DOES UMD IMPLY RMF?

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1. Introduction

We are interested in a Banach space property that is defined by a requirement of an $L_p$ inequality for a certain maximal function. More precisely, given a Banach space $X$, we define the Rademacher maximal function of an $f \in L_1([0,1); X)$ by

$$M_R f(t) = \sup \left\{ \mathbb{E} \left\| \sum_{I \in \mathcal{D}} \varepsilon_I \lambda_I (f)_I \right\|^2 : \sum_{I \in \mathcal{D}} |\lambda_I|^2 \leq 1 \right\}, \quad t \in [0,1),$$

where $\varepsilon_I$ are Rademacher variables and $(f)_I$ stands for the average of $f$ over a dyadic interval $I \subset [0,1)$. Then $X$ is said to have RMF if $\|M_R f\|_{L_p} \lesssim \|f\|_{L_p(X)}$ for some (equivalently, for all) $p \in (1,\infty)$. This notion was first presented by Hytönen, McIntosh and Portal in [1]. Most of what is written in these notes appeared already in there (Section 7 and Appendix C).

We would like to understand how RMF relates to notion of UMD, which asks for unconditionality of the Haar decomposition, i.e. that for all (non-random) signs $\varepsilon_I \in \{-1,1\}$,

$$\left\| \sum_{I \subset [0,1)} \varepsilon_I (f,h_I)_I h_I \right\|_{L_p(X)} \lesssim \left\| \sum_{I \subset [0,1)} (f,h_I)_I h_I \right\|_{L_p(X)},$$

where $h_I = |I|^{-1/2}(1_I - 1_{I^c})$ is the Haar wavelet on the dyadic interval $I$ with left and right halves $I_-$ and $I_+$.

**Question 1.** Does UMD imply RMF?

It is not difficult to see that $\ell_p$ has both RMF and UMD when $1 < p < \infty$ (see Prop. 2). Actually, RMF follows immediately if the space has type 2. As is known, UMD spaces are reflexive, and so, recalling the examples of non-reflexive spaces with type 2 by James [2] and by Pisier and Xu [3], we infer that the converse to Question 1 is not true.

On the other hand, $\ell_1$ has neither RMF nor UMD (see Prop. 3). Moreover, these are ‘super-properties’ in the sense that if $X$ has such a property, and $Y$ is finitely representable in $X$, then also $Y$ has the property. As non-trivial type is equivalent with finite representability of $\ell_1$ in the space, we see that RMF and UMD both imply non-trivial type. We arrive at a strengthening of Question 1:

**Question 2.** Is RMF equivalent with non-trivial type?

Both UMD and non-trivial type are ‘self-dual’ properties meaning that $X^*$ has UMD or non-trivial type whenever $X$ has. A following weakening of the previous two questions is also of interest:

**Question 3.** Is RMF self-dual, i.e. does $X^*$ have RMF whenever $X$ does?

Note again, that by finite representability, $X^{**}$ has RMF whenever $X$ does. In these notes we only consider Question 2 and will not be concerned with the UMD property as such.

2. R-bounds and type

In this section we take a closer look at the quantity that defines the Rademacher maximal function.

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**Definition.** A set $S \subset X$ is said to be $R$-bounded if

$$\left( \mathbb{E} \left\| \sum_{k} \varepsilon_k \lambda_k x_k \right\|^{2} \right)^{1/2} \leq C \left( \sum_{k} \left| \lambda_k \right|^2 \right)^{1/2}$$

for all (finite) collections of vectors $\{x_k\} \subset S$ and scalars $\{\lambda_k\}$. The smallest such $C$ is the R-bound $\mathcal{R}(S)$.

Recall that $X$ is said to have type $p \in [1, 2]$ if

$$\left( \mathbb{E} \left\| \sum_{k} \varepsilon_k x_k \right\|^{2} \right)^{1/2} \lesssim \left( \sum_{k} \left\| x_k \right\|^p \right)^{1/p}$$

for all (finite) collections $\{x_k\} \subset X$. That $X$ has at most type $p$ is equivalent with finite representability of $\ell_p$ in $X$. By finite representability of a Banach space $Y$ in a Banach space $X$ we mean the following: For every $\varepsilon > 0$ and every finite dimensional subspace $E$ of $Y$ there exists a subspace $F$ of $X$ and an isomorphism $\Phi : E \to F$ such that $\|\Phi\| \cdot \|\Phi^{-1}\| < 1 + \varepsilon$. As is well-known, $\ell_2$ is finitely representable in every Banach space and every Banach space is finitely representable in $c_0$.

