Time Frequency Analysis - Winter 2012

Exercise Set 2

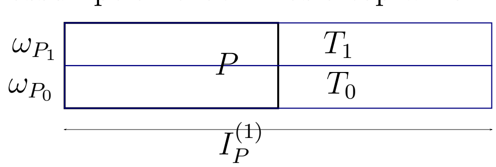
2.1. **Prove that bitiles satisfy $P \leq P'$ if and only if $P_u \leq P'_u$ or $P_d \leq P'_d$. Check also that $\leq$ (for either tiles or bitiles) is a partial order, i.e.: $P \leq P' \leq P$ if and only if $P = P'$, and $P \leq P' \leq P''$ implies $P \leq P''$.** Suppose first that $P \leq P'$ thus $I_P \subset I_{P'}$ and $\omega_{P'} \subset \omega_P$. If $|\omega_P| = |\omega_{P'}|$ we necessarily have that $P = P'$ in which case we have that $P_u \leq P'_u$ *and* $P_d \leq P'_d$. If $|\omega_{P'}| < |\omega_P|$ then we either have that $\omega_{P'_u} \subset \omega_{P'} \subset \omega_{P_u}$ or that $\omega_{P'_d} \subset \omega_{P'} \subset \omega_{P_d}$. In the first case we get that $P_u \leq P'_u$ and in the second that $P_d \leq P'_d$. For the opposite implication if $P_z \leq P'_z$ where $z \in \{u, d\}$. Then $I_P \subset I_{P'}$ and $\omega_{P'_z} \subset \omega_{P_z}$. If $|\omega_{P'_z}| < |\omega_{P_z}|$ then we necessarily have that $|\omega_{P'_z}| \leq \frac{1}{2}|\omega_{P_z}|$ which implies that $\omega_{P'} \subset \omega_{P_z} \subset \omega_P$. If $|\omega_{P'_z}| = |\omega_{P_z}|$ then we must have that $P_z = P'_z$ and thus $P = P'$. In either case we get that $P \leq P'$.

To prove that $\leq$ is a partial order first suppose that $P \leq P' \leq P$. Then $I_P \subset I_{P'} \subset I_P \Rightarrow I_P = I_{P'}$ and also $\omega_{P'} \subset \omega_P \subset \omega_{P'} \Rightarrow \omega_P = \omega_{P'}$. Thus $P = P'$. The opposite implication here is obvious.

If $P \leq P' \leq P''$ then $I_P \subset I_{P'} \subset I_{P''}$ and $\omega_{P''} \subset \omega_{P'} \subset \omega_P$ which shows that $P \leq P''$.

2.2. **Prove that a non-empty tree $\mathbb{T}$ which has a top $T \in \mathbb{T}$ has a minimal top with respect to the partial order $\leq$ of bitiles.** We need to show that if $\mathbb{T}$ is a non-empty tree with a top $T_1 \in \mathbb{T}$ then there exists a minimal top T such that $T \leq T'$ for all other tops T'.

The following example is instructive. If $\mathbb{T}$ is a very simple tree consisting of exactly one bitile P then P itself is a top of $\mathbb{T}$. However it is pretty easy to construct other examples of tops as follows. If $P = I_P \times \omega_P$, consider the parent $I_P^{(1)}$ of I_P and the children ω_{P_0} and ω_{P_1} of P. Then the bitiles $T_i = I_P^{(1)} \times \omega_{P_i}$ are both tops of $\mathbb{T}$ and are disjoint so it is not possible to compare them. However in this case it is obvious that P itself is a minimal top since $P \leq T_0, T_1$. On the other hand if T is a top which is part of the tree and T' is a top which as we just saw cannot be part of the tree then since $T \in \mathbb{T}$ and T' is a top we have that $T \leq T'$ so that T is minimal. Here the assumption that $\mathbb{T}$ has a top which is part of the tree greatly simplified the argument.



