar, Lacey, Nazarov, Sadosky, Sawyer, Shen, Treil, Volberg, Uriarte-Tuero .....)

Outline

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2. Toeplitz products on the Bergman space
3. A dyadic model and a two-weight result for $P_B^+$
4. Counterexamples
Let $\mathbb{D}$ be the unit disk. Recall the Bergman space
\[
A^p(\mathbb{D}) = L^p(\mathbb{D}) \cap \text{Hol}(\mathbb{D}) \quad (1 \leq p \leq \infty)
\]
and the Bergman projection
\[
P_B : L^2(\mathbb{D}) \to A^2(\mathbb{D}),
\]
\[
P_B f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{(1 - z\xi)^2} dA(\xi).
\]
Let $f \in A^2$. It is well-known that the Toeplitz operator

$$T_f : A^2 \rightarrow A^2, \quad h \mapsto P_B fh \quad (h \in A^2)$$

is bounded, if and only if $f \in H^\infty$.

**Question**

For which $f, g \in A^2$ is the Toeplitz product

$$T_g T_f^* : A^2 \rightarrow A^2$$

bounded?
Cruz-Uribe 1992, Zheng/Stroethoff 1999:

\[ M_f \]  \[ \downarrow \]  \[ L^2 \left( \frac{1}{|f|^2}, \mathbb{D} \right) \]  \[ \xrightarrow{T_g T_f^*} \]  \[ A^2(\mathbb{D}) \]  \[ \uparrow \]  \[ M_g \]  \[ \xrightarrow{P_B} \]  \[ A^2(|g|^2, \mathbb{D}) \]
Cruz-Uribe 1992, Zheng/Stroethoff 1999:

\[
M_f \quad \downarrow \quad T_g T_f^* \quad \uparrow \quad M_g
\]

\[
L^2\left(\frac{1}{|f|^2}, \mathbb{D}\right) \quad \xrightarrow{P_B} \quad A^2\left(|g|^2, \mathbb{D}\right)
\]

Two-weighted Bergman projection

When is

\[
P_B : L^2\left(\frac{1}{|f|^2}, \mathbb{D}\right) \longrightarrow L^2\left(|g|^2, \mathbb{D}\right)
\]

bounded?
The Bergman projection and positive Bergman projection

**Theorem**

\[ P_B : L^p(\mathbb{D}) \rightarrow A^p(\mathbb{D}) \text{ bounded for } 1 < p < \infty. \]
The Bergman projection and positive Bergman projection

**Theorem**

\[ P_B : L^p(\mathbb{D}) \rightarrow A^p(\mathbb{D}) \text{ bounded for } 1 < p < \infty. \]

Moreover,

**Theorem**

\[ P_B^+ : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D}) \text{ bounded for } 1 < p < \infty, \]

where

\[ P_B^+ f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{|1 - z\bar{\xi}|^2} dA(\xi). \]
Question

Let $w \geq 0$ be a weight function on $\mathbb{D}$. For which weights $w$ is $P_B : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D})$ bounded?
Weighted Bergman projection

**Question**

Let $w \geq 0$ be a weight function on $\mathbb{D}$. For which weights $w$ is $P_B : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D})$ bounded?

**Definition**

Let $1 < p < \infty$. Then $w \in B_p$, if

$$B_p(w) := \sup_{i \subseteq \mathbb{T}} \left( \frac{1}{|Q_i|} \int_{Q_i} w(z) dA(z) \right) \left( \frac{1}{|Q_i|} \int_{Q_i} w^{1-p'}(z) dA(z) \right)^{\frac{1}{p'-1}} < \infty,$$

where $Q_i$ is the Carleson box

$$Q_i = \{ r\xi : 1 - |l| \leq r < 1, \xi \in l \}.$$
Theorem (Bonami, Békollé)

Let $1 < p < \infty$. Then

$$P_B : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D}) \text{ bounded}$$

$$\iff P_B^+ : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D}) \text{ bounded}$$

$$\iff w \in B_p.$$

”Little” cancellation of the operator!
Joint Berezin $B_2$ conjecture

\[ P_B : L^p_w(\mathbb{D}) \to L^p_v(\mathbb{D}) \text{ bounded } \iff \sup_{z \in \mathbb{D}} B(v)(z)B(w^{-1})(z) < \infty \]

Here,

\[ B(f)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(z)}{1 - \bar{w}z^4} dA(w). \]
Sarason conjecture for Bergman space

**Joint Berezin $B_2$ conjecture**

\[ P_B : L^p_w(\mathbb{D}) \to L^p_v(\mathbb{D}) \text{ bounded} \iff \sup_{z \in \mathbb{D}} B(v)(z)B(w^{-1})(z) < \infty? \]

Here,

\[ B(f)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(z)}{|1 - \bar{w}z|^4} dA(w). \]

