Time Frequency Analysis - Winter 2012

Exercise Set 5

5.1. **Investigate the commutation relations between S_ξ and T_y, M_η, $D\lambda^2$: Find a ξ' (possibly different in the different identities below) so that**

$$S_\xi T_y = T_y S_{\xi'}, \quad S_\xi M_\eta = M_\eta S_{\xi'}, \quad S_\xi D^2_\lambda = D^2_\lambda S_{\xi'}.$$

Finally, find a value of $\eta = \eta(\xi, \lambda)$ such that $S_\xi D^2_\lambda M_\eta = D^2_\lambda M_\eta S_\xi$. We begin by recalling the definition of the operator S_ξ:

$$S_\xi f(x) = \int_{-\infty}^{\xi} \hat{f}(\eta) e^{2\pi i x \eta} d\eta.$$

We now have

$$\begin{aligned}
S_\xi T_y f(x) &= \int_{-\infty}^{\xi} \widehat{T_y f}(\eta) e^{2\pi i \eta x} d\eta = \int_{-\infty}^{\xi} M_{-y} \hat{f}(\eta) e^{2\pi i x \eta} d\eta \\
&= \int_{-\infty}^{\xi} \hat{f}(\eta) e^{2\pi i \eta (x-y)} d\eta = S_\xi f(x-y) = T_y S_\xi f(x). \\
S_\xi M_\eta f(x) &= \int_{-\infty}^{\xi} \widehat{M_\eta f}(s) e^{2\pi i s x} ds = \int_{-\infty}^{\xi} T_\eta \hat{f}(s) e^{2\pi i s x} ds \\
&= \int_{-\infty}^{\xi} \hat{f}(s-\eta) e^{2\pi i s x} ds = \int_{-\infty}^{\xi-\eta} \hat{f}(s) e^{2\pi i (\eta+s) x} ds \\
&= M_\eta S_{\xi-\eta} f(x). \\
S_\xi D^2_\lambda f &= \int_{-\infty}^{\xi} \widehat{D^2_\lambda f}(\eta) e^{2\pi i \eta x} d\eta = \int_{-\infty}^{\xi} D^2_{\lambda^{-1}} \hat{f}(\eta) e^{2\pi i \eta x} dx \\
&= \int_{-\infty}^{\xi} \lambda^{\frac{1}{2}} \hat{f}(\lambda \eta) e^{2\pi i \eta x} d\eta = \lambda^{\frac{1}{2}} \int_{-\infty}^{\lambda\xi} \hat{f}(\eta) e^{2\pi i x \eta / \lambda} \frac{d\eta}{\lambda} \\
&= \lambda^{-\frac{1}{2}} \int_{-\infty}^{\lambda\xi} \hat{f}(\eta) e^{2\pi i \eta \frac{x}{\lambda}} d\eta = D^2_\lambda S_{\lambda\xi} f(x).
\end{aligned}$$

Using the previous identities we can write

$$S_\xi D^2_\lambda M_\eta f(x) = D^2_\lambda S_{\lambda\xi} M_\eta f(x) = D^2_\lambda M_\eta S_{\lambda\xi - \eta} f(x),$$

so we get the desired identity with $\eta = \lambda\xi - \xi = \eta(\xi, \lambda)$.

5.2. **We have used several times the identity $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, where $\langle f, g\rangle = \int_{\mathbb{R}} f\bar{g}$ is the L^2 inner product. Prove this identity in the following two ways:**
(a) Write the identity $\|h\|_{L^2} = \|\hat{h}\|_{L^2}$ for $h = f + ug$, where $u \in \{1, -1, i, -i\}$.
(b) In the identity $\int_{\mathbb{R}} f\hat{h} = \int_{\mathbb{R}} \hat{f} h$ substitute $f = \bar{\hat{f}}$, and manipulate the right

hand side. (a) We have

$$\|f+ug\|_2^2 = \int_{\mathbb{R}} (f+ug)\overline{(f+ug)} = \int_{\mathbb{R}} (|f|^2 + |u|^2|g|^2 + f\bar{u}\bar{g} + ug\bar{f})$$
$$\int_{\mathbb{R}} |f|^2 + \int_{\mathbb{R}} |g|^2 + \bar{u}\int_{\mathbb{R}} f\bar{g} + u\int_{\mathbb{R}} g\bar{f}.$$

Using the same expansion for $\widehat{f+ug}$ and Plancherel's theorem $\|f+ug\|_2^2 = \|\widehat{f+ug}\|_2^2$ we get

$$\bar{u}\int_{\mathbb{R}} f\bar{g} + u\int_{\mathbb{R}} \bar{f}g = \bar{u}\int_{\mathbb{R}} \hat{f}\bar{\hat{g}} + u\int_{\mathbb{R}} \bar{\hat{f}}\hat{g}$$

For $u=1$ we get

$$\int_{\mathbb{R}} f\bar{g} + \int \bar{f}g = \int \hat{f}\bar{\hat{g}} + \int_{\mathbb{R}} \bar{\hat{f}}\hat{g},$$

while for $u=i$

$$-i\int_{\mathbb{R}} f\bar{g} + i\int \bar{f}g = -i\int \hat{f}\bar{\hat{g}} + i\int_{\mathbb{R}} \bar{\hat{f}}\hat{g}.$$

Multiplying the second identity by i and adding them together gives the claim.

