Time-frequency analysis — Winter 2012

2. Exercise set

2.1. Prove that bitiles satisfy $P \le P'$ if and only if $P_u \le P'_u$ or $P_d \le P'_d$. Check also that $\le$ (for either tiles or bitiles) is a partial order, i.e.: $P \le P' \le P$ if and only if $P = P'$, and $P \le P' \le P''$ implies $P \le P''$.

2.2. Let $\mathbb{T}$ be a non-empty tree. Prove that, among there is a minimal top time interval I such that:

- $I = I_T$ for some top T of $\mathbb{T}$, and
- if T' is another top of $\mathbb{T}$, then $I_T \subseteq I_{T'}$.

However, show by an example that there can be different tops with the same minimal time interval but disjoint frequency intervals.

2.3. Recall that $S_N f = \sum_{P \in \mathbb{P}} \langle f, w_{P_d} \rangle w_{P_d}$, where $\mathbb{P} = \{P \text{ bitile} : I_P \subseteq [0,1) : \omega_{P_u} \ni N\}$. Prove that $\mathbb{P}$ is an up-tree and find its minimal top.

2.4. Recall the lemma: for $p \in (1,\infty)$ and $g \in L^1_{\text{loc}}$, we have $\|g\|_{L^{p,\infty}} \lesssim A$ if and only if $|\int_E g| \lesssim A|E|^{1/p'}$ for all bounded sets E. Recall that "$\Leftarrow$" was proven on the lecture; now prove "$\Rightarrow$". (Hint: for $g \ge 0$, $\int_E g = \int_0^\infty |E \cap \{g > t\}| \, \mathrm{d}t$.) Only one of the implications holds for $p = 1$ — investigate which one?

2.5. The Haar functions are defined by $h_I(x) := |I|^{-1/2} 1_I(x) r_0(x/|I|)$ for dyadic intervals I. Write h_I in the Walsh formalism as some w_P. Using properties of the Walsh wave packets, prove that $\{h_I : I \subseteq [0,1) : |I| > 2^{-k}\}$ is an orthonormal basis of $\{f \in L^2(0,1) : \int f = 0,\ f \text{ is constant on } 2^{-k}[j, j+1),\ j = 0, 1, \ldots, 2^k - 1\}$. (Hint: Which domain in the phase plane $\mathbb{R}^2_+$ do the corresponding tiles P cover? — It is also possible to do the exercise directly without using the wave packet formalism, but try to find an 'elegant' proof using the tools that we already have developed.)