**Sarason Conjecture**

\[ T_g T_f^* : A^2 \to A^2 \text{ bounded} \iff \sup_{z \in \mathbb{D}} B(|f|^2)(z)B(|g|^2)(z) < \infty? \]
Joint Berezin $B_2$ conjecture

$$P_B : L^p_w(\mathbb{D}) \to L^p_v(\mathbb{D}) \text{ bounded } \iff \sup_{z \in \mathbb{D}} \mathcal{B}(v)(z)\mathcal{B}(w^{-1})(z) < \infty?$$

Here,

$$\mathcal{B}(f)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(z)}{|1 - \bar{w}z|^4} dA(w).$$

Sarason Conjecture

$$T_g T_f^* : A^2 \to A^2 \text{ bounded } \iff \sup_{z \in \mathbb{D}} \mathcal{B}(|f|^2)(z)\mathcal{B}(|g|^2)(z) < \infty?$$

Answer: open.
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The dyadic model for $P_B^+$ on $\mathbb{C}^+$

Now we consider the Bergman projection $P_B$ and $P_B^+$ on $\mathbb{C}^+$,

$$P_B f(z) = \int_{\mathbb{C}^+} \frac{f(\xi)}{(z - \bar{\xi})^2} dA(\xi).$$

Let

$$Q = \sum_{K \in \mathcal{D}} \frac{1}{|K|^2} \chi_{Q_K} \otimes \chi_{Q_K}.$$

For each $\beta \in \{0, 1\}^\mathbb{Z}$, consider the dyadic grid $\mathcal{D}_\beta$ of intervals of the form $I + \sum_{i>j} 2^{-i} \beta_i$, where $I$ is an dyadic interval of length $2^{-j}$ in $\mathcal{D}_0$. 
A dyadic model and a two-weight result for $P^+_R$

Random grids

Lemma

For $\beta \in \{0,1\}^\mathbb{Z}$, let $Q^\beta = \sum_{K \in \mathcal{D}_\beta} \frac{1}{|K|^2} \chi_{Q_K} \otimes \chi_{Q_K}$. Let $\mu$ denote the canonical probability measure on $\{0,1\}^\mathbb{Z}$ such that the coordinates $\beta_j$ are independent with $\mu(\beta_j = 0) = \mu(\beta_j = 1) = 1/2$. Let $K_\beta(z,w)$ denote the kernel of $Q_\beta$ and $K_B^+(z,w)$ denote the kernel of $P_B^+$. Then for each $\beta \in \{0,1\}^\mathbb{Z}$,

$$K_\beta(z,w) \lesssim K_B^+(z,w) \lesssim \int_{\{0,1\}^\mathbb{Z}} K_{\beta'}(z,w) d\mu(\beta') \quad (z,w \in \mathbb{C}^+)$$
The test function result for $Q$

**Theorem**

Let $u, v$ be weights on $\mathbb{D}$. Then the following are equivalent:

1. $Q : L^2_u(\mathbb{D}) \rightarrow L^2_v(\mathbb{D})$ bounded
2. There exist $K_1, K_2, K_3$ such that
   1. joint $B_2$ condition:
      \[
      \sup_{I \subset \mathbb{T} \text{ interval}} \left( \frac{1}{|Q_i|} \int_{Q_i} u^{-1}(z) dA(z) \right) \left( \frac{1}{|Q_i|} \int_{Q_i} v(z) dA(z) \right) \leq K_1
      \]
   2. test condition $\| Q_{in} u^{-1} \chi_{Q_i} \|_{L^2_v} \leq K_2 \| \chi_{Q_i} \|_{L^2_{u^{-1}}}$
   3. adjoint test condition $\| Q_{in} v \chi_{Q_i} \|_{L^2_{u^{-1}}} \leq K_3 \| \chi_{Q_i} \|_{L^2_v}$

Moreover, in this case $\| Q \|_{L^2_u \rightarrow L^2_v} \lesssim K_1 + K_2 + K_3$.

**Proof:** Adaptation of Eric Sawyer’s test function result for positive kernels.
A dyadic model and a two-weight result for $P^+_R$

A two-weight result for $P^+_B$

Theorem

Let $u, v$ be weights on $\mathbb{D}$. Then the following are equivalent:

1. $P^+_B : L^2_u(\mathbb{D}) \to L^2_v(\mathbb{D})$ bounded

2. There exist $K_1, K_2, K_3$ such that
   1. joint $B_2$ condition:

   $\sup_{I \subset \mathbb{T} \text{ interval}} \left( \frac{1}{|Q_i|} \int_{Q_i} u^{-1}(z)dA(z) \right) \left( \frac{1}{|Q_i|} \int_{Q_i} v(z)dA(z) \right) \leq K_1$

   2. test condition $\|\chi_{Q_i} P^+_B u^{-1} \chi_{Q_i}\|_{L^2_v} \leq K_2 \|\chi_{Q_i}\|_{L^2_{u^{-1}}}$

   3. adjoint test condition $\|\chi_{Q_i} P^+_B v \chi_{Q_i}\|_{L^2_{u^{-1}}} \leq K_3 \|\chi_{Q_i}\|_{L^2_v}$

