Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 9
Solutions

1. a) Suppose G is a compact group and $f \in Map(G, \mathbb{R})$. Let V_f be the vector subspace of $Map(G, \mathbb{R})$ spanned by right translations of f i.e. by the set $\{R_g \mid g \in G\}$.
Then the action $R\colon G \times Map(G, \mathbb{R}) \to Map(G, \mathbb{R})$, $R(g, f') = R_g f'$ (which is continuous with respect to the sup-norm, see exercise 7.3) restricts to the action $R\colon R \times V_f \to V_f$.

   Now suppose V_f is finite-dimensional. Then the action $R\colon G \times V_f \to V_f$ is continuous with respect to the Euclidean topology in V_f, since all norms in a finite-dimensional space define Euclidean topology (this is proved in Topology I course). Hence R defines a continuous linear representation of G on $GL(V_f)$ (see exercise 8. 4).
Prove that f is a matrix coefficient of this representation (Hint: mapping $V_f \to \mathbb{R}, f' \mapsto f'(e)$ is linear). For the definition of the matrix coefficient see exercise 8.5).

   b) Likewise assume $f \in Map(G, \mathbb{R})$ is such that the vector subspace W_f of $Map(G, \mathbb{R})$ spanned by left translations of f i.e. by the set $\{L_g \mid g \in G\}$ is finite-dimensional. Then $L\colon G \times W_f \to W_f$, $L(g, f') = L_{g^{-1}} f'$ is then a continuous (with respect to the Euclidean topology of W_f) linear action of G on finite-dimensional W_f (Check this. Why we use inverse element g^{-1} in the definition of L, instead of simply g, like we did with action R?). Hence it defines a continuous linear representation of G in $GL(W_f)$. Prove that the mapping $G \to \mathbb{R}$, $g \mapsto f(g^{-1})$ is a matrix coefficient of this representation.

   Solution: a) We claim that $L\colon V_f \to \mathbb{R}$, $L(f') = f'(e)$ is linear. Indeed

$$L(af' + bf'') = (af' + bf'')(e) = af'(e) + bf''(e) = aL(f') + bL(f'').$$

   Suppose $g \in G$. Then

$$L(R_g f) = (R_g f)(e) = f(eg) = f(g).$$

   Since $f \in V_f$ and $L\colon V_f \to \mathbb{R}$, this proves that f is a matrix coefficient of this representation.

   b) L is action, since $L(e, f') = L_e(f') = f'$ and

$$L(g, L(g', f')) = L_{g^{-1}}(L(g', f)) = L_{g^{-1}}(L_{g'^{-1}} f') = (L_{g^{-1}} \circ L_{g'^{-1}})(f') =$$

$$L_{g'^{-1} g^{-1}}(f') = L_{(gg')^{-1}} f' = L(gg', f').$$

   Here we used the fact that $L_g \circ L_h = L_{hg}$ (exercise 7.1).
Now the mapping $L'\colon W_f \to \mathbb{R}$, $L'(f') = f'(e)$ is linear - prove is the same as

in a). For every $g \in G$ we have

$$L'(L(g,f)) = L(g,f)(e) = L_{g^{-1}}f(e) = f(eg^{-1}) = f(g^{-1}).$$

Hence $g \mapsto f(g^{-1})$ is a matrix coefficient of this representation.

2. a) Suppose V is a finite-dimensional vector space, and let $e_1, \dots, e_n$ be basis of V.
The dual space of V is defined as

$$V^* = \{L \colon V \to \mathbb{R} \text{ is linear}\}.$$

V^* is a vector space in a natural way (how?). Recall from linear algebra how the following facts are proved.
(i) Suppose $t_1, \dots, t_n \in \mathbb{R}$ are arbitrary. Then there exists unique $L \in V^*$ such that $L(e_i) = t_i$, $i = 1, \dots, n$.
(ii) By (i) there exists for every $j \in \{1, \dots, n\}$ an element $\varepsilon^j \in V^*$ such that $\varepsilon^j(e_i) = \delta_{ij}$. The set $\{\varepsilon^1, \dots, \varepsilon^n\}$ is a basis of V^*. In particular $\dim V^* = \dim V$.
(iii) Suppose $A \in (V^*)^*$ i.e. a linear mapping $A \colon V^* \to \mathbb{R}$. Then there exists unique $v \in V$ such that $A = A_v$, where $A_v(L) = L(v)$. (Hint: prove that $v \mapsto A_v$ is an injective linear mapping $V \to (V^*)^*$).

b) Suppose $\phi \colon G \to GL(V)$ is a continuous linear representation of a topological group G in a finite-dimensional space V. Define $\hat{\phi} \colon G \to GL(V^*)$ by

$$\hat{\phi}(L)(v) = L(\phi(g^{-1}(v)), L \in V^*, v \in V.$$

Prove that $\hat{\phi}$ is a continuous linear representation of G in V^*.

c) Suppose G is compact and f is a matrix coefficient of the representation $\phi \colon G \to GL(V)$ as in b). Prove that the mapping $g \mapsto f(g^{-1})$ is a matrix representation of $\hat{\phi}$.

