Time-frequency analysis — Winter 2012

6. Exercise set

6.1. Prove the following estimate, which was needed to complete the proof on the lectures:

$$\sum_{k=0}^{\infty} \int_{I_T^c} \Big(1 + 2^k \frac{\operatorname{dist}(x, I_T)}{|I_T|}\Big)^{-9} \,\mathrm{d}x \lesssim |I_T|.$$

Here I_T is an interval, and $\operatorname{dist}(x, I_T)$ designates the distance of the point $x \in I_T^c$ from this interval.

6.2. Prove the other estimate needed in the lectures:

$$|\langle \phi_P, \phi_{P'} \rangle| \lesssim \Big(\frac{|I_P|}{|I_{P'}|}\Big)^{1/2} \|v_{I_P} 1_{I_{P'}}\|_1, \qquad v_I(x) := \frac{1}{|I|}\Big(1 + \frac{|x - c(I)|}{|I|}\Big)^{-10},$$

[Hint: Recall that $|\phi_P| \lesssim |I_P|^{1/2} v_{I_P}$, and such a bound is true even if the power 10 in the definition of v_I is replaced by any bigger number.]

6.3. In the lectures we considered a collection $\mathscr{T}$ of trees with the following property:

$(*)$ If $P \in \mathbb{T} \in \mathscr{T}$ and $P' \in \mathbb{T}' \in \mathscr{T}$ satisfy $\omega_P \subseteq \omega_{P'_d}$, then $I_{P'} \cap I_{\mathbb{T}} = \varnothing$.

Prove that under this assumption, every tree $\mathbb{T} \in \mathscr{T}$ can be divided into up-trees $\mathbb{T}_j$, whose top time-intervals $I_{\mathbb{T}_j}$ are pairwise disjoint. [Hint: Let T_j be the maximal tiles in $\mathbb{T}$ and define the subtrees $\mathbb{T}_j := \{P \in \mathbb{T} : P \leq T_j\}$. Check that these give the required decomposition, in particular, they are up-trees. Note that in the assumption $(*)$ we also allow the case that $\mathbb{T}' = \mathbb{T}$.]

6.4. Recall from the lectures the sums

$$S := \Big(\sum_{P \in \mathbb{P}} |\langle f, \phi_P \rangle|^2\Big)^{1/2}, \qquad S_2 := \sum_{\substack{P, P' \in \mathbb{P} \\ \omega_P \subseteq \omega_{P'_d}}} \langle f, \phi_P \rangle \langle \phi_P, \phi_{P'} \rangle \langle \phi_{P'}, f \rangle,$$

where $\mathbb{P} = \bigcup_j \mathbb{T}_j$ is a union of trees. In the lectures, we derived the bounds

$$S^2 \lesssim \sqrt{S^2 + S_2}\|f\|_2, \qquad S_2 \lesssim AS, \qquad A := \sup_{P \in \mathbb{P}} \frac{|\langle f, \phi_P \rangle|}{|I_P|^{1/2}} \Big(\sum_j |I_{\mathbb{T}_j}|\Big)^{1/2}.$$

Now, prove the alternative bound $S_2 \lesssim A^2$, use this to get a bound for S, and then derive by this alternative way the estimate of the Energy lemma,

$$\sum_j |I_{\mathbb{T}_j}| \lesssim \mathscr{E}^{-2} \|f\|_2^2.$$

6.5. This result will be needed on the final week of lectures: Let $\mathbb{P}$ be a finite collection of tiles. Let $\mathscr{J}$ be the collection of all maximal dyadic intervals J with the property that $3J$ (the interval with the same centre and triple the length of J) does *not* contain any I_P with $P \in \mathbb{P}$. Prove that $\mathscr{J}$ is a partition (a pairwise disjoint cover) of $\mathbb{R}$.