The general case of a tree $\mathbb{T}$ with a top $T \in \mathbb{T}$ is roughly the same as the special case of the picture above. Observe that a top of a tree which is itself part of the tree must necessarily be unique. Indeed, if $T, T' \in \mathbb{T}$ and they are both tops then we must have $T \leq T' \leq T$ so that $T = T'$. Also, for every other top T' of $\mathbb{T}$ (which cannot be part of the tree as we just saw) we have that $P \leq T'$ for every $P \in \mathbb{T}$ thus $T \leq T'$ since $T \in \mathbb{T}$ so that T is minimal.

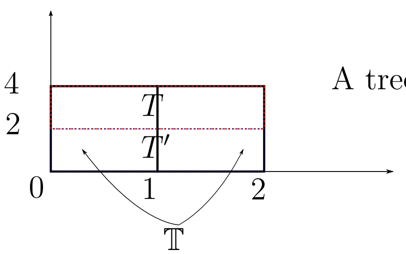
The claim is wrong in general, that is, a tree $\mathbb{T}$ that does not have a top $T \in \mathbb{T}$ does not necessarily have a minimal top. To see this suppose that a tree $\mathbb{T}$ has a top $T \notin \mathbb{T}$ and that no bitile in $\mathbb{T}$ is a top of $\mathbb{T}$. We have that $\omega_T \subseteq \cap_{P\in\mathbb{T}}\omega_P$. Now we claim that there is a tile $P^{\min} \in \mathbb{T}$ such that $\omega_{P^{\min}} = \cap_{P\in\mathbb{T}}\omega_P$. Indeed, define $P^{\min}$ to be the tile with minimum $|\omega_P|$. This is easily seen to be unique and obviously it contains $\cap_{P\in\mathbb{T}}\omega_P$. Now if $P' \in \mathbb{T}$ is any other tile we have that $P^{\min} \cap P' \neq \emptyset$ so one contains the other. Since $\omega_{P^{\min}}$ has minimum length we must have that $\omega_{P^{\min}} \subset \omega_{P'}$ so that $\omega_{P^{\min}} \subset \cap_{P\in\mathbb{T}}\omega_P$.

Now let T be any top of $\mathbb{T}$. If we had that $|\omega_T| = |\omega_{P^{\min}}|$ then this would mean that $I_{P^{\min}} = I_T$ since $I_{P^{\min}} \subset I_T$ and both have the same length in this case. Thus $T = P^{\min}$ which is a contradiction since we suppose that $\mathbb{T}$ does not have a top belonging to the tree. Thus we necessarily have that either $\omega_T \subset \omega_{P_u^{\min}}$ or that $\omega_T \subset \omega_{P_d^{\min}}$. In either case we conclude that ω_T has a dyadic sibling contained in $\omega_{P^{\min}}$. Now it is not hard see that the bitile $T' \overset{\text{def}}{=} I_T \times \omega_T'$ is also a top of $\mathbb{T}$. Indeed, we have that

$$I_P \subset I_T = I_{P^{\min}} \text{ and } \omega_T' \subset \omega_{P_{\min}} \subset \omega_P \text{ for all } P \in \mathbb{T}.$$

However the tops T, T' are disjoint so they cannot be comparable.

This general argument is a bit complicated. It's much simpler to construct a particular example. Suppose that $\mathbb{T}$ consists of just two disjoint bitiles, $P_1 = [0,1) \times [0,4)$ and $P_2 = [1,2) \times [0,4)$. Then every top T must have time interval containing the interval $[0,2)$. Thus the only candidates for tops that are minimal are the bitiles $T = [0,2) \times [2,4)$ and $T' = [0,2) \times [0,2)$. However these tiles are not comparable since they are disjoint.



A tree $\mathbb{T}$ without a minimal top.

