Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 9
26-30.03.2012

1. a) Suppose G is a compact group and $f \in Map(G, \mathbb{R})$. Let V_f be the vector subspace of $Map(G, \mathbb{R})$ spanned by right translations of f i.e. by the set $\{R_g \mid g \in G\}$.
Then the action $R\colon G \times Map(G, \mathbb{R}) \to Map(G, \mathbb{R})$, $R(g, f') = R_g f'$ (which is continuous with respect to the sup-norm, see exercise 7.3) restricts to the action $R\colon R \times V_f \to V_f$.

Now suppose V_f is finite-dimensional. Then the action $R\colon G \times V_f \to V_f$ is continuous with respect to the Euclidean topology in V_f, since all norms in a finite-dimensional space define Euclidean topology (this is proved in Topology I course). Hence R defines a continuous linear representation of G on $GL(V_f)$ (see exercise 8. 4).
Prove that f is a matrix coefficient of this representation (Hint: mapping $V_f \to \mathbb{R}, f' \mapsto f'(e)$ is linear). For the definition of the matrix coefficient see exercise 8.5).

b) Likewise assume $f \in Map(G, \mathbb{R})$ is such that the vector subspace W_f of $Map(G, \mathbb{R})$ spanned by left translations of f i.e. by the set $\{L_g \mid g \in G\}$ is finite-dimensional. Then $L\colon G \times W_f \to W_f$, $L(g, f') = L_{g^{-1}} f'$ is then a continuous (with respect to the Euclidean topology of W_f) linear action of G on finite-dimensional W_f (Check this. Why we use inverse element g^{-1} in the definition of L, instead of simply g, like we did with action R?). Hence it defines a continuous linear representation of G in $GL(W_f)$. Prove that the mapping $G \to \mathbb{R}$, $g \mapsto f(g^{-1})$ is a matrix coefficient of this representation.

2. a) Suppose V is a finite-dimensional vector space, and let $e_1, \ldots, e_n$ be basis of V.
The dual space of V is defined as

$$V^* = \{L\colon V \to \mathbb{R} \text{ is linear}\}.$$

V^* is a vector space in a natural way (how?). Recall from linear algebra how the following facts are proved.
(i) Suppose $t_1, \ldots, t_n \in \mathbb{R}$ are arbitrary. Then there exists unique $L \in V^*$ such that $L(e_i) = t_i$, $i = 1, \ldots, n$.
(ii) By (i) there exists for every $j \in \{1, \ldots, n\}$ an element $\varepsilon^j \in V^*$ such that $\varepsilon^j(e_i) = \delta_{ij}$. The set $\{\varepsilon^1, \ldots, \varepsilon^n\}$ is a basis of V^*. In particular $\dim V^* = \dim V$.
(iii) Suppose $A \in (V^*)^*$ i.e. a linear mapping $A\colon V^* \to \mathbb{R}$. Then there exists unique $v \in V$ such that $A = A_v$, where $A_v(L) = L(v)$. (Hint: prove that

$v \mapsto A_v$ is an injective linear mapping $V \to (V^*)^*$).

b) Suppose $\phi\colon G \to GL(V)$ is a continuous linear representation of a topological group G in a finite-dimensional space V. Define $\hat{\phi}\colon G \to GL(V^*)$ by

$$\hat{\phi}(L)(v) = L(\phi(g^{-1}(v)), L \in V^*, v \in V.$$

Prove that $\hat{\phi}$ is a continuous linear representation of G in V^*.

c) Suppose G is compact and f is a matrix coefficient of the representation $\phi\colon G \to GL(V)$ as in b). Prove that the mapping $g \mapsto f(g^{-1})$ is a matrix representation of $\hat{\phi}$.

3. Combine the previous exercises and the exercise 8.6 to prove the following important result.

   Suppose $f \in Map(G, \mathbb{R})$, G compact group. Then the following conditions are equivalent:
   i) f is a matrix coefficient of some continuous linear representation $G \to GL(V)$, V finite-dimensional vector space.
   ii) The vector subspace of $Map(G, \mathbb{R})$ spanned by the set $\{R_g \mid g \in G\}$ is finite-dimensional.
   iii) The vector subspace of $Map(G, \mathbb{R})$ spanned by the set $\{L_g \mid g \in G\}$ is finite-dimensional.

4. Suppose V, V' are both finite-dimensional irreducible linear G-spaces. Suppose $L\colon V \to V'$ is linear G-mapping. Prove that either $L = 0$ or L is an isomorphism of vector spaces. (Hint: consider $\operatorname{Ker} L$ and $\operatorname{Im} L$).

5. a) Suppose $(V, \langle\rangle)$ is a finite-dimensional inner-product space and $L \in V^*$. Prove that there exists unique $v \in V$ such that

$$L(w) = \langle v, w\rangle \text{ for all } w \in W.$$

   Conclude the following: suppose G is a compact group and V is a finite-dimensional linear G-space. Let $\langle,\rangle$ be G-invariant inner product in V. Prove that every matrix coefficient of the corresponding representation can be written in the form

$$f(g) = \langle gv, w\rangle$$

   for some $v, w \in V$.

   b) Suppose $\mathbb{R}^n, \mathbb{R}^m$ are G-spaces, G-compact, $w \in V, w' \in V'$. Prove that the mapping defined by

$$L(v) = \int_G \langle gv, w\rangle g^{-1} w' dg$$

   is a linear G-mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$. Here $\langle,\rangle$ is a G-invariant inner product in V.

c) Suppose V, V' are finite-dimensional irreducible G-spaces, G-compact and suppose V and V' are non-equivalent. Suppose f is a matrix coefficient for the representation in V and f' is a matrix coefficient for the representation in V'. Prove that

$$\int_G ff' = 0.$$

(Hint: You may assume $V = \mathbb{R}^n, V' = \mathbb{R}^m$. Define L as above, represent f and f' as in a). Show that then $\int_G ff' = \langle Lv, w\rangle$, so if its not 0, L is not zero mapping, which contradicts previous exercise).

6. Suppose $V = \mathbb{R}^n$ is a linear G-space, G compact. Define $L\colon V \to V$ by

$$L(v) = \int_G gv\ dg.$$

Prove that L is linear, $L(V) = V^G$ and $L(v) = v$ for all $v \in V^G$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.