Time Frequency Analysis - Winter 2012

Exercise Set 4

4.1. **Prove the following relations for the modulation, translation and dilation operators** $M_y f(x) \stackrel{\text{def}}{=} e^{2\pi ixy} f(y)$**,** $T_y f(x) = f(x-y)$**,** $D^p_\lambda f(x) = \lambda^{-\frac{1}{p}} f(x/\lambda)$**, where** $y \in \mathbb{R}$ **and** $\lambda > 0$**:**

$$\widehat{M_y f} = T_y \hat{f}, \quad \widehat{T_y f} = M_{-y} \hat{f}, \quad \widehat{D^p_\lambda f} = D^{p'}_{\frac{1}{\lambda}} \hat{f}.$$

Here p' **is the dual exponent of** p**:** $\frac{1}{p} + \frac{1}{p'} = 1$**.** We have

$$\widehat{M_y f}(\xi) = \int_{\mathbb{R}} e^{-2\pi ix\xi} e^{2\pi ixy} f(x) dx = \int_{\mathbb{R}} e^{-2\pi i(\xi-y)x} f(x) dx = \hat{f}(\xi - y) = T_y \hat{f}(\xi),$$

$$\widehat{T_y f}(\xi) = \int_{\mathbb{R}} e^{-2\pi ix\xi} f(x-y) dy = \int_{\mathbb{R}} e^{-2\pi i(x+y)\xi} f(x) dx = e^{-2\pi iy\xi} \hat{f}(\xi) = M_{-y} \hat{f}(\xi),$$

$$\widehat{D^p_\lambda f}(\xi) = \frac{1}{\lambda^{\frac{1}{p}}} \int_{\mathbb{R}} e^{-2\pi ix\xi} f(x/\lambda) dx = \frac{1}{\lambda^{\frac{1}{p}}} \int_{\mathbb{R}} e^{2\pi i(\lambda x)\xi} f(x) \lambda dx = \lambda^{1-\frac{1}{p}} \hat{f}(\lambda\xi) = D^{p'}_{\lambda^{-1}} \hat{f}(\xi).$$

4.2. **Find the Fourier transforms of** $x^\alpha f(x)$ **and** $\partial^\beta_x f(x)$ **in terms of** $\hat{f}$**, and show that the Fourier transform maps the Schwartz space**

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}, \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta_x f(x)| < \infty\}$$

into itself, i.e. if $f \in \mathcal{S}(\mathbb{R})$ **then also** $\hat{f} \in \mathcal{S}(\mathbb{R})$**.** We first perform the calculation for $\alpha = 1$ since it's more transparent. Noting that $xe^{-2\pi ix\xi} = (-\frac{1}{2\pi i}\frac{d}{d\xi})e^{-2\pi ix\xi}$ we have

$$\begin{aligned}\widehat{xf}(\xi) &= \int_{\mathbb{R}} e^{-2\pi ix\xi} x f(x) dx = \int_{\mathbb{R}} \big(- \frac{1}{2\pi i}\frac{d}{d\xi}\big) e^{-2\pi ix\xi} f(x) dx \\ &= \big(- \frac{1}{2\pi i}\frac{d}{d\xi}\big) \hat{f}(\xi),\end{aligned}$$

where some justification is needed for the last equality but everything works fine for 'nice' functions. One can perform the same calculation for general α noting that $x^\alpha e^{-2\pi ix\xi} = (-\frac{1}{2\pi i}\frac{d}{d\xi})^\alpha e^{-2\pi ix\xi}$ or by iterating the previous result. Thus

$$\mathcal{F}((-2\pi ix)^\alpha f)(\xi) = \partial^\alpha_\xi \hat{f}(\xi).$$

The other calculation is similar. We write

$$\mathcal{F}(\partial_x f)(\xi) = \int_{\mathbb{R}} e^{-2\pi ix\xi} \partial_x f(x) dx = \int_{\mathbb{R}} \Big[\partial_x (e^{-2\pi ix\xi} f(x)) - \partial_x e^{-2\pi ix\xi} f(x) \Big] dx.$$