Solution: a) The sum of two linear mappings $L, L' \colon V \to \mathbb{R}$ and scalar multiplication are defined pointwise,

$$(L + L')(v) = L(v) + L'(v),$$

$$(cL)(v) = cL(v).$$

We leave it to the reader to verify that these operations are well-defined (i.e. result is always also a linear mapping $V \to \mathbb{R}$ and make V^* a vector space.
i) Suppose $v \in V$. Since $e_1, \dots, e_n$ is a basis there exists unique linear representation of the form

$$v = a_1e_1 + a_2e_2 + \ldots + a_ne_n,$$

where $a_i, i = 1, \dots, n$ are real numbers. Thus if $L \in V^*$ is such that $L(e_i) = t_i$ are fixed, we must (by linearity) have

$$L(v) = \sum_{i=1}^{n} a_i L(v_i),$$

so L is unique. Conversely it is routine exercise in linear algebra to see that L defined by this formula is linear.
(ii) Suppose $L \in V^*$ is a linear combination of the form

$$L = a_1\varepsilon_1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n.$$

Then evaluating at e_i gives us $L(e_i) = a_i$ for all i. Hence the representation is unique and the set $\{\varepsilon^1, \ldots, \varepsilon^n\}$ is thus linearly independent. Conversely if we let $a_i = L(e_i)$ as above and define a mapping $L' = a_1\varepsilon_1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n$, we see that L and L' agree on the basis vectors, so by a) they must be the same. Hence $\{\varepsilon^1, \ldots, \varepsilon^n\}$ is actually a basis. In particular V and V^* have the same dimension.

(iii) We define mapping $A\colon V \to (V^*)^*$ by $A(v) = A_v$, where $A_v(L) = L(v)$. It is easy to verify that this mapping is linear. Moreover it is injective, since if $v \in V$ is such that $L(v) = 0$ for all $L \in V^*$, then $v = 0$. This is true since we can take as L mapping ε^i for every i and hence if $v = a_1e_1 + a_2e_2 + \ldots + a_ne_n$, $a_i = \varepsilon^i(v) = 0$, so v must be zero.
Now by a) $\dim V = \dim V^* = \dim V^{**}$, hence an injective linear mapping A must be also surjective. This proves the claim.

b) First we check that $\hat{\phi}$ is a linear representation algebraically. First of all every mapping $\hat{\phi}(g)$ is a linear mapping $V^* \to V^*$, which is an easy check.
If $g, g' \in G$, then

$$\hat{\phi}(gg')(L) = L \circ \phi((gg')^{-1}) = L \circ \phi(g'^{-1}g^{-1}) = L \circ (\phi(g'^{-1}) \circ \phi(g^{-1})) =$$

$$= (L \circ \phi(g'^{-1})) \circ \phi(g^{-1}) = \hat{\phi}(g')(L) \circ \phi(g^{-1}) = \hat{\phi}(g)(\hat{\phi}(g')(L)) = (\hat{\phi}(g) \circ \hat{\phi}(g'))(L),$$

so

$$\hat{\phi}(gg') = \hat{\phi}(g) \circ \hat{\phi}(g'),$$

also clearly $\hat{\phi}(e) = \mathrm{id}$.
Hence $\hat{\phi}\colon G \to GL(V^*)$ is a well-defined homomorphism of groups.