Finally let $\mathbb{T}$ be a tree and consider the top with minimal $|I_T|$ (maybe it is not unique). Then for any other top T' we have that $I_T \cap I_T' \neq \emptyset$ so on contains the other. Since $|I_T| \leq |I_{T'}|$ we must have that $I_T \subseteq I_{T'}$ so that T has minimal time interval with respect to"$\subseteq$".

2.3. **Recall that $S_N f = \sum_{P\in\mathbb{P}} \langle f, w_{P_d}\rangle w_{P_d}$, where $\mathbb{P} = \{P$ bitile: $I_P \subseteq [0,1), \omega_{P_u} \ni N\}$. Prove that $\mathbb{P}$ is an up-tree and find its minimal top.** Let $I \overset{\text{def}}{=} [0,1)$ and consider the dyadic intervals $\omega_j \overset{\text{def}}{=} [4j, 4j+4)$ for $j = 1, 2, \ldots$. Now observe that the ω_j's are disjoint and thus there is a unique $\omega_J \ni N$. Then the rectangle $P_J \overset{\text{def}}{=} [0,1) \times \omega_J$ is a bitile and in fact we have that $\omega_J \in \mathbb{P}$. Now suppose that $P \in \mathbb{P}$ and $P \neq P_J$. We must have that I_P is properly contained in $[0,1)$ so that

$|\omega_P| > 4 = |\omega_{P_J}|$. Since $N \in \omega_P \cap \omega_{P_J}$ we conclude that $\omega_{P_J} \subset \omega_P$. Thus $P \leq P_J$. So P_J is a top of $\mathbb{P}$ and since $P_J \in \mathbb{P}$, P_J is the minimal top of $\mathbb{P}$.

2.4. **Let $g \in L^1_{\text{loc}}$ and $p \in (1, \infty)$. Show the "$\Rightarrow$" direction of the equivalence**

$$\|g\|_{L^{p,\infty}} \lesssim A \Leftrightarrow \left| \int_E g \right| \lesssim A|E|^{\frac{1}{p'}},$$

for for all bounded sets E. Only one of the implications holds for $p = 1$. Investigate which one? Suppose that $\|g\|_{L^{p,\infty}} \lesssim A$. We have

$$\begin{aligned} \int_E |g(x)|dx &= p \int_0^\infty |\{x \in E : |g(x)| > \lambda\}| d\lambda \\ &\lesssim \int_0^b |E| d\lambda + A^p \int_b^\infty \frac{1}{\lambda^p} d\lambda \end{aligned}$$

for any constant $b > 0$. Observe that the second term above is finite if and only if $p > 1$. Thus

$$\int_E |g(x)|dx \leq b|E| + A^p \frac{b^{-p+1}}{p-1}$$

Optimizing in b (take $b = A(p-1)^{-\frac{1}{p}}|E|^{-\frac{1}{p}}$) we get

$$\int_E |g(x)|dx \leq \frac{A}{(p-1)^{\frac{1}{p}}} |E|^{\frac{1}{p'}}.$$

This simple argument fails for $p = 1$ (and the constant blows up) so we suspect that this is actually the direction that fails for $p = 1$. Indeed let for example $g(x) = x^{-1}\mathbf{1}_{[0,1)}(x)$. We have for $\lambda > 0$:

$$|\{|g| > \lambda\}| = |\{x \in [0,1) : |g(x)| > \lambda\}| = |[0, \lambda^{-1})| = \max(\frac{1}{\lambda}, 1),$$

so that $\|g\|_{L^{1,\infty}} \lesssim 1$. Now for $E = [0,1)$ we have

$$\int_E |g(x)|dx \geq \int_1^\infty |\{x \in E : |g(x)| > \lambda\}| d\lambda = \int_1^\infty \frac{d\lambda}{\lambda} = +\infty.$$