**Remarks.**

- Khintchine-Kalnane inequality allows us to replace $2$ by any other $p \in [1, \infty)$ when handling randomized sums:

$$\left( \mathbb{E} \left\| \sum_{k} \varepsilon_k x_k \right\|^{2} \right)^{1/2} \approx \left( \mathbb{E} \left\| \sum_{k} \varepsilon_k x_k \right\|^{p} \right)^{1/p}$$

- Note that R-bounds satisfy the following ‘triangle inequality’: For $S, S' \subset X$ one has

$$\left| \mathcal{R}(S) - \mathcal{R}(S') \right| \leq \mathcal{R}(S + S') \leq \mathcal{R}(S) + \mathcal{R}(S').$$

- Convex combinations and limit points can always be added without increasing the R-bound: For any $S \subset X$ we have

$$\mathcal{R}\left( \text{conv}(S) \right) \leq \mathcal{R}(S),$$

and this extends to absolute convex hulls.

- R-bounds always exceed uniform bounds, that is,

$$\sup_{x \in S} \|x\| \leq \mathcal{R}(S).$$

Moreover, that $\mathcal{R}(S) \lesssim \sup_{x \in S} \|x\|$ for all $S \subset X$ is equivalent with $X$ having type $2$.

- R-bounds of unions are controlled by R-bounds of their constituents in the following way: If $X$ has type $p \in [1, 2)$, then

$$\mathcal{R}\left( \bigcup_{m} S_m \right) \leq \left( \sum_{m} \mathcal{R}(S_m) \right)^{2p/(2-p)} \left( \sum_{m} \left\| S_m \right\|^{2} \right)^{(2-p)/2p}.$$

In particular, for any sequence $(x_k)_{k=1}^{\infty}$ in any space $X$ one has

$$\mathcal{R}(x_1, x_2, \ldots) \leq \left( \sum_{k=1}^{\infty} \|x_k\|^2 \right)^{1/2}.$$

It is perhaps instructive to consider the modulus

$$\gamma_X(N) = \sup_{x_1, \ldots, x_N \in B_X} \mathcal{R}(x_1, \ldots, x_N), \quad N \in \mathbb{N}.$$  

**Proposition 1.** If $X$ has type $p \in [1, 2]$ but no greater, then $\gamma_X(N) = N^{1/p-1/2}$ for all $N$.

**Proof.** If $X$ has type $p$ and $x_1, \ldots, x_N \in B_X$, then

$$\mathcal{R}(x_1, \ldots, x_N) \leq \left( \sum_{k=1}^{N} \|x_k\|^{2} \right)^{(2-p)/2p} \leq N^{1/p-1/2}.$$

On the other hand, the canonical basis vectors $e_k$ in $\ell_p$ satisfy

$$\left( \mathbb{E} \left\| \sum_{k=1}^{N} \varepsilon_k e_k \right\|^{2} \right)^{1/2} \lesssim N^{1/p} = N^{1/p-1/2} \left( \sum_{k=1}^{N} 1 \right)^{1/2}.$$
so that \( \mathcal{R}(e_1, \ldots, e_N) \geq N^{1/p - 1/2} \). Hence, if \( X \) has no greater type than \( p \), then \( \ell_p \) is finitely representable in \( X \) and so \( \gamma_X(N) \geq N^{1/p - 1/2} \).

\[ \square \]

**Remark.** For \( X = \ell_p \) we have \( \gamma_{\ell_p}(N) = \mathcal{R}(B_{\ell_p}^N) \), where \( \ell_p^N \) is the span of \( \{e_1, \ldots, e_N\} \) in \( \ell_p \), as usual. Indeed, for \( p = 1 \) this is evident already from the fact that \( B_{\ell_1}^N = \text{absconv}\{e_1, \ldots, e_N\} \).

More generally, if \( 1 \leq p \leq 2 \), we have for any \( \{x_k\} \subset B_{\ell_p}^N \) that

\[ \left( \mathbb{E} \left\| \sum_k \varepsilon_k \lambda_k x_k \right\|^2_{\ell_p^N} \right)^{1/2} \leq \left( \mathbb{E} \left\| \sum_k \varepsilon_k \lambda_k x_k \right\|^2_{\ell_p^2} \right)^{1/2} \leq \left( \sum_k \left\| \lambda_k x_k \right\|^2_{\ell_p^2} \right)^{1/2}, \]

since \( \|x_k\|_{\ell_p^N} \leq \|x_k\|_{\ell_p^2} \leq 1 \) for each \( k \). Thus \( \mathcal{R}(B_{\ell_p}^N) \leq N^{1/p - 1/2} = \gamma_{\ell_p}(N) \), while the reverse inequality is evident.