By exercise 8.4 it is enough to find a basis in V^* such that entries $\hat{\phi}(g)_{ij}$ of the matrix of representation are continuous with respect to that basis. Let $e_1, \ldots, e_n$ be a basis of V and let $\{\varepsilon^1, \ldots, \varepsilon^n\}$ be a dual basis of V^*. It is easy to verify that with respect to this basis $\hat{\phi}(g)_{ij} = \hat{\phi}(g)(\varepsilon^j)(e_i) = \varepsilon^j(\phi(g^{-1})(e_i) = \phi(g^{-1})_{ji}$. In other words the matrix of $\hat{\phi}(g)$ is a transpose of the matrix $\phi(g^{-1}$. Since the entries of the matrix $\phi(g)$ are continuous functions of g (because ϕ is continuous) and $g \mapsto g^{-1}$ is continuous as well, it follows that $\hat{\phi}$ is continuous.

c) Suppose $f(g) = L(\phi(g))v$ for some $L \in V^*$ and $v \in V$. Now $A_v\colon V^* \to \mathbb{R}$ is linear, so

$$f(g^{-1}) = L \circ \phi(g^{-1})(v) = (\hat{\phi}(g)(L))(v) = A_v(\hat{\phi}(g)L),$$

so by definition $g \mapsto f(g^{-1})$ is a matrix coefficient of representation $\hat{\phi}$.

3. Combine the previous exercises and the exercise 8.6 to prove the following important result.

Suppose $f \in Map(G, \mathbb{R})$, G compact group. Then the following conditions are equivalent:
i) f is a matrix coefficient of some continuous linear representation $G \to GL(V)$, V finite-dimensional vector space.
ii) The vector subspace of $Map(G, \mathbb{R})$ spanned by the set $\{R_g \mid g \in G\}$ is finite-dimensional.
iii) The vector subspace of $Map(G, \mathbb{R})$ spanned by the set $\{L_g \mid g \in G\}$ is finite-dimensional.

Solution: In the Exercise 8.6 we have already shown that i) implies ii) and iii). In the exercise 1 we have shown that ii) implies i). Finally suppose iii). Then by exercise 1 mapping $\hat{f}$ defined by $\hat{\phi}(g) = f(g^{-1})$ is a matrix coefficient of a certain linear representation ϕ in a finite-dimensional space V. By the previous exercise the mapping $g \mapsto \hat{f}(g^{-1})$ is a matrix coefficient of a dual representation in V^*. But this mapping is precisely f and dual representation is continuous linear representation in a finite-dimensional vector space, as the previous exercise shows.

4. Suppose V, V' are both finite-dimensional irreducible linear G-spaces. Suppose $L\colon V \to V'$ is linear G-mapping. Prove that either $L = 0$ or L is an isomorphism of vector spaces. (Hint: consider $\operatorname{Ker} L$ and $\operatorname{Im} L$).

   Solution: Suppose $x \in \operatorname{Ker} L$. Then
   $$L(gx) = gL(x) = g \cdot 0 = 0,$$
   since L is G-equivariant and $g\colon V \to V$ is linear. Hence $\operatorname{Ker} L$ is a linear G-subspace of V. Since V is irreducible, $\operatorname{Ker} L = \{0\}$ or $\operatorname{Ker} L = V$. In the latter case $L = 0$ and we are done. In case $\operatorname{Ker} L = \{0\}$ we see that L is injective, and we continue by considering $\operatorname{Im} L$. Suppose $y = L(x)$ for some $x \in V$. Then
   $$gy = gL(x) = L(gx),$$
   so $\operatorname{Im} L$ is also linear G-space. As above we conclude that $\operatorname{Im} L = 0$ or $\operatorname{Im} L = V'$. In the first case $L = 0$ again and second case means that L is a surjection.
   Hence either $L = 0$ or L is a bijection.

5. a) Suppose $(V, \langle\rangle)$ is a finite-dimensional inner-product space and $L \in V^*$. Prove that there exists unique $v \in V$ such that
   $$L(w) = \langle v, w\rangle \text{ for all } w \in V.$$
   Conclude the following: suppose G is a compact group and V is a finite-dimensional linear G-space. Let $\langle , \rangle$ be G-invariant inner product in V. Prove that every matrix coefficient of the corresponding representation can be written in the form
   $$f(g) = \langle gv, w\rangle$$
   for some $v, w \in V$.

b) Suppose $\mathbb{R}^n, \mathbb{R}^m$ are G-spaces, G-compact, $w \in V, w' \in V'$. Prove that the mapping defined by

$$L(v) = \int_G \langle gv, w\rangle g^{-1}w'dg$$

is a linear G-mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$. Here $\langle , \rangle$ is a G-invariant inner product in V.

c) Suppose V, V' are finite-dimensional irreducible G-spaces, G-compact and suppose V and V' are non-equivalent. Suppose f is a matrix coefficient for the representation in V and f' is a matrix coefficient for the representation in V'. Prove that

$$\int_G ff' = 0.$$

(Hint: You may assume $V = \mathbb{R}^n, V' = \mathbb{R}^m$. Define L as above, represent f and f' as in a). Show that then $\int_G ff' = \langle Lv, w\rangle$, so if its not 0, L is not zero mapping, which contradicts previous exercise).