The other direction works and the proof is the same as for $p > 1$. First assume that $g \geq 0$. Take any compact $B \subset \{g(x) > \lambda\}$ with $|B| < +\infty$. We have $g/\lambda > 1$ on B so:

$$|B| = \int_B dx < \int_B \frac{g(x)}{\lambda} = \frac{1}{\lambda} \int_B g(x)dx \lesssim \frac{1}{\lambda} A$$

by the hypothesis (note that $p' = \infty$). Taking the supremum over compact $B \subset \{g(x) > \lambda\}$ we get $\|g\|_{L^{1,\infty}} \lesssim A$. For general g write it as

$$g \leq \operatorname{Re} g^+ - \operatorname{Re} g^- + i \operatorname{Im} g^+ - i \operatorname{Im} g^-$$

so it is enough to prove the estimate $\|g_i\|_{L^{1,\infty}} \lesssim A$ for each one of the terms g_i appearing in the previous sum. Now observe that each one of them is non-negative and the hypothesis is satisfied for each one of them.

2.5. **Let** $h_I(x) \stackrel{\text{def}}{=} |I|^{-\frac{1}{2}}(\mathbf{1}_I(x) r_0(x/|I|))$ **denote the Haar function adopted to** I**. Write** h_I **in the Walsh formalism as some** w_P**. Using properties of the Walsh wave packets prove that** $\{h_I : I \subseteq [0,1) : |I| > 2^{-k}\}$ **is an orthonormal basis of** $\{f \in L^2(0,1) : \int f = 0, f$ **is constant on** $2^{-k}[j, j+1),\ j = 0, 1, 2, \ldots 2^{k-1}$**.**
We have that $r_0(x) = w_{2^0}(x) = w_1(x)$ thus we immediately see that

$$h_I(x) = w_{I \times |I|^{-1}[1,2)}(x).$$

Consider now the set of tiles

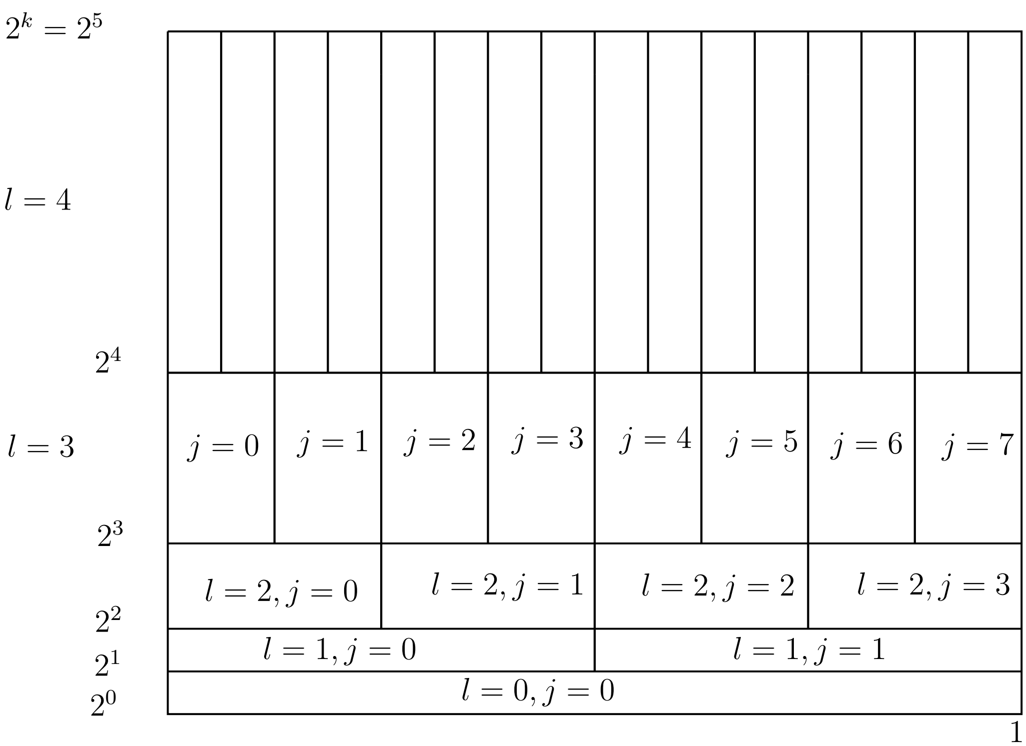
$$\mathbb{P}_1 \stackrel{\text{def}}{=} \{P = I_P \times \omega_P : I_P \subseteq [0,1), |I_P| > 2^{-k}, \omega_P = |I_P|^{-1}[1,2]\}.$$

