Time-frequency analysis — Winter 2012

3. Exercise set

3.1. Let $\mathbb{P}_1$, $\mathbb{P}_2$ be finite collections of tiles. Recall that if $\bigcup_{P\in\mathbb{P}_1} P = \bigcup_{P\in\mathbb{P}_2} P$, then also $\operatorname{span}\{w_P : P\in\mathbb{P}_1\} = \operatorname{span}\{w_P : P\in\mathbb{P}_2\}$. Give an example to show that it is possible to have the second "=", even if the first "=" is replaced by "$\subsetneq$". [Hint: it suffices to consider collections $\mathbb{P}_i$ of just two tiles.]

3.2. Let $\mathbb{P}$ be a finite collection of bitiles with $\operatorname{density}(\mathbb{P})\cdot\operatorname{energy}(\mathbb{P}) = 0$. Check that then $\displaystyle\sum_{P\in\mathbb{P}}\langle f, w_{P_d}\rangle\langle w_{P_d}, 1_{E_{P_u}}\rangle = 0$. Do this both directly from the definition, and by using the Tree Lemma. [Hint: every collection $\{P\}$ of a single bitile is a tree.]

3.3. Prove the following corollary of the Density Lemma and the Energy Lemma: If $\mathbb{P}_n$ is a finite collection of bitiles with

$$(*)\qquad \operatorname{density}(\mathbb{P}_n)\leq 4^n|E|,\qquad \operatorname{energy}(\mathbb{P}_n)\leq 2^n\|f\|_{L^2}^2,$$

then we can decompose it as $\mathbb{P}_n = \mathbb{P}_{n-1}\cup\bigcup_j \mathbb{T}_{n,j}$, where $\mathbb{P}_{n-1}$ satisfies $(*)$ with $n-1$ in place of n, each $\mathbb{T}_{n,j}$ is a tree with top $T_{n,j}$, and the top time intervals satisfy $\displaystyle\sum_j |I_{T_{n,j}}|\leq C4^{-n}$.

3.4. Let $\mathbb{P}$ be a finite collection of tiles with the property that $\displaystyle\sum_{P\in\mathbb{P}} 1_P\leq n$, i.e., at any point of the phase plane, at most n tiles $P\in\mathbb{P}$ intersect. Prove that there exists a decomposition $\mathbb{P}=\bigcup_{j=1}^n\mathbb{P}_j$, where each $\mathbb{P}_j$ is a pairwise disjoint collection of tiles. Give also an example to show that the claim is not true if the tiles are replaced by arbitrary sets.

[Hint: By induction, it is enough to decompose $\mathbb{P}=\mathbb{P}_1\cup\mathbb{P}'$, where $\mathbb{P}_1$ is pairwise disjoint, and $\mathbb{P}'$ has maximum overlap $n-1$. Here is a simple algorithm: Let P_1 be the *longest* (say, in horizontal direction) tile in $\mathbb{P}$ with an n-intersection. Remove P_1 from $\mathbb{P}$, add P_1 to $\mathbb{P}$, and repeat as long as possible. Prove by counterassumption that the constructed $\mathbb{P}_1$ is pairwise disjoint.]

3.5. Follow the hints to give a (bit unusual) proof of the bound $\|Mf\|_{L^2}\leq 2\|f\|_{L^2}$, where $\displaystyle Mf(x)=\sup_{J\in\mathscr{D}}\frac{1_J(x)}{|J|}\int_J|f|$ is the dyadic maximal function:

It suffices (why?) to consider nonnegative f, and an arbitrary linearization $\displaystyle\tilde{M}f=\sum_{J\in\mathscr{D}}\frac{1_{E(J)}}{|J|}\int_J f$, where the subsets $E(J)\subseteq J$ are pairwise disjoint. Let $\tilde{M}^*$ be the Hilbert space adjoint of $\tilde{M}$, and check that $\tilde{M}^*\tilde{M}f\leq 2\tilde{M}f$ pointwise. Recall (or just take for granted) that the operator norm of any linear operator A on a Hilbert space satisfies $\|A\|=\|A^*\|=\|A^*A\|^{1/2}$.