(b) We have already proved that

$$\int_{\mathbb{R}} f\hat{h} = \int_{\mathbb{R}} \hat{f}h.$$

Applying this identity to $\bar{\hat{f}}$ in place of f we get

$$\langle \hat{h}, \hat{f}\rangle = \int_{\mathbb{R}} \hat{h}\bar{\hat{f}} = \int_{\mathbb{R}} \hat{\bar{\hat{f}}}h = \int_{\mathbb{R}} \bar{\hat{\hat{f}}}h = \int_{\mathbb{R}} h\bar{f} = \langle h, f\rangle.$$

5.3. **Let $\mathbb{T}$ be an up-tree of tiles. Show that it can be divided into 20 subcollections $\mathbb{T}_i$ so that if $P, P' \in \mathbb{T}_i$ for the same i, then $\phi_P, \phi_{P'}$ are orthogonal to each other.** Let $P = I_P \times \omega_P$ be a tile. Remember that

$$\phi_P(x) \stackrel{\text{def}}{=} M_{c(\omega_d)} T_{c(I)} D^2_{|I|}\phi,$$

and that $\operatorname{supp}\hat{\phi} \subset [-\frac{1}{20}, \frac{1}{20}]$. For fixed $k \in \mathbb{Z}$ consider all the tiles in $\mathbb{T}$ such that $|\omega_P| = 2^k$ so that $|I_P| = 2^{-k}$. The centers of the time intervals of any two tiles $P, P' \in \mathbb{T}$ satisfy $c(I_P) - c(I_{P'})/|I| \in \mathbb{Z}$. Call two tiles in $\mathbb{T}$ equivalent if $|\omega_P| = |\omega_{P'}| = 2^k$ and $c(I_P) - c(I_{P'}) \equiv 0 \mod 20$. Call $\mathbb{T}_k^{(1)}, \mathbb{T}_k^{(2)}, \ldots, \mathbb{T}^{(20)}$, the different equivalent classes. Finally consider the subcollections $\mathbb{T}_j \stackrel{\text{def}}{=} \cup_{k\in\mathbb{Z}} \mathbb{T}_k^{(j)}$ for $j = 1, 2, \ldots, 20$. The enumeration of the equivalence classes is irrelevant so it can be arbitrary. Now let $P \in \mathbb{T}_j, P' \in \mathbb{T}_{j'}$ for some j, j'. If P, P' correspond to the same k then they satisfy $|\omega_P| = |\omega_{P'}| = 2^k$. In this case the centers of the time intervals satisfy $c(I_P) - c(I_{P'}) = 20n2^{-k}$. By the calculation in the proof of Proposition 5.2 in the notes we get $\langle \phi_P, \phi_{P'}\rangle = 0$. In the complementary case we have $|\omega_P| \neq |\omega'_P|$ so suppose $|\omega_P| < |\omega_{P'}|$. Since both tiles were part of an up-tree, the upper frequency intervals intersect so we must have $\omega_{P_u} \subsetneq \omega_{P'_u}$ so in fact the whole frequency interval satisfies $\omega_P \subset \omega_{P'_u}$. However the frequency support of each ϕ_P is contained in ω_{P_d}:

$$\mathrm{supp}(\hat{\phi}_P) \subset c(\omega_d) + |\omega_P|[-\frac{1}{20}, \frac{1}{20}] \subset \omega_{P_d}.$$

Thus in this case as well we have $\langle \phi_P, \phi_{P'} \rangle = \langle \widehat{\phi_P}, \widehat{\phi_{P'}} \rangle = 0$.

5.4. **Prove the following fact that was implicitly used in transforming the probabilistic expectation E into the Lebesgue integral over** $[0,1]$**: For numbers** $0 \le a \le b \le 1$**, we have**

$$\mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in [a,b)\big) = b - a,$$

where β_j **are independent random variables with** $\mathbf{P}(\beta_j = 0) = \mathbf{P}(\beta_j = 1) = 1/2$**.** First of all observe that the probability that $\sum_{j=1}^{\infty} 2^{-j}\beta_j$ attains any single value is zero. Indeed, for any given $a \in [0,1)$ we either have that the value a is never attained, thus the probability is zero, or there is a deterministic sequence $\{\epsilon_j\}_{j=1}^{\infty} \subset \{0,1\}^{\mathbb{Z}}$, depending on a, such that

$$\sum_{j=1}^{\infty} 2^{-j}\beta_j = a \Leftrightarrow \beta_j = \epsilon_j \quad \text{for all} \quad j.$$

Thus

$$\{\sum_{j=1}^{\infty} 2^{-j}\beta_j = a\} \subseteq \bigcap_{j=1}^{N} \{\beta_j = \epsilon_j\}$$

for all positive integers N. We conclude that in any case

$$\mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j = a\big) \le \prod_{j=1}^{N} \mathbf{P}\big(\beta_j = \epsilon_j\big) = \frac{1}{2^N},$$

by the independence of the β_j's. Since this holds for arbitrary any we get $\mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j = a\big) = 0$.