Assuming that f vanishes in some appropriate sense at infinity we get

$$\mathcal{F}(\partial_x f)(\xi) = -\int_{\mathbb{R}} (-2\pi i\xi) e^{-2\pi ix\xi} f(x) dx = (2\pi i\xi) \hat{f}(\xi).$$

The calculation for general α is similar only one has to integrate by parts α times instead of just one. Also note that in general we will need that the derivatives of f up to order $\alpha - 1$ also vanish at infinity. We get in general

$$\mathcal{F}(\partial_x^\beta f)(\xi) = (2\pi i\xi)^\beta \hat{f}(\xi).$$

Thus the Fourier transform turns multiplication by the free variable to differentiation and vice versa, it transforms differentiation to multiplication by the corresponding monomial.

Now let $f \in \mathcal{S}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{N}$. First observe that the functions $\partial_x^\alpha x^\beta f$, $x^\alpha \partial_x^\beta f$ are also Scwhartz functions (the Schwartz class is 'closed' under the operations ∂_x, multiplication by x). Thus for any $\alpha, \beta \in \mathbb{N}$ we have by the previous calculations that

$$|\partial_\xi^\alpha \xi^\beta \hat{f}(\xi)| \simeq_{\alpha,\beta} |\mathcal{F}(x^\alpha \partial_x^\beta f)(\xi)| \leq \|x^\alpha \partial_x^\beta f\|_{L^1(\mathbb{R})} < \infty,$$

where we used the general estimate $\sup_{\xi\in\mathbb{R}} |\hat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R})}$.

4.3. **Let ϕ be a 'nice' function. Prove that for all $x \in \mathbb{R}$ and $\epsilon > 0$,**

$$|D_\epsilon^1 \phi * f(x)| \leq C_\phi M f(x),$$

where C_ϕ is some constant depending only on ϕ and M is the Hardy-Littlewood maximal function

$$Mf(x) = \sup_I \frac{\mathbf{1}_I(x)}{|I|} \int_I |f(y)| dy.$$

Formulate more precisely the assumption that ϕ is 'nice' so that this estimate works. It clearly suffices to assume that $f \geq 0$. An easy estimate can be obtained if ϕ is bounded and has compact support contained say in some ball $B(0, R)$. We have

$$\begin{aligned}
|D_\epsilon^1 \phi * f(x)| &= \frac{1}{\epsilon} \int_{\mathbb{R}} \phi(y/\epsilon) f(x-y) dy = \int_{\mathbb{R}} \phi(y) f(x - \epsilon y) dy \\
&\leq \|\phi\|_\infty \int_{|y|<R} f(x-\epsilon y) dy = \|\phi\|_\infty \frac{1}{\epsilon} \int_{|y|<\epsilon R} f(x-y) dy \\
&= \|\phi\|_\infty \frac{2R}{2\epsilon R} \int_{|y|<\epsilon R} f(x-y) dy \leq 2R \|\phi\|_\infty M f(x) \overset{\text{def}}{=} C_\phi M f(x).
\end{aligned}$$

Note that in this case $\phi \in L^1(\mathbb{R})$ and $\|\phi\|_1 \leq 2R\|\phi\|_\infty$. With a little more care one can just assume that $\phi \in L^1(\mathbb{R})$ plus some additional technical hypotheses. Let ϕ be a simple function of finite measure suport, $\phi = \sum_{k=1}^K c_k \mathbf{1}_{I_k}$ where $I_k = (-a_k, b_k)$ with $c_k, a_k, b_k > 0$. Then

$$D_\epsilon^1 * f(x) = \sum_k c_k \int_{I_k} f(x-\epsilon y) dy = \frac{1}{\epsilon} \sum_k c_k \int_{\epsilon I_k} f(x-y) dy$$