Moreover, in this case $\|P^+_B\|_{L^2_u \to L^2_v} \lesssim K_1 + K_2 + K_3$. 
Proof of the two-weight result

Proof:

test conditions for $P_B^+$ $\Rightarrow$ test conditions for all $Q^\beta$

$\Rightarrow$ all $Q^\beta$ are uniformly bounded $\Rightarrow$ $P_B^+$ is bounded
A corollary

Theorem (Aleman, P., Reguera)

Let $w \in B_2$ and let $P^+_B$ be the positive Bergman projection. Then

$$\| P^+_B \|_{L^2_w(D) \rightarrow L^2_w(D)} \leq C B_2(w),$$

with $C$ independent of the weight $w$. 
A corollary

Theorem (Aleman, P., Reguera)

Let \( w \in B_2 \) and let \( P_B^+ \) be the positive Bergman projection. Then

\[
\| P_B^+ \|_{L_w^2 (\mathbb{D}) \to L_w^2 (\mathbb{D})} \leq C \, B_2 (w),
\]

with \( C \) independent of the weight \( w \).

Corollary

\[
\| P_B \|_{L_w^2 (\mathbb{D}) \to L_w^2 (\mathbb{D})} \leq C \, B_2 (w)
\]
A corollary

**Theorem (Aleman, P., Reguera)**

Let $w \in B_2$ and let $P_B^+$ be the positive Bergman projection. Then

$$\|P_B^+\|_{L^2_w(\mathbb{D}) \to L^2_w(\mathbb{D})} \leq C B_2(w),$$

with $C$ independent of the weight $w$.

**Corollary**

$$\|P_B\|_{L^2_w(\mathbb{D}) \to L^2_w(\mathbb{D})} \leq C B_2(w)$$

The result is sharp (Aleman and Constantin).
A corollary

**Theorem (Aleman, P., Reguera)**

*Let* $w \in B_2$ *and let* $P_B^+$ *be the positive Bergman projection. Then*

$$
\| P_B^+ \|_{L_w^2(\mathbb{D}) \rightarrow L_w^2(\mathbb{D})} \leq C B_2(w),
$$

*with* $C$ *independent of the weight* $w$.

**Corollary**

$$
\| P_B \|_{L_w^2(\mathbb{D}) \rightarrow L_w^2(\mathbb{D})} \leq C B_2(w)
$$

The result is sharp (Aleman and Constantin).

Proof: Set $u = w$ and check that the constants $K_2, K_3$ in the testing of $Q$ depend linearly on the $B_2$ constant (as in the paper of Lacey, Petermichl, and Reguera).
Some questions

Question

Let $f, g \in A^2$. Is the Toeplitz product $T_g T_f^* : A^2 \to A^2$ is bounded, if and only if $M_g P_B^+ M_f : A^2 \to L^2(\mathbb{D})$ is bounded?
A dyadic model and a two-weight result for $P_R^+$

Some questions

Question

Let $f, g \in A^2$. Is the Toeplitz product $T_g T_f^* : A^2 \to A^2$ is bounded, if and only if $M_g P_B^+ M_f : A^2 \to L^2(\mathbb{D})$ is bounded?

Notice that the weights given by Bergman space functions on the disk are far from general, in contrast to weights given by Hardy space functions on the circle. They have to be at least subharmonic.
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A counterexample

There exist weights \((u, w)\) satisfying the joint \(B_2\) condition, such that 

\[ P_B^+ : L^2_u(\mathbb{C}^+) \to L^2_w(\mathbb{C}^+) \]

is not bounded.
Proof of the counterexample

Let $Q_1$ be a Carleson cube of side length $2^{-N}$ inside the unit cube $Q_0$, $\sigma = u^{-1} = \chi_{Q_1}2^{2N}$ and $w = \sum_{j=0}^{N} 2^{-2N+2j} \chi_{T_{Q_1(j)}}$.

It is easy to see that $u$ and $w$ satisfy joint $B_2$ condition. However

$$\|Q_{in}(\sigma \chi_{Q_0})\|_{L_w^2}^2 = \| \sum_{j=0}^{N} \chi_{Q_1(j)} \langle \sigma \rangle_{Q_1(j)} \|_{L_w^2}^2 \geq \sum_{j=0}^{N} |Q_{1(j)} \langle w \rangle_{T_{Q_1(j)}} \langle \sigma \rangle_{Q_1(j)}^2 \geq \sum_{j=0}^{N} |Q_{1(j)} \langle \sigma \rangle_{Q_1(j)} = N.$$ 

Note that one of the weights is "far from subharmonic".
Counterexample for $P_B$

The joint $B_2$ condition is not sufficient for the boundedness of $P_B$ for general weights $(u, w)$.
Counterexample for $P_B$

The joint $B_2$ condition is not sufficient for the boundedness of $P_B$ for general weights $(u, w)$.

Remark

*The joint Berezin $B_2$ condition is also not sufficient for the boundedness of $P_B^+$ for general weights $(u, w)$.*
THANK YOU!