### 3. THE RMF PROPERTY

By definition, \( X \) has RMF if the Rademacher maximal function

\[ M_R f(t) = \mathcal{R}(\langle f \rangle_I : I \ni t, I \subset [0, 1) \text{ dyadic}) \text{, where } \langle f \rangle_I = \frac{1}{|I|} \int_I f(s) \, ds, \]

satisfies \( \|M_R f\|_{L_p} \lesssim \|f\|_{L_p(X)} \) for some (equivalently, for all) \( p \in (1, \infty) \). Note, that if \( X \) has type \( 2 \), then \( R \)-bounds are controlled by uniform bounds and \( M_R f \) is controlled pointwise by the standard dyadic maximal function

\[ M f(t) = \sup\{\|\langle f \rangle_I\| : I \ni t, I \subset [0, 1) \text{ dyadic}\}, \]

which satisfies \( \|M f\|_{L_p} \lesssim \|f\|_{L_p(X)} \) for all \( p \in (1, \infty) \).

**Proposition 2.** \( \ell_p \) has RMF when \( 1 < p < \infty \).

**Proof.** The proof relies on the fact that the norm of an \( f = (f_k)_k \in L_p(\ell_p) \) is easy to compute by Fubini's theorem:

\[ \|f\|_{L_p(\ell_p)}^p = \sum_k \|f_k\|_{L_p}^p. \]

Averages can be calculated coordinatewise, i.e. \( \langle f \rangle_I = \langle \langle f_k \rangle_I \rangle_k \), and hence

\[ \mathbb{E} \left\| \sum_{I \ni t} \varepsilon_I \lambda_I \langle f \rangle_I \right\|^p_{\ell_p} = \sum_k \mathbb{E} \left\| \sum_{I \ni t} \varepsilon_I \lambda_I \langle f_k \rangle_I \right\|^p_{\ell_p} \approx \sum_k \left( \sum_{I \ni t} |\lambda_I \langle f_k \rangle_I|^2 \right)^{p/2} \leq \sum_k \sup_{I \ni t} |\langle f_k \rangle_I|^p \left( \sum_{I \ni t} |\lambda_I|^2 \right)^{p/2} \]

so that

\[ M_R f(t)^p = \mathcal{R}(\langle f \rangle_I : I \ni t)^p \lesssim \sum_k \sup_{I \ni t} |\langle f_k \rangle_I|^p = \sum_k M f_k(t)^p. \]

Consequently, using the \( L_p \) inequality for the standard dyadic maximal function,

\[ \|M_R f\|_{L_p}^p \lesssim \sum_k \|M f_k\|_{L_p}^p \lesssim \sum_k \|f_k\|_{L_p}^p = \|f\|_{L_p(\ell_p)}^p, \]

\[ \square \]
For the RMF property it suffices to consider simple functions. In particular, we may restrict our attention to functions that are constant on dyadic intervals of length $2^{-N}$ in which case also their maximal functions are constant on these intervals. Moreover, for every such interval $J$ there is a permutation $\pi$ of $\{1, \ldots, 2^N\}$ so that

$$\{(f)_I : I \supset J \} = \left\{ \frac{1}{2^k} \sum_{j=1}^{2^k} x_{\pi(j)} : 0 \leq k \leq N \right\},$$

where $x_j$ is the value of $f$ on $[(j-1)2^{-N}, j2^{-N})$. We can then state the RMF property as the requirement that the modulus

$$\text{RMF}_X(N) = \sup \left\{ \sum_{\pi} \mathcal{B}\left( \frac{1}{2^k} \sum_{j=1}^{2^k} x_{\pi(j)} : 0 \leq k \leq N \right)^p : \sum_{j=1}^{2^N} \|x_j\|^p \leq 1 \right\}$$

remains bounded for $N \in \mathbb{N}$. Here the $2^N$ permutations $\pi$ are understood to arise as above.

The smaller modulus

$$\beta_X(N) = \sup_{x_1, \ldots, x_{2^N} \in B_X} \min_{\pi} \mathcal{B}\left( \frac{1}{2^k} \sum_{j=1}^{2^k} x_{\pi(j)} : 0 \leq k \leq N \right)$$

is perhaps more tractable and provides a finitary version of the quantity

$$\sup \{ \inf M_R f : \|f\| \leq 1 \text{ a.e.} \}.$$

**Proposition 3.**

$$\beta_{\ell_1}(N) \gtrsim \sqrt{\log N}.$$  

Consequently, $\ell_1$ does not have RMF.