where if $I_k = (a,b)$ then $\epsilon I_k = (\epsilon a, \epsilon b)$. Thus
$$D^1_\epsilon * f(x) = \sum_k c_k \frac{|I_k|}{\epsilon |I_k|} \int_{x-\epsilon I_k} f(y)dy$$
Now $0 \in I_k \Rightarrow x \in x - \epsilon I_k$ for all k so that
$$D^1_\epsilon * f(x) \le \sum_k c_k |I_k| Mf(x) = \|\phi\|_{L^1} Mf(x).$$
The functions ϕ of the form considered approximate positive L^1 functions such that $0 \in \operatorname{supp}(\phi)$ so the result extends to all these functions ϕ. There is no claim that this is the most general set of hypotheses.

4.4. **Write down the proof of the Heisenberg uncertainty principle for general x_o and ξ_o. Investigate which functions give equality.** Assume $f \in \mathcal{S}(\mathbb{R})$. We have
$$\begin{aligned}(\xi - \xi_o)\hat{f}(\xi) &= (\xi - \xi_o)\int_{\mathbb{R}} f(x) e^{-i2\pi i x\xi} dx = (\xi - \xi_0)\int_{\mathbb{R}} f(x) e^{-i2\pi i x(\xi-\xi_o)} e^{-i2\pi x \xi_o} dx \\
&= \int_{\mathbb{R}} M_{-\xi_o} f(x) (-\frac{\partial_x}{2\pi i}) e^{-i2\pi i x(\xi-\xi_o)} dx \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}} \partial_x M_{-\xi_o} f(x) e^{-i2\pi i x(\xi-\xi_o)} dx \\
&= T_{\xi_o}\mathcal{F}(\frac{\partial_x}{2\pi i} M_{-\xi_o} f)(\xi) \\
&= T_{\xi_o}\mathcal{F}(M_{-\xi_o}(\frac{\partial_x}{2\pi i} - \xi_o) f).\end{aligned}$$
Thus
$$\left(\int_{\mathbb{R}} (x-x_o)^2 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (\xi-\xi_o)^2 |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} = \|(x-x_o)f(x)\|_2 \|(\frac{\partial_x}{2\pi i} - \xi_o) f\|_2.$$
Consider $P_{x_o} f(x) = (x - x_o) f(x)$ the position operator and the momentum operator $Q_{\xi_o} f(x) = \frac{1}{2\pi i}(\partial_x - 2\pi i \xi_o) f(x)$. The commutator is
$$\begin{aligned}[P_{x_o}, Q_{\xi_o}] &= \frac{1}{2\pi i}\big[(x-x_o)(\partial_x - 2\pi i\xi_o) f - (\partial_x - 2\pi i \xi_o)(x - x_o) f\big] \\
&= \frac{1}{2\pi i}\big[(x-x_o)\partial_x f - (x-x_o)\partial_x f - f\big] \\
&= \frac{i}{2\pi} f.\end{aligned}$$
Since modulation does not change the L^2-norm we get
$$\begin{aligned}\frac{1}{2\pi}\|f\|_2^2 &= |\langle f, [P_{x_o}, Q_{\xi_o}] f\rangle| = |\langle f, P_{x_o} Q_{\xi_o} f\rangle - \langle f, Q_{\xi_o} P_{x_o} f\rangle| \\
&= |\langle P_{x_o} f, Q_{\xi_o} f\rangle - \langle Q_{\xi_o} f, P_{x_o} f\rangle| = |\langle P_{x_o} f, Q_{\xi_o} f\rangle - \overline{\langle P_{x_o} f, Q_{\xi_o} f\rangle}| \\
&= |2i \operatorname{Im}\langle P_{x_o} f, Q_{\xi_o} f\rangle| \le 2\|P_{x_o} f\|_2 \|Q_{\xi_o} f\|_2,\end{aligned}$$

which is the uncertainty principle.