Consider the case where $[a,b) = [0,1/2)$. Then we have

$$0 \le \sum_{j=1}^{\infty} 2^{-j}\beta_j \le 1/2 \Leftrightarrow \beta_1 = 0,$$

so in this case the claim is immediate. Now let $[a,b)$ be any dyadic interval of the form $2^{-k}[j, j+1)$. We have

$$\sum_{j=k+1}^{\infty} 2^{-j}\beta_j \le \sum_{j=k+1}^{\infty} 2^{-j} \le 2^{-k},$$

and $\sum_{j=1}^{k} 2^{-j}\beta_j$ is some number of the form $\ell 2^{-k}$ with $\ell \leq 2^k - 1$ where the exact value of ℓ depends only on the values of $\beta_1, \ldots, \beta_k$. Independently of what this exact value is, there exists some sequence of numbers $\epsilon_1, \ldots, \epsilon_k \in \{0,1\}$ such that

$$\mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in 2^{-k}[j, j+1)\big) = \mathbf{P}\left(\bigcap_{j=1}^{k}\{\beta_j = \epsilon_j\}\right) = 2^{-k} = |[a,b)|,$$

by the independence of the β_j's. Finally any interval $[a,b)$ can be written as a (possibly infinite) disjoint union of dyadic intervals except maybe the endpoints, that is

$$(a,b) = \cup_{m=1}^{\infty}\Delta_m$$

where the Δ_j's are dyadic and disjoint. Indeed one considers the maximal dyadic intervals in (a,b) so they are disjoint by construction. Now since (a,b) is open any point $x \in (a,b)$ is contained in some dyadic interval and thus in some maximal dyadic interval. The claim follows easily since

$$\begin{aligned}
\mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in [a,b)\big) &= \mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in (a,b)\big) = \mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in \cup_m \Delta_m\big) \\
&= \sum_m \mathbf{P}\big(\sum_{j=1}^{\infty} 2^{-j}\beta_j \in \Delta_m\big) = \sum_m |\Delta_m| \\
&= |[a,b)| = b - a.
\end{aligned}$$

5.5. **Given a function ψ and an interval J, denote $\psi_J \stackrel{\text{def}}{=} T_{c(J)} D^2_{|J|}\psi$. Let I be a fixed standard dyadic interval. Prove that**

$$\mathbf{E}\psi_{I+\beta}(x) = (\psi * \mathbf{1}_{[0,1)})\mathbf{1}_I(x),$$

where $\mathbf{E}$ is the expectation over the random choice of the shift parameter $\beta \in \{0,1\}^{\mathbb{Z}}$. First of all let us look at $\psi_{I+\beta}$ for a fixed β. We have

$$\psi_{I+\beta}(x) = T_{c(I+\beta)} D^2_{|I|}\psi = T_{c(I+\beta)} \frac{1}{|I|^{\frac{1}{2}}}\psi(\frac{x}{|I|})$$

As in the proof of Theorem 6.1 in the notes we write for $|I| = 2^{-k}$:

$$\begin{aligned}
c(I+\beta) - c(I) &= \sum_{2^{-j} < 2^{-k}} \beta_j 2^{-j} = \sum_{j \geq k+1} \beta_j 2^{-j} = \sum_{s=1}^{\infty} \beta_{s+k} 2^{-(s+k)} \\
&= |I| \sum_{j=1}^{\infty} \beta_{j+k} 2^{-s},
\end{aligned}$$

and the binary series above is uniformly distributed in the interval $[0,1)$ as β_j's are chosen randomly in $\{0,1\}^{\mathbb{Z}}$ by exercise 5.4. Thus

$$\begin{aligned}\mathbf{E}\psi_{I+\beta} &= \frac{1}{|I|^{\frac{1}{2}}}\int_0^1 \psi\big(\frac{x-c(I)-u|I|}{|I|}\big)du = \frac{1}{|I|^{\frac{1}{2}}}\psi * \mathbf{1}_{[0,1)}\big(\frac{x-c(I)}{|I|}\big)\\ &= D^2_{|I|}\psi * \mathbf{1}_{[0,1)}(x-c(I)) = T_{c(I)}D^2_{|I|}\psi * \mathbf{1}_{[0,1)}(x)\\ &= (\psi * \mathbf{1}_{[0,1)})_I(x).\end{aligned}$$