Solution: a) For every $v \in V$ define a mapping $L_v\colon V \to \mathbb{R}$ by $L_v(w) = \langle v, w\rangle$. From the properties of the inner product it follows that L_v is linear, hence $L_v \in V^*$ and we can define a mapping $A\colon V \to V^*$ by $A(v) = L_v$. We are done once we prove that A is a bijection. Since $\dim V = \dim V^* < \infty$, it is enough to prove that A is a linear injection. First we prove that A is linear. Suppose $v, v' \in V, c, d \in \mathbb{R}$. Then, by the bilinearity of the inner product,

$$A(cv + dv')(w) = \langle cv + dv', w\rangle = c\langle v, w\rangle + d\langle v', w\rangle = (cA(v) + dA(v'))(w),$$

so $A(cd + dv') = cA(v) + dA(v')$. Suppose $v \in V$ is such that $A(v) = 0$. Then

$$\langle v, v\rangle = A_v(v) = 0,$$

so again by the properties of the inner product $v \neq 0$. Hence A is injective and we are done.

b)L is well-defined, since the integrand is clearly continuous.

L is linear:

$$L(cv + dv') = \int_G \langle g(cv + dv'), w\rangle g^{-1}w'dg = \int_G \langle cgv + dgv', w\rangle g^{-1}w'dg =$$

$$\int_G (c\langle gv, w\rangle + d\langle gv', w\rangle)g^{-1}w'dg = c\int_G \langle gv, w\rangle g^{-1}w'dg + d\int_G \langle gv', w\rangle g^{-1}w'dg = cL(v) + dL(v'),$$

since inner product is bilinear, every g acts lineary and Haar integral is linear.

Finally L is G-equivariant:

$$L(g'v) = \int_G \langle gg'v, w\rangle g^{-1}w'dg = \int_G \langle gv, w\rangle g'g^{-1}w'dg = g'\int_G \langle gv), w\rangle g^{-1}w'dg = g'L(v).$$

Here we first made a translation change of the form $g \mapsto gg'^{-1}$ in the Haar integral and then used the fact that g' is linear, so commutes with integral.

c)We may assume $V = \mathbb{R}^n$ and $V' = \mathbb{R}^m$, so that we can integrate V and V'-valued functions. By the definition of the matrix coefficient and a) we see that there exists $v, w \in V, v', w' \in V'$, such that

$$f(g) = \langle v, gw \rangle,$$
$$f'(g) = \langle v', gw' \rangle = \langle g^{-1}v', w \rangle,$$

where inner products are G-invariant inner products in V and V'. Now

$$\int_G ff'dg = \int_G \langle v, gw \rangle \langle g^{-1}v', w \rangle dg = \langle \int_G \langle v, gw \rangle g^{-1}v'dg, w' \rangle.$$

Here we used the fact that inner product with fixed w' in V' is linear, so commutes with intergal. Now we can write

$$\int_G ff'dg = \langle L(v), w' \rangle,$$

where $L(v) = \int_G \langle v, gw \rangle g^{-1}v'$ is linear G-mapping by b). By the exercise 4 L is either 0 or an isomorphism. But if its isomorphism, it means that representations are equivalent, contrary to assumptions. Hence $L = 0$, so

$$\int_G ff'dg = \langle L(v), w' \rangle = 0.$$

6. Suppose $V = \mathbb{R}^n$ is a linear G-space, G compact. Define $L \colon V \to V$ by

$$L(v) = \int_G gv\ dg.$$

Prove that L is linear, $L(V) = V^G$ and $L(v) = v$ for all $v \in V^G$.

Solution: Linearity of L follows from linearity of integral and linearity of each $g \colon V \to V$. Suppose $g' \in G$. Then

$$g'L(v) = g' \int_G gv = \int_G g'(gv) = \int_G (g'g)v = \int_G gv = L(v)$$

by invariance of intergal and linearity of g', which implies that it commutes with integral. Hence $L(v) \in V^G$ for all $v \in V$.
Conversely if $v \in V^G$, then $gv = v$ for all $v \in V$, so

$$L(v) = \int_G gv\ dg = \int_G v\ dg = v.$$

In particular $L(V) = V^G$ and $L(v) = v$ for all $v \in V^G$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.