Let us take a look at the area this set of tiles covers in the phase plane. All the time-intervals are contained in $[0,1)$ and the interval $[0,1)$ itself is admissible in this collection. The smallest time-intervals in this collection are the dyadic intervals of length 2^{-k+1}. On the other hand all the time intervals have length $\geq 2^{-k+1}$ so the frequency intervals have length at most 2^{k-1} and at least 1. Thus all the frequency intervals in $\mathbb{P}$ are of the form

$$\{2^l[1,2), \quad l = 0, 1, 2, \ldots, k-2\} = \{[2^l, 2^{l+1}), \quad l = 0, 1, 2, \ldots, k-2)\}.$$

Observe that they are disjoint and each one starts where the previous one ends. We have that

$$\mathbb{P}_1 = \{2^{-l}[j, j+1) \times [2^l, 2^{l+1}) : l = 0, 1, 2, \ldots, k-1,\ j = 0, 1, \ldots, 2^l - 1\}.$$



Thus the tiles in $\mathbb{P}_1$ are pairwise disjoint and cover the area $[0,1) \times [1, 2^{k-1}]$ in the phase plane $\mathbb{R}^2_+$.

Now that we have a pretty good picture of the wave packets in $\mathbb{P}_1$ let's move the subspace of $L^2(0,1)$ described as

$$L_k^2 \stackrel{\text{def}}{=} \left\{ f \in L^2(0,1) : \int f = 0, f \text{ is constant on } 2^{-k}[j, j+1),\ \ j = 0, 1, 2, \ldots 2^k - 1 \right\}$$

Clearly all these functions are linear combinations of the form

$$\sum_{j=0}^{2^k-1} c_j \mathbf{1}_{2^{-k}[j,j+1)}$$

subject to the condition

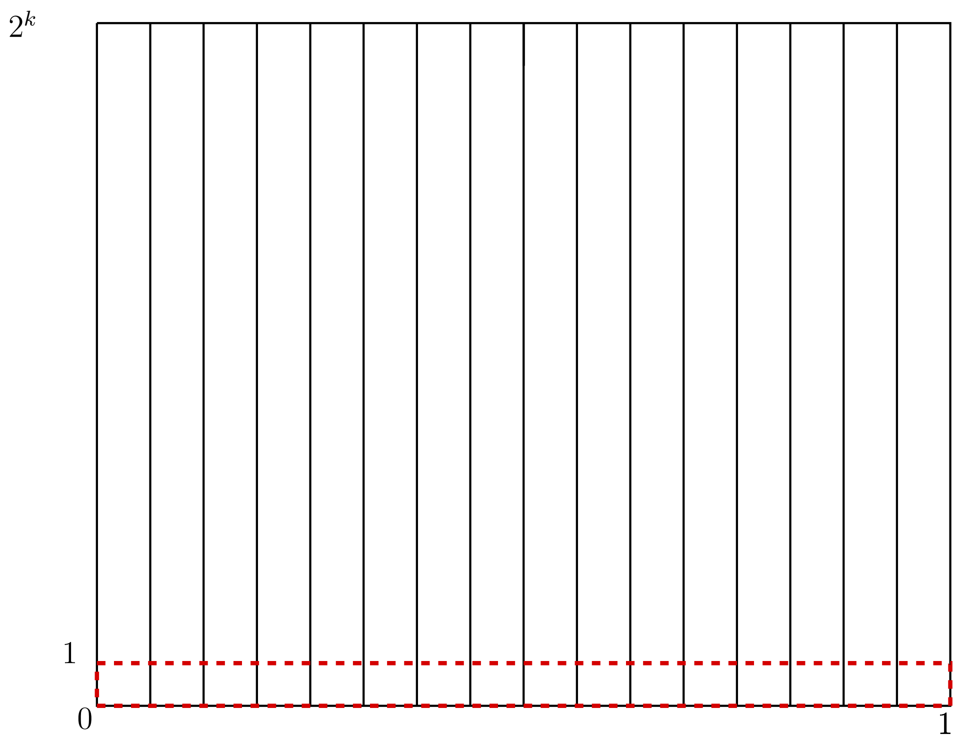
$$0 = \int \sum_{j=0}^{2^k-1} c_j \mathbf{1}_{2^{-k}[j,j+1)} = 2^{-k} \sum_{j=0}^{2^k-1} c_j = 0.$$