We have used the inequality $|\operatorname{Im}(iz)| \leq |z|$ which becomes an equality if $|\operatorname{Re}(z)| = |z|$ that is when $z \in \mathbb{R}$. We have also used the Cauchy-Schwarz inequality for two functions in L^2 which becomes an equality exactly when one function is a multiple of the other (the functions are 'co-linear'). Recalling the functions we used Cauchy-Scharz for we conclude that we must have

$$P_{x_o} f = \lambda Q_{\xi_o} f \Leftrightarrow (x - x_o) f(x) = \lambda \frac{1}{2\pi i}(\partial_x - 2\pi i \xi_o) f(x)$$
$$\Leftrightarrow (x - x_o + \lambda \xi_o) f(x) = \frac{\lambda}{2\pi i} \partial_x f(x)$$

Here λ must be purely imaginary $\lambda = i\beta$ for $\beta \in \mathbb{R}$. We get

$$2\pi(x - x_o + i\beta\xi_o) f(x) = \beta \partial_x f(x) \Rightarrow \partial f / f = \frac{2\pi}{\beta}(x - x_o) + 2\pi i \beta \xi_o$$

We conclude that

$$f(x) = c e^{\frac{2\pi}{2\beta}(x - x_o)^2} e^{2\pi\beta\xi_o x}.$$

Observe that we must have $\beta < 0$ in order to get a Schwartz function so it

4.5. **Prove the existence of a symmetric non-negative $\phi_0 \in C^\infty$ which is strictly positive on $(-1,1)$, zero outside, and satisfies**

$$\sum_{k \in \mathbb{Z}} \phi_0(x + k) \equiv 1.$$

Show that for such ϕ_0 the function $\phi \stackrel{\text{def}}{=} \sqrt{\phi_0}$ is also C^∞, and therefore satisfies the properties required for the basic wave packet. Let

$$\phi_1(x) \stackrel{\text{def}}{=} \mathbf{1}_{(0,\infty)} e^{-\frac{1}{x}}, \quad \phi_2(x) \stackrel{\text{def}}{=} \phi_1(x)\phi_1(\frac{1}{3} - x).$$

In order to check that ϕ_1 is C^∞ it suffices to do so at 0 (everywhere else it is obvious). First check that it is continuous at zero, then

$$\lim_{x \to 0^+} \frac{\phi(x) - 0}{x} = \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{x \to +\infty} y e^{-y} = 0.$$

You can use induction on k to show that ϕ_1 is C^k for every k. One way to do that is to show that for every $k \in N$ and $x >$ we have $\phi_1^{(k)}(x) = P_k(\frac{1}{x}) e^{-\frac{1}{x}}$ where P_k is a polynomial of degree $2k$. Then you can show that $\lim_{x \to 0^+} \phi_1^{(k)}(x) = 0$ by essentially using the fact that e^{-y} decays faster than any polynomial power as $x \to +\infty$. So ϕ_1 and thus ϕ_2 are C^∞-functions. Also since $\phi_1(x) \equiv 0$ for $x \leq 0$ (together with all its derivatives) we conclude that ϕ_2 is supported on $(0, 1/3)$.

Now we define

$$\phi_3(x) \stackrel{\text{def}}{=} \int_{-\infty}^{x} \phi_2(y) dy, \quad \phi_4(x) \stackrel{\text{def}}{=} c - \phi_3(1 - x).$$