**Proof.** We test the modulus $\beta_{\ell_1}(N)$ by the canonical basis vectors $e_j$, whose R-bounds are equal in all permutations, and show that

$$\mathcal{B}\left( \frac{1}{2^k} \sum_{j=1}^{2^k} e_j : 0 \leq k \leq N \right) \gtrsim \sqrt{\log N}.$$

Assuming that $N = 2^m$ we choose $\lambda_{2^i} = 1$, for $i = 1, \ldots, m$, and $\lambda_k = 0$ otherwise. Now

$$E \left\| \sum_{k=0}^{N} \varepsilon_k \lambda_k \frac{1}{2^k} \sum_{j=1}^{2^k} e_j \right\|_{\ell_1} = E \left\| \sum_{i=1}^{m} \varepsilon_i \frac{1}{2^{2^i}} \sum_{j=1}^{2^{2^i}} e_j \right\|_{\ell_1}$$

$$\geq E \left\| \sum_{i=1}^{m} \varepsilon_i \frac{1}{2^{2^i}} \sum_{j=2^{2^{i-1}}}^{2^{2^i}} e_j \right\|_{\ell_1} - \sum_{i=1}^{m} \left\| \frac{1}{2^{2^i}} \sum_{j=1}^{2^{2^{i-1}}} e_j \right\|_{\ell_1}$$

$$= \sum_{i=1}^{m} \frac{2^{2^i} - 2^{2^{i-1}}}{2^{2^i}} - \sum_{i=1}^{m} \frac{2^{2^{i-1}}}{2^{2^i}}$$

$$\geq \frac{m}{2} - 1 \gtrsim \sqrt{\log N} \left( \sum_{k=0}^{N} |\lambda_k|^2 \right)^{1/2},$$

as required. \(\square\)

**Remarks.**

- What fails in a similar argument with $p > 1$ is that

$$E \left\| \sum_{i=1}^{m} \varepsilon_i \frac{1}{2^{2^i}} \sum_{j=2^{2^{i-1}}}^{2^{2^i}} e_j \right\|_{\ell_p} = \left( \sum_{i=1}^{m} \frac{2^{2^i} - 2^{2^{i-1}}}{2^{2^i}} \right)^{1/p} \leq \left( \sum_{i=1}^{\infty} 2^{2(1-p)2^i} \right)^{1/p} \lesssim 1.$$

We of course have

$$\mathcal{B}\left( \frac{1}{2^k} \sum_{j=1}^{2^k} e_j : 0 \leq k \leq N \right) \leq \beta_{\ell_p}(N) \leq \text{RMF}_{\ell_p}(N) \lesssim 1,$$
but we can also argue directly using the fact that $\ell_p$ has type $p$:  
\[
\mathbb{E} \left\| \sum_{k=0}^N \varepsilon_k \lambda_k \frac{1}{2^k} \sum_{j=1}^{2^k} e_j \right\| \leq \left( \sum_{k=0}^N |\lambda_k|^p \left\| \frac{1}{2^k} \sum_{j=1}^{2^k} e_j \right\|_{p^*}^p \right)^{1/p} 
= \left( \sum_{k=0}^N |\lambda_k|^p 2^{k(1-p)} \right)^{1/p} \lesssim \left( \sum_{k=0}^N |\lambda_k|^2 \right)^{1/2}.
\]

- Averages being particular kinds of convex combinations, it is clear that for any $x_1, \ldots, x_{2^N} \in B_X$ and any $\pi$ we have 
\[
\mathcal{A} \left( \frac{1}{2^k} \sum_{j=1}^{2^k} x_{\pi(j)} : 0 \leq k \leq N \right) \leq \mathcal{A}(x_1, \ldots, x_{2^N}) \leq \gamma_X(2^N),
\]
but $\gamma_X(2^N) \to \infty$ unless $X$ has type 2. On the other hand, any prescribed set of vectors \{\(y_0, \ldots, y_N\)\} $\subset B_X$ can be obtained as averages: By setting 
\[
x_1 = y_0, \quad x_{2^j-1+1} = \cdots = x_{2^j} = 2y_j - y_{j-1}, \quad j = 1, \ldots, N
\]and $\|x_j\| \leq 3$ for all $j = 1, \ldots, 2^N$. This shows that the minimum over permutations is necessary in $\beta_X(N)$, as 
\[
\sup_{x_1, \ldots, x_{2^N} \in B_X} \mathcal{A} \left( \frac{1}{2^k} \sum_{j=1}^{2^k} x_j : 0 \leq k \leq N \right) \approx \gamma_X(2^N).
\]

Problems.

- Is the restriction to the smaller modulus justified, i.e. does $\text{RMF}_X(N) \lesssim \beta_X(N)$ hold?
- What use can be made of the permutations $\pi$? Can we see that 
\[
\beta_{\ell_p}(N) \lesssim \mathcal{A} \left( \frac{1}{2^k} \sum_{j=1}^{2^k} e_j : 0 \leq k \leq N \right)
\]
without resorting to Proposition 2?

References

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