Denoting $I_j \stackrel{\text{def}}{=} 2^{-k}[j, j+1) \subset [0, 1)$ for $1 \leq j \leq 2^k - 1$ observe that we have

$$\mathbf{1}_{2^{-k}[j,j+1)} = |I_j|^{\frac{1}{2}}\ |I_j|^{-\frac{1}{2}} \mathbf{1}_{I_j} w_0(\cdot/|I_j|) = |I_j|^{\frac{1}{2}} w_{P_j},$$

where $P_j \stackrel{\text{def}}{=} 2^{-k}[j, j+1) \times 2^k[0, 1)$. Thus L_k^2 is contained in the linear span of the wave packets in the collection

$$\mathbb{P}_2 \stackrel{\text{def}}{=} \{2^{-k}[j, j+1) \times 2^k[0, 1) : j = 0, 1, 2, \ldots, 2^k - 1\}.$$



More concisely we can write

$$L^2_k = \text{span}\{w_P : P \in \mathbb{P}_2\} \cap \{f \in L^2 : \int f = 0\}.$$

The area in the phase plane covered by the tiles in $\mathbb{P}_2$ is the square $[0,1) \times [0, 2^k) = [0,1) \times [0,1) \cup [0,1) \times [1, 2^k)$. Write $P_0 \stackrel{\text{def}}{=} [0,1) \times [0,1)$. By the results in the lecture notes and in particular Lemma 1.5 and Proposition 1.3 we conclude that

$$\text{span}\{w_P : P \in \mathbb{P}_2\} = \text{span}\{w_P : P \in \mathbb{P}_1 \cup P_0\}$$

and that the set $\{w_P : P \in \mathbb{P}_1 \cup P_0\}$ is an orthonormal basis of $\text{span}\{w_P : P \in \mathbb{P}_2\}$. Now observe that any $f \in L^2_k$ is orthogonal to w_{P_0} since $\int f = 0$ and w_{P_0} is just $\mathbf{1}_{[0,1)}$! Since obviously $L^2_k \subseteq \text{span}\{w_P : P \in \mathbb{P}_2\}$ we have for every $f \in L^2_k$ that

$$f = \sum_{P \in \mathbb{P}_1} \langle f, w_P \rangle w_P + \langle f, w_{P_0} \rangle w_{P_0}.$$

and by the zero-mean property we actually get that

$$0 = \int f = \int \sum_{P \in \mathbb{P}_1} \langle f, w_P \rangle w_P + \int \langle f, w_{P_0} \rangle w_{P_0} = 0 + \langle f, w_{P_0} \rangle.$$

Thus every $f \in L^2_k$ can be written in the form $f = \sum_{P \in \mathbb{P}_1} c_P w_P$ and of course the set $\{w_P : P \in \mathbb{P}_1\}$ is linearly independent.