Since ϕ_3 vanishes outside $(0, 1/3)$ we have

$$\phi_3(x) = \int_0^x \phi_2(y)dy$$

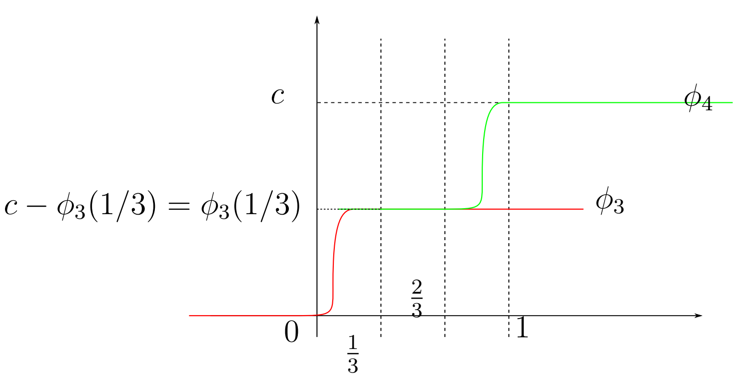
is supported on $(0, \infty)$. Since ϕ_2 vanishes for $x \geq 1/3$ we conclude that $\phi_3(x)$ is constant for $x \geq 1/3$:

$$\int_0^{\frac{1}{3}} \phi_2(x)dx = \phi_3(1/3), \quad x \geq 1/3.$$

Thus ϕ_3 is a smooth 'pulse', identically zero for $x \leq 0$, identically $\phi_3(1/3)$ for $x \geq 1/3$ and C^∞ in the transition interval $(0, 1/3)$.

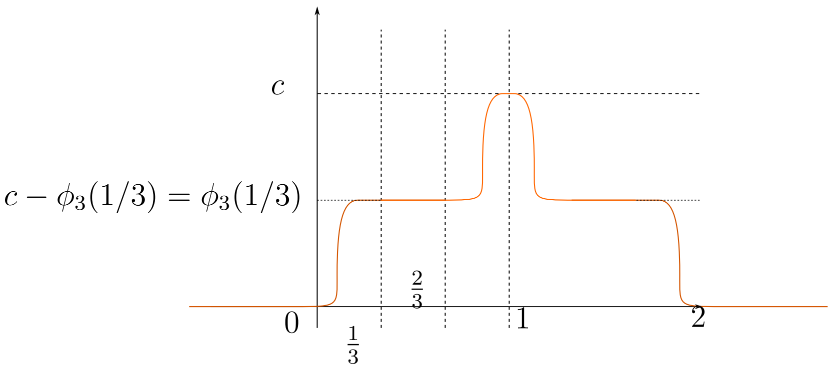
Let us now study the function $\phi_4(x) = c - \phi_3(1-x)$. For $1 - x < 0 \Leftrightarrow x > 1$ we have that $\phi_3(1-x) = 0$ thus $\phi_4(x) = c$ for $x > 1$. For $1 - x > 1/3 \Leftrightarrow x < 2/3$ we have that $\phi_3(1-x) = \phi_3(1/3)$ thus $\phi_4(x) = c - \phi_3(1/3)$ is constant as well. Thus ϕ_4 is also a smooth pulse, identically equal to $c - \phi_3(1/3)$ for $x \leq 2/3$, identically equal to c for $x \geq 1$ and smooth in the transition integral $(2/3, 1)$.

We combine these two functions in the function ϕ_5 as follows. If $x < \frac{1}{3}$ we set $\phi_5(x) \stackrel{\text{def}}{=} \phi_3(x)$. If $x > \frac{2}{3}$ we set $\phi_5(x) \stackrel{\text{def}}{=} \phi_4(x)$. In the interval $(1/3, 2/3)$ both functions ϕ_3 are defined and constant so we just make sure that these constants agree. Indeed for $x \in (1/3, 2/3)$ we have $\phi_3(x) \equiv \phi(1/3)$ and $\phi_4(x) = c - \phi_3(1/3)$. Choose $c = 2\phi_3(1/3)$ so that these values agree. Since we 'glue' the function together in the interval where they are both constant and equal the resulting function ϕ_5 is also smooth. The function ϕ_5 looks like a 'double smooth pulse', identicallye equal to 0 for $x < 0$, identically equal to $\phi_3(1/3)$ for $1/3 \leq x \leq 2/3$ and identically equal to $2\phi_3(1/3)$ for $x \geq 1$, and smooth in all the transition intervals $(0, 1/3)$ and $(2/3, 1)$.



