(Strong) $A_\infty$-weights on metric spaces

Riikka Korte  riikka.korte@helsinki.fi
August 30, 2012

University of Helsinki
Department of Mathematics and Statistics
Outline

Introduction

Strong $A_\infty$-weights

$A_\infty$-weights in metric spaces

$SA_\infty(X) \subset A_\infty(X)$
Reference:

The talk is mainly based on the following paper:

- O. E. Kansanen and R. K.,
Introduction

- $SA_\infty$-weights were introduced by David and Semmes when trying to characterize the subclass of $A_\infty$-weights that are comparable to the Jacobian determinants of quasiconformal mappings in $\mathbb{R}^n$. 

The inclusions above hold also in Ahlfors-regular metric spaces supporting a $(1,1)$-Poincaré inequality. In $\mathbb{R}^n$ there are several characterizations for $A_\infty$-weights that are not necessarily equivalent on metric spaces (Strömberg-Torchinsky).
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Setting

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  \[ \mu(B(x, 2r)) \leq c_D \mu(B(x, r)). \]
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■ (1, 1)-Poincaré inequality

\[
\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq c_P r \int_{B(x, \lambda r)} g_u \, d\mu,
\]

where \(g_u\) is an upper gradient of \(u\) (corresponds to \(|\nabla u|\)).
Let $\nu$ be a doubling measure on $X$. We associate with $\nu$ the quasi-distance $\delta_\nu(x, y)$

$$\delta_\nu(x, y) = [\nu(B(x, d(x, y))) + \nu(B(y, d(x, y)))]^{1/Q}.$$
Strong $A_\infty$-weights

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- $\nu$ is a *metric doubling measure* if there exists a distance function $\delta$ such that

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$\omega \in L^1_{loc}(X)$ is a strong $A_\infty$-weight ($\omega \in SA_\infty(X)$), if

$$d_\nu = \omega d_\mu.$$
$A_\infty$-weights in metric spaces: possible definitions

1. There are $0 < \varepsilon, \delta < 1$ s.t. for all $E \subseteq B$,

$$\mu(E) < \varepsilon \mu(B) \implies \nu(E) < (1 - \delta)\nu(B)$$
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2. There are \( c > 0 \) and \( p \geq 1 \) s.t. for all \( E \subseteq B \),
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3. $\omega$ satisfies the reverse Hölder inequality.

In (3) – (5), we assume that $\nu$ is a weighted measure with respect to $\mu$ and that there exists $\omega$ s.t.
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Comparison of definitions for $A_{\infty}(X)$

- If $\mu$ is doubling, then we have

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- If also $\nu$ is doubling, then (1) $\implies$ (4).

**Corollary**

*If both $\nu$ and $\mu$ are doubling, then the conditions (1)–(5) are equivalent.*
Example

- $X = \{ x \in \mathbb{R}^n : x_1 x_2 = 0 \}, \ n \geq 2$ with

$$d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

$$\mu = \mathcal{L}^{n-1}$$
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- Satisfies (2) with \( p = 1 \) and \( c = 2 \).

- \( \nu \) is not doubling and therefore cannot satisfy (4).
Main result

Theorem
Suppose that $\nu$ is a metric doubling measure. Then $\nu$ has an $A_\infty$ density i.e.

$$\frac{\nu(E)}{\nu(B)} \leq c \left( \frac{\mu(E)}{\mu(B)} \right)^\delta.$$
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Corollary
Every strong $A_\infty$-weight is an $A_p$-weight for some $p < \infty$. 
Proof of the Main Theorem (1)

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- Show that weights $\omega_t$ related to $\nu_t$ satisfy reverse Hölder’s inequality with uniform constants. ($\mathbb{R}^n$: integrate along lines parallel to coordinate axes)

- Show that $\omega_t$ converges nicely enough to some weight $\omega$. (We will skip at least this part in this talk.)
Proof of the Main Theorem (2)

We construct the measure $\nu_t, t > 0$.

- Fix $t > 0$. Take a cover $\{B_i = B(x_i, t)\}$ of $X$ and a partition of unity $\{\phi_i^t\}$ subordinate to the cover.
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- Fix $t > 0$. Take a cover $\{B_i = B(x_i, t)\}$ of $X$ and a partition of unity $\{\phi_i^t\}$ subordinate to the cover.

- Set

$$a_i^t := \frac{\int_X \phi_i^t d\nu}{\int_X \phi_i^t d\mu} \quad \text{then} \quad a_i^t \approx \frac{\nu(B_i)}{\mu(B_i)}$$
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- Define

$$\nu_t(A) := \sum_i a^t_i \int_A \phi^t_i d\mu$$

Then

$$\omega_t = \sum_i a^t_i \phi^t_i$$
Proof of the Main Theorem (3)

Given a curve family $\Gamma$, we define the 1-\textit{modulus} by

$$\text{mod}(\Gamma) = \inf_{\rho} \int \rho \, d\mu,$$

where $\rho$ satisfies for all $\gamma \in \Gamma$

$$\int_{\gamma} \rho \, ds \geq 1.$$
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Lemma

Let $\Gamma$ be a curve family consisting of all curves joining $B(x_0, r_0)$ and $X \setminus B(x_0, 2r_0)$. Then there exists $C = C(c_D, c_P)$ s.t.

$$\text{mod}(\Gamma) \geq C \frac{\mu(B(x_0, r_0))}{r_0}.$$
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Proof of the Main Theorem (4)

- The case $t \gtrsim r_0$ is easy so we may assume $t \ll r_0$
- Take $\gamma \in \Gamma$ as above. $\nu$ is metric doubling measure $\Rightarrow$

$$\int_\gamma \omega_t^{1/Q} ds \gtrsim \delta_\nu(\gamma(0), \gamma(L)) \approx \nu(B(x_0, r_0))^{1/Q}$$
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  $$\int_\gamma \omega_s^{1/Q} ds \gtrapprox \delta_{\nu}(\gamma(0), \gamma(L)) \approx \nu(B(x_0, r_0))^{1/Q}$$
- Thus $\rho$ is a good test function for the modulus if
  $$\rho = \frac{C}{\nu(B(x_0, r_0))^{1/Q} \omega_t^{1/Q} \chi_{B(x_0, 2r_0)}}$$
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  \[ \rho = \frac{C}{\nu(B(x_0, r_0))^{1/Q}} \omega_t^{1/Q} \chi_B(x_0, 2r_0) \]
- Combined with $\omega_t(B(x_0, 2r_0)) \lesssim \nu(B(x_0, 2r_0 + 4t))$ this gives
  \[ \left( \int_B \omega_t d\mu \right)^{1/Q} \leq C \int_B \omega_t^{1/Q} d\mu \]
Proof of the Main Theorem (4)

- The case $t \gtrsim r_0$ is easy so we may assume $t << r_0$
- Take $\gamma \in \Gamma$ as above. $\nu$ is metric doubling measure $\Rightarrow$
  $$\int_{\gamma} \omega_t^{1/Q} ds \gtrapprox \delta_{\nu}(\gamma(0), \gamma(L)) \approx \nu(B(x_0, r_0))^{1/Q}$$
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  $$\left( \int_B \omega_t d\mu \right)^{1/Q} \leq C \int_B \omega_t^{1/Q} d\mu$$
- Gehring lemma implies that
  $$\left( \int_B \omega_t^{1+\varepsilon} d\mu \right)^{1/(1+\varepsilon)} \leq C \int_B \omega_t d\mu.$$