Finally, we reflect the function ϕ_5 with respect to 1, translate to center it at 0, and normalize to make its value equal to 1 at 0. In formulas this means we define

$$\begin{aligned}\phi_0(x) &= \frac{1}{(\phi_5(1))^2}\phi_5(x+1)\phi_5(2-(x+1)) \\ &= \frac{1}{(2\phi_3(1/3))^2}\phi_5(x+1)\phi_5(1-x).\end{aligned}$$



We need to check that this forms a partition of unity. We want to study the function

$$\psi(x) \overset{\text{def}}{=} \sum_{k\in\mathbb{Z}} \phi_0(x+k) = \frac{1}{(2\phi_3(1/3))^2} \sum_{k\in\mathbb{Z}} \phi_5(x+k+1)\phi_5(1-k-x).$$

Since ψ is 1-periodic so it suffices to consider everything in the interval $(0,1)$. Remember that ϕ_0 was supported in $(-1,1)$, for $x \in (0,1)$ only the terms corresponding to $k=0$ and $k=-1$ will contribute to the sum. We can thus simplify a bit

$$\psi(x) = \frac{1}{(2\phi_3(x))^2}(\phi_5(x+1)\phi_5(1-x) + \phi_5(x)\phi_5(2-x))$$

Now $x \in (0,1)$ implies that $x+1 > 1$ and $2-x > 1$ so that $\phi_5(x+1) = \phi_5(2-x) = 2\phi_3(1/3)$ by the construction of ϕ_5. We end up with

$$\psi(x) = \frac{1}{2\phi_3(1/3)}(\phi_5(x) + \phi_5(1-x)).$$

For $1/3 \le x \le 2/3$ we have that $1-x$ is in the same interval so $\phi_5(x) = \phi_5(1-x) = 2\phi(1/3)$ thus $\psi(x) = 1$. If $x < 1/3$ then $1-x > 2/3$ so we have that $\psi(x) = \frac{1}{2\phi_3(1/3)}(\phi_3(x) + \phi_4(1-x)) = \frac{1}{2\phi_3(1/3)}(\phi_3(x) + c - \phi_3(x)) = 1$, remembering that $c = 2\phi_3(1/3)$. The corresponding calculation is valid for $x \in (2/3, 1)$ so we get that $\psi(x) \equiv 1$ as we wanted to show.

In general now suppose that ϕ is a C^∞ function, strictly positive on some interval (a,b) and identically zero on $\mathbb{R} \setminus (a,b)$. For such a function we claim that $\psi \overset{\text{def}}{=} \sqrt{\phi}$ is also C^∞. This is obvious for $x \notin [a,b]$ since the function ϕ is identically zero there and also obvious for $x \in (a,b)$ thus we only need to examine what happens at an endpoint say a. Let us assume, without loss of generality, that $a = 0$ so that the function ϕ is identically zero for $x < 0$ and strictly positive for $x > 0$. We have by Talyor's theorem and the fact that ϕ is C^∞ that

$$\lim_{\delta\to 0+} \phi(\delta)/\delta^k = 0 \quad \forall k.$$

In particular $\lim_{\delta\to 0^+} \frac{\sqrt{\phi(\delta)} - \sqrt{\phi(0)}}{\delta} = \lim_{\delta\to 0^+} \sqrt{\frac{\phi(\delta)}{\delta^2}} = 0$. Obviously we have that $\sqrt{\phi(\delta)}/\delta \equiv 0$ for $\delta < 0$ so we see that ψ is differentiable at 0. However now observe that ϕ' is also C^∞ so the same argument applies for ϕ' in the place of ϕ. We can then complete the proof by induction.